EM 3 Section 19: Waves in Conductors: Skin Effect

19. 1. Recap: Waves in conductors

Last time we derived the equation

$$\nabla^2 E = \mu \varepsilon \frac{\partial^2 E}{\partial t^2} + \mu \sigma \frac{\partial E}{\partial t}$$  \hspace{1cm} (1)

where $\sigma$ is the conductivity. Substituting a plane wave ansatz

$$E = \tilde{E}_0 \exp(i(\tilde{k}z - \omega t))$$  \hspace{1cm} (2)

yields

$$\tilde{k}^2 = \mu \varepsilon \omega^2 + i \mu \sigma \omega .$$  \hspace{1cm} (3)

To solve this we have to take a complex wavenumber

$$\tilde{k} = k + i \kappa$$  \hspace{1cm} (4)

Equating the real and imaginary parts in (3) yields

$$k^2 - \kappa^2 = \mu \varepsilon \omega^2$$  \hspace{1cm} (5)

$$2 k \kappa = \mu \sigma \omega .$$  \hspace{1cm} (6)

The second equation can be solved for $\kappa = \frac{\mu \sigma \omega}{2k}$ then eliminating $\kappa$ from (5) yields

$$k^4 - \left( \frac{\mu \sigma \omega}{2k} \right)^2 = \mu \varepsilon \omega^2 k^2$$  \hspace{1cm} (7)

This is a quadratic in $k^2$ with solution

$$k^2 = \frac{1}{2} \mu \varepsilon \omega^2 + \frac{1}{2} \left( (\mu \varepsilon \omega^2)^2 + (\mu \sigma \omega)^2 \right)^{1/2}$$

$$= \frac{\mu \varepsilon \omega^2}{2} \left[ \left( 1 + \left( \frac{\sigma}{\varepsilon \omega} \right)^2 \right)^{1/2} + 1 \right]$$  \hspace{1cm} (7)

(we have taken the positive square root so that the solution for $k^2$ is positive). Then we can use (5) to obtain

$$\kappa^2 = \frac{\mu \varepsilon \omega^2}{2} \left[ \left( 1 + \left( \frac{\sigma}{\varepsilon \omega} \right)^2 \right)^{1/2} - 1 \right] .$$  \hspace{1cm} (8)

Now the complex wavenumber (4) implies

$$E = \tilde{E}_0 e^{-\kappa z} e^{i(kz - \omega t)} .$$  \hspace{1cm} (9)

The first exponential decays with $z$ and causes attenuation of the wave. The characterisitic distance over which the wave decays is known as the skin depth and is given by

$$\delta = \frac{1}{\kappa}$$  \hspace{1cm} (10)
Thus the skin depth is the typical distance a wave penetrates into a conductor.

In the result (8) the ratio $\frac{\sigma}{\epsilon \omega}$ is significant. $1/\omega$ has the dimensions of time as does $\epsilon/\sigma$. Thus this quantity is a ratio of two timescales.

19. 2. **Good and poor conductors**

In order to understand the timescale $\epsilon/\sigma$ let us return to the continuity equation for free charge

$$\frac{\partial \rho_f}{\partial t} = -\nabla \cdot J_f$$

Using Ohm’s law and Gauss’s law (plus linear media property)

$$\nabla \cdot J_f = \sigma \nabla \cdot E = \frac{\sigma}{\epsilon} \nabla \cdot D = \frac{\sigma}{\epsilon} \rho_f .$$

So finally

$$\frac{\partial \rho_f}{\partial t} = -\frac{\sigma}{\epsilon} \rho_f ,$$

which has solution

$$\rho_f(t) = \rho_f(0)e^{-(\epsilon/\sigma)t} .$$

So the free charge density decays on a timescale $\tau = \frac{\epsilon}{\sigma}$ which is the **relaxation time**. If this is small then any free excess charge is quickly rearranged away and the medium is a good conductor. A perfect conductor would have this timescale tending to zero i.e. $\sigma \to \infty$.

On the other hand if $\tau$ is large, free charge hangs around for a long time and the medium is a poor conductor.

Let us return to the quantity $\frac{\sigma}{\epsilon \omega}$ that appears in (8) which we may write using the relaxation time $\tau$ and period $T = 2\pi/\omega$ as

$$\frac{\sigma}{\epsilon \omega} = \frac{1}{\tau} \frac{T}{2\pi} \frac{1}{\tau} .$$

we see that is (roughly) the ratio of the oscillation period of the wave to the charge relaxation time in the conductor. If, for a given frequency $\omega$, this ratio is large the medium is a good conductor, whereas if the ratio is small the medium is a poor conductor for that frequency.

In the tutorial you are invited to work out the different limits. One finds from (8) that the skin depth

$$\delta \simeq \left(\frac{2}{\mu \omega \sigma}\right)^{1/2} \text{ for } \sigma \gg \epsilon \omega$$

$$\delta \simeq \left(\frac{4 \epsilon}{\mu \sigma^2}\right)^{1/2} \text{ for } \sigma \ll \epsilon \omega$$

Thus the skin depth is much smaller for a good conductor. Also note that for a poor conductor the behaviour does not depend on frequency.

Typical metals are good conductors up to about 1 MHz

$\delta \simeq 1\text{cm at 50 Hz (mains frequency)}$

$\delta \simeq 10 \mu\text{m at 50 MHz}$
Consequences / Applications of Skin effect

- shielding of sensitive electronics (metal casework)
- power lines and cable design: conductors > 1cm thick are wasted since the current resides only in the skin layer around the outside and there is a ‘dead zone’ in the centre
- submarines can’t use radio
- mobile phones don’t work inside metal boxes (so paint concert halls with metal paint?)
- microwave oven doors: metal mesh stops radiation escaping, holes ≪ λ are OK

19. 3. Phase lag of magnetic field

MI and MII imply further constraints on our wave. As usual

\[ i \tilde{k} \cdot \tilde{E}_0 = 0 \quad i \tilde{k} \cdot \tilde{B}_0 = 0 \]

Take the direction of propagation \( \tilde{k} \) in the \( \varepsilon_z \) direction and \( \tilde{E}_0 \) in the \( \varepsilon_x \) direction. Substituting in MIII

\[ i \tilde{k} \times \tilde{E}_0 = i \omega \tilde{B}_0 \]

\[ \Rightarrow \tilde{B}_0 = \frac{i \tilde{k} \tilde{E}_0}{\omega \varepsilon_y} \cdot \] (12)

However, \( \tilde{k} \) is complex so \( \tilde{E}_0 \) and \( \tilde{B}_0 \) will also be complex. Let us write

\[ \tilde{k} = Re^{i\phi} \]

Then using (7,8)

\[ R = \left( k^2 + \kappa^2 \right)^{1/2} = (\mu e \omega^2)^{1/2} \left( 1 + \frac{\sigma}{\varepsilon \omega} \right)^{1/4} \]

\[ \phi = \tan^{-1} \left( \frac{\kappa}{k} \right) = \tan^{-1} \left[ \frac{1 + \left( \frac{\sigma}{\varepsilon \omega} \right)^2}{1 + \left( \frac{\sigma}{\varepsilon \omega} \right)^2} \right]^{1/2} \]

For a good conductor

\[ \phi \rightarrow \tan^{-1}[1] = \pi/4 \]

and

\[ \tilde{k} \simeq (\mu \omega \sigma)^{1/2} e^{i\pi/4} \] (13)

The vectors \( \tilde{E}_0, \tilde{B}_0 \) are also complex. Let us write

\[ \tilde{E}_0 = E_0 e^{i\delta_x} \quad \tilde{B}_0 = B_0 e^{i\delta_y} \] (14)
Putting these in (12) yields

\[ B_0 e^{i\delta_B} = \frac{R e^{i\phi}}{\omega} E_0 e^{i\delta_E} \]  
(15)

\[ \Rightarrow \delta_B - \delta_E = \phi \]  
(16)

Condition (16) means that the magnetic field lags behind the electric field by angle \( \phi \).

Finally taking the real part to get real fields we have

\[ E = E_0 e^{-\kappa z} \cos(kz - \omega t + \delta_E) \hat{e}_x \]  
(17)

\[ B = B_0 e^{-\kappa z} \cos(kz - \omega t + \delta_E + \phi) \hat{e}_y \]  
(18)

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**Figure 1:** Electric and magnetic fields and the skin depth (Griffiths fig 9.18)

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19. 4. **Intrinsic Impedance**

As we have seen

\[ \hat{E} = \varepsilon_0 \hat{E}_0 e^{i(kz - \omega t)} \quad ; \quad \hat{B} = \varepsilon_0 \hat{B}_0 e^{i(kz - \omega t)} \]

where \( \hat{E}_0 \) and \( \hat{B}_0 \) are complex

Whereas in vacuum \( E \) and \( H = B/\mu_0 \) are *in phase*, here there are not. The complex number

\[ Z \equiv \frac{\hat{E}_0}{\hat{H}_0} \]

(19)

is the **Intrinsic Impedance** of the medium. One can think of it as the generalised resistance (when \( Z \) is real it reduces to the resistance). Dimensions are \( \Omega \) (Ohms): check units \( E = V/m; \, H = A/m \implies E/H = V/A = \Omega \)

In a vacuum

\[ \frac{E_0}{H_0} = \frac{E_0 \mu_0}{B_0} = c \mu_0 \equiv Z_{\text{vac}} = 377\Omega \]

This is real since \( E, H \) are in phase

In a dielectric

\[ \frac{E_0}{H_0} = \frac{E_0 \mu}{B_0} = \left( \frac{\mu_r}{\varepsilon_r} \right)^{1/2} Z_{\text{vac}} \]

As we have seen in a good conductor we have \( \tilde{k} \approx \sqrt{i\mu \omega \sigma} \) (13)

\[ Z = \frac{\tilde{E}_0}{\tilde{H}_0} = \frac{\tilde{E}_0 \mu}{\tilde{B}_0} = \frac{\omega \mu}{k} \approx \left( \frac{\mu \omega}{\sigma} \right)^{1/2} e^{-i\pi/4} \]

which is complex.