EM 3 Section 9: Applications of Ampère's Law; Magnetic Vector Potential

9. 1. Applications of Ampère's Law

$$\boxed{\oint_C \underline{B}\cdot \mathrm{d}\underline{l} = \mu_0 \int_A \underline{J}\cdot \mathrm{d}\underline{S} = \mu_0 \mathrm{I}} \tag{1}$$

Like Gauss' law for electric fields Ampère's law is the most efficient way of calculating magnetic fields *when the system has some symmetry.* The symmetries which work are

- Infinite straight lines (see straight wire example from last lecture)
- Infinite planes (see next example of current sheet)
- Infinite solenoids (see tutorial 5.1)
- Toroids (see toroidal example below)

The difficult part is working out the *direction* of the magnetic field; after that Ampere's law readily gives the answer by choosing the Amperian loop appropriately.

Field of an infinite slab of current (*Griffiths Example 5.8*)

An infinite sheet of conductor of thickness d, carries a uniform current density $\underline{J}$ parallel to the surface of the sheet. Let us take $\underline{e}_z$ normal to the sheet and choose $\underline{e}_x$ to be along the

Figure 1: Infinite current sheets and Amperian loops (Griffiths Fig 5.33)

direction of the current. By B-S law $\underline{B}$ field has to be perpendicular to $\underline{J}$ (i.e. in $y--z$ plane). Now the symmetry of the infinite plane means that any component of $\underline{B}$ in the $\underline{e}_z$ direction cancels. Thus the planar symmetry implies that $\underline{B}$ is in the $\underline{e}_y$ direction i.e. $\|$ to plane and $\perp$ to current.

We take the integral round a rectangular loop $\|$ to the y–z plane loop of length l and height h enclosing the sheet. The magnetic field *outside* the slab is then:

$$2|B_y|l = \mu_0 J l d \qquad |B_y| = \frac{\mu_0 J d}{2}$$

This is a uniform magnetic field but *note that the directions of the field on the two sides of the sheet are opposite to each other!*

$$\underline{B} = -\frac{\mu_0 J d}{2}\underline{e}_y \quad \text{for} \quad z > d/2 \qquad \underline{B} = +\frac{\mu_0 J d}{2}\underline{e}_y \quad \text{for} \quad z < -d/2 \tag{2}$$

The field *inside* the conducting sheet can also be calculated by choosing a loop in the y–z plane that straddles the surface of the sheet. Then using the above result for the portion outside yields that *inside* the slab

$$B_y = -\mu_0 J z \qquad |z| < d/2$$

where $z = 0$ is at the centre of the sheet. (Exercise)

Field of a toroid (*Griffiths Example 5.10***)**

A toroid consists of a set of coils of radius R, carrying a current I, and formed into a larger circle of radius a, so that they look like a doughnut. There are n coils per unit length around the larger circle. The toroidal symmetry is a little subtle: there is clearly symmetry with

Figure 2: Doughnut shaped toroid (see Griffiths Fig 5.39 for more general toroid)

respect to rotation about z axis (no dependence on ϕ) but also since the current always flows in the $\underline{e}_\rho - \underline{e}_z$ plane one can deduce from the BS law that the field must always be in the $\underline{e}_\phi$ direction i.e. it is circumferential (since other components cancel).

Griffiths Ex 5.10 gives a proof of this for a toroid of arbitrary cross-section.

Then we take our Amperian loops to be circles $\perp$ to $\underline{e}_z$. If the circle is not enclosed by the toroid, the current which cuts the circle is zero. Therefore $\underline{B} = 0$ outside the toroid.

If the Amperian loop is a circle *enclosed by the toroid* of radial distance from z-axis ρ, then

$$B_\phi 2\pi\rho = \mu_0 n 2\pi a \mathrm{I}$$

Note that the rhs is constant since the same number of turns is always enclosed by such a loop. The field inside the toroid coils is:

$$B_\phi = \frac{\mu_0 n \mathrm{I} a}{\rho} \tag{3}$$

Note that this is not uniform, but depends on ρ the radial distance from the z-axis.

9. 2. The Magnetic Vector Potential

Just as the theorem of 1.7 was the heart of Electrostatics the following theorem is the heart of magnetostatics:

Theorem The following three statements concerning a vector field $\underline{B}$ over some region in space are equivalent

1. $\underline{\nabla} \cdot \underline{B} = 0$ *the vector field is "solenoidal"*
2. $\underline{B} = \underline{\nabla} \times \underline{A}$ *the vector field may be written as the curl of a* **vector potential**
3. the surface integral of the field $\int_S \underline{B} \cdot \mathrm{d}\underline{S}$ is independent of the shape of the surface S for a given boundary curve; a consequence is $\oint_A \underline{B} \cdot \mathrm{d}\underline{S} = 0$ for any *closed surface* A

We do not prove all the equivalences (see Griffiths 1.6) but it is clear that 2. implies 1. since 'div curl =0'

$$\underline{\nabla} \cdot \underline{B} = \underline{\nabla} \cdot (\underline{\nabla} \times \underline{A}) = 0 \tag{4}$$

Thus starting from the key property of the magnetic field is $\underline{\nabla} \cdot \underline{B} = 0$ (no monopoles), we find from 2. that we may always write the magnetic field as the curl of a vector potential $\underline{A}$

$$\boxed{\underline{B} = \underline{\nabla} \times \underline{A}} \tag{5}$$

This is our key result (**c.f.** $\underline{E} = -\underline{\nabla} V$ for a static electric field).

Finally 3. gives the integral form of Gauss' law for magnetic fields

Using Stokes' theorem

$$\Phi_B \equiv \int_S \underline{B} \cdot \mathrm{d}\underline{S} = \oint_C \underline{A} \cdot \mathrm{d}\underline{l} \tag{6}$$

The magnetic flux through a surface is given by the integral of the magnetic vector potential around the loop enclosing that surface.

9. 3. Poisson's equation for the vector potential

Ampère's law can be written in the form:

$$\underline{\nabla} \times \underline{B} = \underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = \mu_0 \underline{J}$$

Using a vector operator identity for "curlcurl" (see lecture 1) this becomes:

$$\nabla^2 \underline{A} - \underline{\nabla}(\underline{\nabla} \cdot \underline{A}) = -\mu_0 \underline{J} \tag{7}$$

In the same way as we are free to *choose* the value of the scalar potential in electrostatics to be $V(\infty) = 0$, we are free to *choose* the divergence of the magnetic vector potential.
This property is known as gauge invariance.

The choice of $\underline{\nabla} \cdot \underline{A} = 0$ is known as the **Coulomb gauge**. It leads from (7) to Poisson's equation for the magnetic vector potential:

$$\boxed{\nabla^2 \underline{A} = -\mu_0 \underline{J}} \tag{8}$$

Equation 8 implies three equations, one for each component of the vector potential:

$$\nabla^2 A_x = -\mu_0 J_x \qquad \nabla^2 A_y = -\mu_0 J_y \qquad \nabla^2 A_z = -\mu_0 J_z \tag{9}$$

Assuming that $\underline{J}$ goes to zero at infinity we can read off the solution using our knowledge of the solution of Poisson's equation for such a boundary condition

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \int \frac{\underline{J}(\underline{r}')}{|\underline{r} - \underline{r}'|} \mathrm{d}V' \tag{10}$$

The equivalent of the expression the electrostatic potential from a charge can be written down for the magnetic vector potential at $\underline{r}$ due to a current element $I\underline{\mathrm{d}l}'$ or $\underline{J}\mathrm{d}V'$ at $\underline{r}'$:

$$\underline{\mathrm{d}A}(\underline{r}) = \frac{\mu_0 \mathrm{I}(\underline{r}')\underline{\mathrm{d}l}'}{4\pi|\underline{r} - \underline{r}'|} = \frac{\mu_0 \underline{J}(\underline{r}')\mathrm{d}V'}{4\pi|\underline{r} - \underline{r}'|} \tag{11}$$

Note that the direction of $\underline{\mathrm{d}A}$ is *parallel to the current element whereas* $\underline{\mathrm{d}B}$ is *perpendicular* by B-S law.

Example: vector potential of magnetic dipole (see tutorial)

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \frac{\underline{m} \times \hat{\underline{r}}}{r^2} \tag{12}$$

9. 4. Pause for thought and summary of statics

Electrostatics: Stationary charges $\dfrac{\partial \rho}{\partial t} = 0$ are source of electric fields

$$\underline{\nabla} \cdot \underline{E} = \frac{\rho}{\epsilon_0} \qquad \text{M1} \tag{13}$$

Coulomb's law (field due to point charge) leads to

$$\underline{\nabla} \times \underline{E} = 0 \quad \text{MIII} \quad \Rightarrow \quad \underline{E} = -\underline{\nabla} V$$

In turn the above lead to Poisson's equation for the scalar potential V

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

Magnetostatics: Steady current loops $\dfrac{\partial \underline{J}}{\partial t} = 0$ are source of magnetic fields (no magnetic monopoles).

$$\underline{\nabla} \cdot \underline{B} = 0 \qquad \text{MII} \tag{14}$$

Biot-Savart law (field due to current element) leads to

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J} \quad \text{MIV} \quad \text{and} \quad \underline{B} = \underline{\nabla} \times \underline{A}$$

In turn in the Coulomb gauge the above lead to a vector Poisson equation for $\underline{A}$

$$\nabla^2 \underline{A} = -\mu_0 \underline{J}$$

In the following we shall see how MIII and MIV need to be modified when *time-varying fields* are present.