**EM 3 Section 20: Reflection at boundaries: normal incidence**

*20. 1. Reminder on plane waves and amplitudes*

Consider a plane polarised wave propagating, as usual, in the $\underline{e}_z$ direction

$$\underline{E} = \underline{E}_0 \mathrm{e}^{i(kz-\omega t)} \qquad \underline{E} = \underline{E}_0 \mathrm{e}^{i(kz-\omega t)}$$

As we have seen Maxwell III implies $ik\underline{e}_z \times \underline{E}_0 = i\omega B_0 \underline{e}_y$. Usually we take $\underline{E}_0 = E_0 \underline{e}_x$ (plane polarised in $x$ direction) and

$$\underline{B}_0 = \frac{kE_0}{\omega}\underline{e}_y \ .$$

Now $E_0$,$B_0$ can, in principle, be complex, as they were for waves in a conductor. Previously we indicated this by a tilde e.g. $\tilde{E}_0$ but to lighten notation we won't do that here and instead just refer to $E_0$ as the complex amplitude; the (real) amplitude is then the modulus $|E_0|$ i.e. $E_0 = |E_0|\mathrm{e}^{i\delta_E}$. Recall that the complex impedance is given by the ratio of complex amplitudes

$$Z = \frac{E_0}{H_0} = \frac{\mu E_0}{B_0} \ .$$

As we have seen complex $Z$ allows a *phase shift* between $\underline{E}$ and $\underline{H}$

*20. 2. Waves at interfaces*

Now consider a plane polarised wave propagating in the $\underline{e}_z$ direction *normal incidence* to an interface and call this $\underline{E}_{inc}$. Generally medium 1 has complex impedance $Z = Z_1$ and medium 2 has complex impedance $Z = Z_2$. We take coordinates: $\underline{e}_x$ along $\underline{E}_{inc}$; $\underline{e}_y$ along $\underline{H}_{inc}$; $\underline{e}_z$ along $\underline{k}_1$ (forming a right handed triad).

We place the boundary at $z = 0$ so that the $x$–$y$ plane is the interface between the two media

Figure 1: Wave at interface between two media *similiar to Griffiths fig. 9.13*

*20. 3. Interfaces between two dielectric media*

It is simplest to start by considering two dielectric media where we have seen that

$$Z_i = v_i \mu_i$$

is real and there is no phase lag between $\underline{E}$ and $\underline{H}$

$$\underline{E}_{inc} = E_I \, \underline{e}_x \, e^{i(k_1 z - \omega t)}$$

$$\underline{H}_{inc} = \frac{E_I}{\mu_i v_1} \, \underline{e}_y \, e^{i(k_1 z - \omega t)}$$

Also we can take the amplitude $E_I$ to be real. Likewise for transmitted and reflected waves (see diagram):

$$\underline{E}_{trans} = E_T \, \underline{e}_x \, e^{i(k_2 z - \omega t)}$$

$$\underline{H}_{trans} = \frac{E_T}{\mu_2 v_2} \, \underline{e}_y \, e^{i(k_2 z - \omega t)}$$

$$\underline{E}_{ref} = E_R \, \underline{e}_x \, e^{i(-k_2 z - \omega t)}$$

$$\underline{H}_{ref} = -\frac{E_R}{\mu_1 v_1} \, \underline{e}_y \, e^{i(-k_2 z - \omega t)}$$

**N.B.** The reflected wave propagates in $-$ve $z$ direction hence *sign switch* in the exponential (so that wave speed is $v = -\omega/k$) and *sign switch* in $\underline{H}_{ref}$ (so that $-\underline{e}_z$, $\underline{E}$, $\underline{H}$ form a right-handed triad).

Now invoke continuity conditions (see sections 17 and 18): $\underline{e}_x$ and $\underline{e}_y$ are both *tangential* to interface and tangential components of $\underline{E}$ and $\underline{H}$ are **continuous**. Note that we assume that there *no surface currents or charges* which is usually the case. Then the continuity conditions become

$\underline{E}_{tan} = E_x$ is continuous

$$\Rightarrow E_I + E_R = E_T$$

$\underline{H}_{tan} = H_y$ is continuous

$$\Rightarrow \frac{E_I}{\mu_1 v_1} - \frac{E_R}{\mu_1 v_1} = \frac{E_T}{\mu_2 v_2}$$

Solve for $E_T$ and $E_R$, knowing $E_I$: add the equations to find

$$\frac{2E_I}{\mu_1 v_1} = \left[\frac{1}{\mu_1 v_1} + \frac{1}{\mu_2 v_2}\right] E_T$$

Also recall that

$$v_i = \frac{1}{\sqrt{\mu_i \epsilon_i}} = \frac{c}{n_i}$$

then the **Amplitude transmission coefficient**

$$t \equiv \frac{E_T}{E_I} = \frac{2}{1+\beta}$$

and the **Amplitude reflection coefficient**

$$r \equiv \frac{E_R}{E_I} = \frac{1-\beta}{1+\beta}$$

where $\beta$ is defined as

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2}$$

Now if the permeabilities $\mu_i = \mu_0$ (non-magnetic media) we find

$$\begin{aligned} r &= \frac{v_2 - v_1}{v_1 + v_2} = \frac{n_1 - n_2}{n_1 + n_2} \\ t &= \frac{2v_2}{v_1 + v_2} = \frac{2n_1}{n_1 + n_2} \end{aligned}$$

So the reflected wave is *in phase* if $v_2 > v_2$ but *out of phase* if $v_2 < v_1$. If $v_2 = v_1$ (two media the same) there is no reflected wave as expected.

**Energy flow**

The Poynting vector is given as usual by

$$\underline{S} = \underline{E} \times \underline{H} = \frac{1}{\mu} \underline{E} \times \underline{B}$$

so the energy flux per unit volume averaged over one period or **intensity** of the wave is given by

$$|\langle \underline{S} \rangle| = \frac{1}{\mu} |\langle \underline{E} \times \underline{B} \rangle| = \frac{1}{\mu v} \frac{E_0^2}{2} = \frac{\epsilon v}{2} E_0^2$$

So $R$ the ratio of reflected to incident intensity and $T$ the ratio of transmitted to incident intensity are given by

$$R = r^2 = \left( \frac{n_1 - n_2}{n_1 + n_2} \right)^2 \qquad T = \frac{\epsilon_2 v_2}{\epsilon_1 v_1} t^2 = \frac{4 n_1 n_2}{(n_1 + n_2)^2}$$

**N.B.** since $R + T = 1$ we recover energy conservation.

*20. 4. General waves at interface: normal incidence*

Basically we now repeat the above calculation but for complex impedance so that there may be phase lag between $\underline{E}$ and $\underline{H}$

$$\begin{aligned} \underline{E}_{inc} &= E_I \, \underline{e}_x \, e^{i(k_1 z - \omega t)} \\ \underline{H}_{inc} &= \frac{E_I}{Z_1} \, \underline{e}_y \, e^{i(k_1 z - \omega t)} \\ \underline{E}_{trans} &= E_T \, \underline{e}_x \, e^{i(k_2 z - \omega t)} \\ \underline{H}_{trans} &= \frac{E_T}{Z_2} \, \underline{e}_y \, e^{i(k_2 z - \omega t)} \\ \underline{E}_{ref} &= E_R \, \underline{e}_x \, e^{i(-k_2 z - \omega t)} \\ \underline{H}_{ref} &= -\frac{E_R}{Z_1} \, \underline{e}_y \, e^{i(-k_2 z - \omega t)} \end{aligned}$$

We again assume that there *no surface currents or charges* and the continuity conditions reduce to $\underline{E}_{tan} = E_x$ continuous and $\underline{H}_{tan} = H_y$ continuous

$$\begin{aligned} E_I + E_R &= E_T \\ \frac{E_I}{Z_1} - \frac{E_R}{Z_2} &= \frac{E_T}{Z_2} \end{aligned}$$

Solve for $E_T$ and $E_R$, knowing $E_I$ as before

$$t \equiv \frac{E_T}{E_I} = \frac{2Z_2}{Z_2 + Z_1}$$

$$r \equiv \frac{E_R}{E_I} = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

**N.B.** These are now *complex* quantities

*20. 5. Reflection at Conducting Surface: why metals are shiny*

The $x$–$y$ plane is a boundary between vacuum (medium 1) and a conductor (medium 2).

$$Z_1 = Z_{vac} = 377\Omega$$

$$Z_2 = \sqrt{\frac{-i\mu\omega}{\sigma}} = \frac{1-i}{\sigma\delta}$$

where $\delta = \sqrt{2/\mu\sigma\omega}$ is skin depth

$Z_2$ is complex and $\omega$-dependent. But typical magnitude is tiny... e.g. Cu at $10^{10}$ Hz:

$$|Z_2| = 0.036\Omega = 10^{-4} Z_{vac}$$

and at $10^{15}$ Hz (visible light frequency)

$$|Z_2| = 3.6\Omega = 0.01 Z_{vac}$$

Amplitude reflection (note phase reversal)

$$r = \frac{Z_2 - Z_1}{Z_2 + Z_1} \simeq -1$$

to within (complex) terms of order 1 percent

Near perfect reflection (with phase reversal) is exhibited by good conductor— this explains why metals are shiny.

Physical origin is the skin effect; transmitted wave decays like $e^{-z/\delta}$, almost all the energy you put in comes back out

**Energy Flow**

With complex impedances we need to bit more careful with the Poynting vector. Generally we use the *time-averaged* Poynting vector which is given by

$$\langle \underline{S} \rangle = \hat{\underline{k}} \frac{1}{2} \Re \left( \frac{1}{Z} \right) |E_0|^2$$

and the **intensity** is given by its magnitude

$$|\langle \underline{S} \rangle| = \frac{1}{2} \Re \left( \frac{1}{Z} \right) |E_0|^2$$