EM 3 Section 19: Waves in Conductors: Skin Effect

19. 1. Recap: Waves in conductors

Last time we derived the equation

$$\boxed{\nabla^2 \underline{E} = \mu\epsilon \frac{\partial^2 \underline{E}}{\partial t^2} + \mu\sigma \frac{\partial \underline{E}}{\partial t}} \tag{1}$$

where σ is the conductivity. Substituting a plane wave ansatz

$$\underline{E} = \underline{\tilde{E}}_0 \exp i(\tilde{k}z - \omega t) \tag{2}$$

yields

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega \ . \tag{3}$$

To solve this we have to take a *complex* wavenumber

$$\tilde{k} = k + i\kappa \tag{4}$$

Equating the real and imaginary parts in (3) yields

$$k^2 - \kappa^2 = \mu\epsilon\omega^2 \tag{5}$$

$$2k\kappa = \mu\sigma\omega \ . \tag{6}$$

The second equation can be solved for $\kappa = \dfrac{\mu\sigma\omega}{2k}$ then eliminating κ from (5) yields

$$k^4 - \left(\frac{\mu\sigma\omega}{2}\right)^2 = \mu\epsilon\omega^2 k^2$$

This is a quadratic in k^2 with solution

$$\begin{aligned} k^2 &= \frac{1}{2}\mu\epsilon\omega^2 + \frac{1}{2}\left((\mu\epsilon\omega^2)^2 + (\mu\sigma\omega)^2\right)^{1/2} \\ &= \frac{\mu\epsilon\omega^2}{2}\left[\left(1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2\right)^{1/2} + 1\right] \end{aligned} \tag{7}$$

(we have taken the positive square root so that the solution for k^2 is positive). Then we can use (5) to obtain

$$\kappa^2 = \frac{\mu\epsilon\omega^2}{2}\left[\left(1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2\right)^{1/2} - 1\right] \ . \tag{8}$$

Now the complex wavenumber (4) implies

$$\underline{E} = \underline{\tilde{E}}_0 \mathrm{e}^{-\kappa z} \mathrm{e}^{i(kz - \omega t)} \ . \tag{9}$$

The first exponential decays with z and causes *attenuation* of the wave. The characterisitic distance over which the wave decays is known as the *skin depth* and is given by

$$\boxed{\delta = \frac{1}{\kappa}} \tag{10}$$

Thus the skin depth is the typical distance a wave penetrates into a conductor.

In the result (8) the ratio $\frac{\sigma}{\epsilon\omega}$ is significant. $1/\omega$ has the dimensions of time as does ϵ/σ. Thus this quantity is a ratio of two timescales.

19. 2. Good and poor conductors

In order to understand the timescale ϵ/σ let us return to the continuity equation for free charge

$$\frac{\partial \rho_f}{\partial t} = -\underline{\nabla} \cdot \underline{J}_f \tag{11}$$

Using Ohm's law and Gauss's law (plus linear media property)

$$\underline{\nabla} \cdot \underline{J}_f = \sigma \underline{\nabla} \cdot \underline{E} = \frac{\sigma}{\epsilon} \underline{\nabla} \cdot \underline{D} = \frac{\sigma}{\epsilon} \rho_f \ .$$

So finally

$$\frac{\partial \rho_f}{\partial t} = -\frac{\sigma}{\epsilon} \rho_f$$

which has solution

$$\rho_f(t) = \rho_f(0) \mathrm{e}^{-(\sigma/\epsilon)t} \ .$$

So the free charge density decays on a timescale $\tau = \dfrac{\epsilon}{\sigma}$ which is the *relaxation time*. If this is small then any free excess charge is quickly rearranged away and the medium is a good conductor. A perfect conductor would have this timescale tending to zero i.e. $\sigma \to \infty$.

On the other hand if τ is large, free charge hangs around for a long time and the medium is a poor conductor.

Let us return to the quantity $\frac{\sigma}{\epsilon\omega}$ that appears in (8) which we may write using the relaxation time τ and period $T = 2\pi/\omega$ as

$$\frac{\sigma}{\epsilon\omega} = \frac{1}{2\pi} \frac{T}{\tau} \ .$$

we see that is (roughly) the ratio of the oscillation period of the wave to the charge relaxation time in the conductor. If, for a given frequency ω, this ratio is large the medium is a good conductor, whereas if the ratio is small the medium is a poor conductor for that frequency.

In the tutorial you are invited to work out the different limits. One finds from (8) that the skin depth

$$\delta \simeq \left(\frac{2}{\mu\omega\sigma} \right)^{1/2} \quad \text{for} \quad \sigma \gg \epsilon\omega$$

$$\delta \simeq \left(\frac{4\epsilon}{\mu\sigma^2} \right)^{1/2} \quad \text{for} \quad \sigma \ll \epsilon\omega$$

Thus the skin depth is much smaller for a good conductor. Also note that for a poor conductor the behaviour does not depend on frequency.

Typical metals are good conductors up to about 1 MHz

$\delta \simeq$ 1cm at 50 Hz (mains frequency)

$\delta \simeq$ 10 μm at 50 MHz

Consequences / Applications of Skin effect

- shielding of sensitive electronics (metal casework)
- power lines and cable design: conductors > 1cm thick are wasted since the current resides only in the skin layer around the outside and there is a 'dead zone' in the centre
- submarines can't use radio
- mobile phones don't work inside metal boxes (so paint concert halls with metal paint?)
- microwave oven doors: metal mesh stops radiation escaping, holes $\ll \lambda$ are OK

19. 3. Phase lag of magnetic field

MI and MII imply further constraints on our wave. As usual

$$i\underline{\tilde{k}} \cdot \underline{\tilde{E}}_0 = 0 \qquad i\underline{\tilde{k}} \cdot \underline{\tilde{B}}_0 = 0$$

Take the direction of propagation $\underline{\tilde{k}}$ in the $\underline{e}_z$ direction and $\underline{\tilde{E}}_0$ in the $\underline{e}_x$ direction. Substituting in MIII

$$\begin{aligned} i\underline{\tilde{k}} \times \underline{\tilde{E}}_0 &= i\omega \underline{\tilde{B}}_0 \\ \Rightarrow \quad \underline{\tilde{B}}_0 &= \frac{\tilde{k}\tilde{E}_0}{\omega} \underline{e}_y \, . \end{aligned} \tag{12}$$

However, $\tilde{k}$ is complex so $\tilde{E}_0$ and $\tilde{B}_0$ will also be complex. Let us write

$$\tilde{k} = R\mathrm{e}^{i\phi}$$

Then using (7,8)

$$\begin{aligned} R &= \left(k^2 + \kappa^2\right)^{1/2} = (\mu\epsilon\omega^2)^{1/2} \left(1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2\right)^{1/4} \\ \phi &= \tan^{-1}\left(\frac{\kappa}{k}\right) = \tan^{-1}\left[\frac{\left(1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2\right)^{1/2} - 1}{\left(1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2\right)^{1/2} + 1}\right]^{1/2} \end{aligned}$$

For a good conductor

$$\phi \to \tan^{-1}[1] = \pi/4$$

and

$$\tilde{k} \simeq (\mu\omega\sigma)^{1/2} \mathrm{e}^{i\pi/4} \tag{13}$$

The vectors $\underline{\tilde{E}}_0$, $\underline{\tilde{B}}_0$ are also complex. Let us write

$$\tilde{E}_0 = E_0 \mathrm{e}^{i\delta_E} \qquad \tilde{B}_0 = B_0 \mathrm{e}^{i\delta_B} \tag{14}$$

Putting these in (12) yields

$$B_0 e^{i\delta_B} = \frac{Re^{i\phi}}{\omega} E_0 e^{i\delta_E} \tag{15}$$

$$\Rightarrow \delta_B - \delta_E = \phi \tag{16}$$

Condition (16) means that the magnetic field lags behind the electric field by angle ϕ.

Finally taking the real part to get real fields we have

$$\underline{E} = E_0 e^{-\kappa z} \cos(kz - \omega t + \delta_E)\underline{e}_x \tag{17}$$

$$\underline{B} = B_0 e^{-\kappa z} \cos(kz - \omega t + \delta_E + \phi)\underline{e}_y \tag{18}$$

Figure 1: Electric and magnetic fields and the skin depth *(Griffiths fig 9.18)*

19. 4. Intrinsic Impedance

As we have seen

$$\underline{E} = \underline{e}_x \, \tilde{E}_0 e^{i(kz-\omega t)} \qquad ; \qquad \underline{B} = \underline{e}_y \, \tilde{B}_0 e^{i(kz-\omega t)}$$

where $\tilde{E}_0$ and $\tilde{B}_0$ are complex

Whereas in vacuum $\underline{E}$ and $\underline{H} = \underline{B}/\mu_0$ are *in phase*, here there are not. The complex number

$$\boxed{Z \equiv \frac{\tilde{E}_0}{\tilde{H}_0}} \tag{19}$$

is the **Intrinsic Impedence** of the medium. One can think of it as the generalised resistance (when Z is real it reduces to the resistance). Dimensions are Ω (Ohms): check units $E = V/m; H = A/m \quad \Rightarrow E/H = V/A = \Omega$

In a vacuum

$$\frac{E_0}{H_0} = \frac{E_0 \mu_0}{B_0} = c\mu_0 \equiv Z_{vac} = 377\Omega$$

This is real since $\underline{E}, \underline{H}$ are in phase

In a dielectric

$$\frac{E_0}{H_0} = \frac{E_0 \mu}{B_0} = \left(\frac{\mu_r}{\epsilon_r}\right)^{1/2} Z_{vac}$$

As we have seen in a good conductor we have $\tilde{k} \approx \sqrt{i\mu\omega\sigma}$ (13)

$$Z = \frac{\tilde{E}_0}{\tilde{H}_0} = \frac{\tilde{E}_0 \mu}{\tilde{B}_0} = \frac{\omega\mu}{\tilde{k}} \simeq \left(\frac{\mu\omega}{\sigma}\right)^{1/2} e^{-i\pi/4}$$

which is complex.