EM 3 Section 1: Revision: Whistlestop tour of Vector Calculus

You will have met vector calculus last year in Maths for Physics 4. This year we shall see the true utility and power of vector calculus in formulating electrostatics. You need to revise *div, grad, curl!* The following highlights some keypoints but does not replace your second year notes.

1. 1. Gradient

The gradient operator ("grad") acting on a scalar field $f(\underline{r})$ is a vector which in Cartesian Co-ordinates (x,y,z) reads

$$\boxed{\underline{\nabla} f = \frac{\partial f}{\partial x}\underline{e}_x + \frac{\partial f}{\partial y}\underline{e}_y + \frac{\partial f}{\partial z}\underline{e}_z} \tag{1}$$

Important things to remember:

- $\underline{\nabla} f$ is a **vector** quantity (vectors either underlined or boldface in these notes)
- $\underline{\nabla} f$ points in the direction of maximum increase of f
- $\underline{\nabla} f$ is perpendicular to the level surfaces of f
- For a small change of position $\underline{\mathrm{d}r}$ the change in f is $\mathrm{d}f = \underline{\nabla} f \cdot \underline{\mathrm{d}r}$
- The line integral $\int_A^B \underline{\nabla} f \cdot \underline{\mathrm{d}l} = f_B - f_A$ is independent of the path from A to B

Simple example to be memorised $\underline{\nabla} r = \hat{\underline{r}}$.

Remark Often due to the symmetry of the problem it is convenient to consider other co-ordinate systems such as *spherical polar coordinates* which comprise (r, ϕ, θ) or *cylindrical polar coordinates* which comprise (ρ, ϕ, z) (you should remind yourselves of these co-ordinate systems). In these systems the expression for the gradient (and the other operations below) look more complicated e.g. in spherical polars

$$\underline{\nabla} f = \frac{\partial f}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\underline{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\underline{e}_\phi$$

(where $\underline{e}_r = \hat{\underline{r}}$). But when the system has a spherical symmetry $f = f(r)$ (no θ or ϕ dependence) the gradient is simply $\underline{\nabla} f = \frac{\partial f}{\partial r}\underline{e}_r$.

This is consistent with the *chain rule* which states

$$\boxed{\underline{\nabla} f(r) = \frac{\mathrm{d}f}{\mathrm{d}r}\underline{\nabla} r = \frac{\mathrm{d}f}{\mathrm{d}r}\hat{\underline{r}}} \tag{2}$$

Important example: $\underline{\nabla}\left(\frac{1}{r}\right) = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\right)\hat{\underline{r}} = -\frac{1}{r^2}\hat{\underline{r}}$

1. 2. Divergence and the Divergence Theorem

The divergence ("div") is a scalar product $\underline{\nabla}\cdot$ of the gradient operator with a vector field $\underline{K}$. In Cartesians it reads

$$\boxed{\underline{\nabla}\cdot\underline{K} = \frac{\partial K_x}{\partial x} + \frac{\partial K_y}{\partial y} + \frac{\partial K_z}{\partial z}} \qquad (3)$$

The divergence represents the rate with which flux lines of the vector field $\underline{K}$ are converging towards sinks (negative divergence), or diverging from sources (positive divergence).

Simple example: $\underline{\nabla}\cdot\underline{r} = \dfrac{\partial x}{\partial x} + \dfrac{\partial y}{\partial y} + \dfrac{\partial z}{\partial z} = 3$

The **divergence theorem** states that:

$$\boxed{\oint_A \underline{K}\cdot \mathrm{d}\underline{S} = \int_V \underline{\nabla}\cdot\underline{K}\,\mathrm{d}V} \qquad (4)$$

where A is a closed surface enclosing a volume V, $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ is a volume element (sometimes written d^3r), and $\mathrm{d}\underline{S}$ is a vector element of area (normal to the surface).

This theorem holds for any vector field $\underline{K}$ and any closed surface A.

1. 3. Curl and Stokes' Theorem

The curl operator is a vector product of the gradient operator $\underline{\nabla}\times$ with a vector field $\underline{K}$:

$$\underline{\nabla}\times\underline{K} = \left[\frac{\partial K_z}{\partial y} - \frac{\partial K_y}{\partial z}\right]\underline{e_x} + \left[\frac{\partial K_x}{\partial z} - \frac{\partial K_z}{\partial x}\right]\underline{e_y} + \left[\frac{\partial K_y}{\partial x} - \frac{\partial K_x}{\partial y}\right]\underline{e_z} \qquad (5)$$

or

$$\boxed{\underline{\nabla}\times\underline{K} = \begin{vmatrix} \underline{e_x} & \underline{e_y} & \underline{e_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ K_x & K_y & K_z \end{vmatrix}} \qquad (6)$$

The curl represents the curvature of the vector field $\underline{K}$ around an axis of rotation. Using the corkscrew rule, clockwise(anticlockwise) rotation has a curl in the negative(positive) direction along the axis of rotation.

Simple example to be memorised: $\boxed{\underline{\nabla}\times\underline{r} = 0}$

Stokes's theorem states that:

$$\boxed{\oint_C \underline{K} \cdot \mathrm{d}\underline{l} = \int_A \underline{\nabla} \times \underline{K} \cdot \mathrm{d}\underline{S}} \tag{7}$$

where C is a closed contour bounding a surface A.

This theorem holds for any vector field $\mathbf{K}$ and any closed curve C.

1. 4. Laplacian

The Laplacian of a scalar field is a scalar defined as

$$\nabla^2 f = \underline{\nabla} \cdot (\underline{\nabla} f) \tag{8}$$

and reads in Cartesians

$$\boxed{\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}} \tag{9}$$

1. 5. Useful Identities

These are best proved by suffix notation (see MfP4).

First, there are various product identities. Generally these are as you'd expect,

1. $\underline{\nabla}(\phi f) = \phi \underline{\nabla} f + (\underline{\nabla}\phi) f$
2. $\underline{\nabla} \cdot (\phi\, \underline{A}) = \phi\, \underline{\nabla} \cdot \underline{A} + \underline{A} \cdot \underline{\nabla}\phi$
3. $\underline{\nabla} \times (\phi\, \underline{A}) = \phi\, (\underline{\nabla} \times \underline{A}) + (\underline{\nabla}\phi) \times \underline{A}$

You should be able to write these down.

Others are less obvious and do not need to be memorised:

4. $\underline{\nabla}\, (\underline{A} \cdot \underline{B}) = (\underline{A} \cdot \underline{\nabla})\, \underline{B} + (\underline{B} \cdot \underline{\nabla}) \underline{A} + \underline{A} \times (\underline{\nabla} \times \underline{B}) + \underline{B} \times (\underline{\nabla} \times \underline{A})$
5. $\underline{\nabla} \cdot (\underline{A} \times \underline{B}) = \underline{B} \cdot (\underline{\nabla} \times \underline{A}) - \underline{A} \cdot (\underline{\nabla} \times \underline{B})$
6. $\underline{\nabla} \times (\underline{A} \times \underline{B}) = \underline{A}\, (\underline{\nabla} \cdot \underline{B}) - \underline{B}\, (\underline{\nabla} \cdot \underline{A}) + (\underline{B} \cdot \underline{\nabla})\, \underline{A} - (\underline{A} \cdot \underline{\nabla})\, \underline{B}$

Second there are some simple identities (involving two grads) that prove fundamental to Electromagnetism

- "curl grad = 0"

  $$\underline{\nabla} \times (\underline{\nabla} f) = 0 \tag{10}$$

  where $f(\underline{r})$ is any scalar field.

- "div curl = 0"
$$\underline{\nabla}.(\underline{\nabla} \times \underline{K}) = 0 \tag{11}$$
where $\underline{K}(\underline{r})$ is any vector field.

- "curl curl = grad div - delsquared"
$$\underline{\nabla} \times (\underline{\nabla} \times \underline{K}) = \underline{\nabla}(\underline{\nabla}.\underline{K}) - \nabla^2 \underline{K} \tag{12}$$
Note that:
$$\nabla^2 K_x = \frac{\partial^2 K_x}{\partial x^2} + \frac{\partial^2 K_x}{\partial y^2} + \frac{\partial^2 K_x}{\partial z^2}$$

The first two (10,11) are crucial.

*1. 6. * 3d Taylor expansion*

As noted above the change in f due to a small change of position $\underline{\mathrm{d}r}$ is $\mathrm{d}f = \underline{\nabla} f \cdot \underline{\mathrm{d}r}$

This is actually the first term in the 3d Taylor expansion about a point $\underline{r}'$ which may be neatly written

$$\begin{aligned} f(\underline{r}) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left[(\underline{r} - \underline{r}') \cdot \underline{\nabla}\right]^n f(\underline{r})\Big|_{\underline{r}=\underline{r}'} && (13) \\ &= f(\underline{r}_0) + \sum_{i=1}^{3} (x_i - x_i') \left.\frac{\partial f(\underline{r})}{\partial x_i}\right|_{\underline{r}=\underline{r}'} + \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} (x_i - x_i')(x_j - x_j') \left.\frac{\partial^2 f(\underline{r})}{\partial x_j \partial x_i}\right|_{\underline{r}=\underline{r}'} \cdots && (14) \end{aligned}$$

Often the first two terms $f(\underline{r}) \simeq f(\underline{r}_0) + (\underline{r} - \underline{r}_0) \cdot \underline{\nabla} f(\underline{r})\big|_{\underline{r}=\underline{r}_0}$ is all we require.

1. 7. Important Theorem

The following three statements concerning a vector Field $\underline{F}$ over some region in space are equivalent

1. $\underline{\nabla} \times \underline{F} = 0$ *the vector field is irrotational*
2. $\underline{F} = \underline{\nabla}\phi$ *the vector field may be written as the gradient of a scalar field*
3. the line integral of the field $\int_A^B \underline{F} \cdot \underline{\mathrm{d}l}$ is independent of the path from A to B;
   a consequence is $\oint_C \underline{F} \cdot \underline{\mathrm{d}l} = 0$ for any closed curve C

You should remind yourselves of how each implies the other
e.g. Stokes theorem gives 1. $\Leftrightarrow$ 3.

This theorem is the heart of electrostatics.

EM 3 Section 2: Revision of Electrostatics

2. 1. Charge Density

At the microscopic level charge is a discrete property of elementary particles.
The fundamental charge of an electron is $-e$, where $e = 1.6 \times 10^{-19}$C.

The charge of a proton is $+e$: $q_p + q_e < 10^{-21}e$. Antimatter has the opposite charge to matter: $q_p + q_{\bar{p}} < 10^{-8}e$. $\Rightarrow$ **there is no charge in a vacuum!**

Classical electromagnetism deals with macroscopic charge distributions.
These are defined by a **charge density**, ρ with units Cm^{-3}:

$$\rho(\underline{r}) = [N_p(\underline{r}) - N_e(\underline{r})]e$$

where N are the number densities of protons and electrons.

The total charge in a volume V is obtained by integration:

$$\boxed{Q_V = \int_V \rho(\underline{r})\mathrm{d}V} \tag{1}$$

where dV is a small element of volume $dxdydz$ (sometimes we use dτ and sometimes d^3r).

Line charges have a charge density λ with units Cm^{-1}. Surface charges have a charge density σ with units Cm^{-2}. Again the total charge can be obtained by integration:

$$Q_A = \int_A \sigma dS \qquad Q_L = \int_L \lambda dl \tag{2}$$

2. 2. Point charges and δ-function

In electrostatic problems it is common to introduce **point charges** at a particular position $\underline{r}\,'$. These are represented by a delta function:

$$\boxed{\rho(\underline{r}) = Q\delta(\underline{r} - \underline{r}\,')} \tag{3}$$

where:

$$\int_V \delta(\underline{r} - \underline{r}\,') = 1 \quad \text{if } \underline{r}\,' \text{ in } V \qquad \int_V \delta(\underline{r} - \underline{r}\,') = 0 \quad \text{otherwise} \tag{4}$$

N.B. Here we are using the three dimensional delta-function, in Cartesians

$$\delta(\underline{r} - \underline{r}\,') = \delta(x - x')\delta(y - y')\delta(z - z') \tag{5}$$

for this reason one sometimes writes $\delta^3(\underline{r} - \underline{r}\,')$

2. 3. Coulomb's Law

The force between two point charges is given by

$$\boxed{\underline{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{\underline{r}}} \tag{6}$$

where $\hat{\underline{r}}$ is a unit vector indicating that the force acts along the line connecting the two charges.

The constant ϵ_0 is known as **the permittivity of free space**.

$$\epsilon_0 = 8.85 \times 10^{-12}\mathrm{CN^{-1}m^{-2}} \tag{7}$$

A quite accurate and easily remembered number is:

$$\frac{1}{4\pi\epsilon_0} = 9 \times 10^9\mathrm{Nm^2C^{-1}} \tag{8}$$

We'll take Coulomb's law as the empirical starting point for electrostatics. The inverse square dependence on the separation of the charges is measured to an accuracy of 2 ± 10^{-16} using experiments based on the original experiment by Cavendish (Duffin P.31).

Coulomb forces must be added as vectors using the principle of superposition. Thus the force on a point charge q at $\underline{r}$ due to a charge distribution $\rho(\underline{r}')$ is

$$\underline{F} = \frac{q}{4\pi\epsilon_0}\int \frac{(\underline{r} - \underline{r}')}{|\underline{r} - \underline{r}'|^3}\rho(\underline{r}')\mathrm{d}^3 r' \tag{9}$$

Example: Force due to a Line Charge

As an example consider the force of a line charge λ on a point charge Q. This can be obtained from the sum of the contributions:

$$\underline{\mathrm{d}F} = \frac{Q\lambda \underline{\mathrm{d}l}}{4\pi\epsilon_0 r^2}\underline{\hat{r}}$$

Figure 1: diagram of integrating over line charge elements $\underline{\mathrm{d}l}$ at angle θ to point charge

The components *parallel* to the line charge cancel, so we have to sum the contributions *perpendicular* to the line which are $dF\cos\theta$. This yields

$$F_\perp = \int_{-L/2}^{L/2} \frac{Q\lambda \mathrm{d}l}{4\pi\epsilon_0} \frac{\cos\theta}{(l^2+a^2)^2}$$

This integral is best solved by transforming it into an integral over $d\theta$ rather than dl i.e. we make the substitution:

$$l = a\tan\theta \qquad dl = a\sec^2\theta d\theta \qquad r = (l^2+a^2)^{1/2} = a\sec\theta$$

$$dF_\perp = \frac{Q\lambda}{4\pi\epsilon_0 a}\int_{-\theta_0}^{\theta_0}\cos\theta d\theta = \frac{Q\lambda}{4\pi\epsilon_0 a} 2\sin\theta_0$$

where $L/2 = a\tan\theta_0$ and $\sin\theta_0 = L/2\,(a^2+L^2/4)^{-1/2}$ (see diagram). Thus

$$F_\perp = \frac{Q\lambda L}{4\pi\epsilon_0 a(a^2+L^2/4)^{1/2}} \tag{10}$$

where L is the length of the line charge, a is the distance of Q from the line.

In the limit of an infinite line charge $L\to\infty$:

$$F_\perp = \frac{Q\lambda}{2\pi\epsilon_0 a} \tag{11}$$

Note that the force due to a line charge falls off with distance like $1/a$*!*

In the "far-field" limit where $a \gg L$

$$F_\perp \simeq \frac{Q\lambda L}{2\pi\epsilon_0 a^2} \tag{12}$$

This is the same as the force due to a point charge $q = \lambda L$ at the origin

2. 4. Electric Field

The force that a point charge q experiences is written

$$\boxed{\underline{F} = q\underline{E}} \tag{13}$$

which defines the electric field. The electric field is defined as the force per unit charge experienced by a small static test charge, q, in units of NC^{-1} or more usually Vm^{-1}:

Thus comparing with Coulomb's force law we see that the field at $\underline{r}$ due to a point charge q at the origin is

$$\boxed{\underline{E} = \frac{q}{4\pi\epsilon_0 r^2}\underline{\hat{r}}} \tag{14}$$

which is also known as Coulomb's law.

A positive (negative) point charge is a source (sink) for $\underline{E}$. In a field line diagram field lines

Figure 2: Field lines of electric field emanating from a point charge

begin at positive charges (or infinity) and end at negative charges(or infinity). The density of field lines indicates the strength of the field.

2. 5. Gauss's law for $\underline{E}$

$$\boxed{\oint_A \underline{E} \cdot \mathrm{d}\underline{S} = \frac{Q}{\epsilon_0}} \tag{15}$$

where A is any closed surface the surface, $\mathrm{d}\underline{S}$ is a vector normal to a surface element, and $\epsilon_0 = 8.85 \times 10^{-12}$ F m^{-1}, Q is the total charge enclosed by the surface. $\underline{E}(\underline{r}) \cdot \mathrm{d}\underline{S}$ is the Electric flux through the surface area element at point $\underline{r}$. The left hand side is often written as

$$\Phi_E = \oint_A \underline{E} \cdot \mathrm{d}\underline{S}$$

which is the total flux of the electric field out of the surface..

Thus *the total Electric flux through any closed surface is proportional to the charge enclosed* (not on how it is distributed)

Simple example of Gauss' law

First let us recover Coulomb's law. Consider a point charge $+q$ at the origin. Take the surface as a sphere of radius r. Now by symmetry the field must point radially outwards Thus $\underline{E} = E_r \underline{e}_r$ where E_r has no angular dependence. Then the integral over spherical polar coordinates simplifies considerably

$$\oint_A \underline{E} \cdot \mathrm{d}\underline{S} = \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta r^2 \sin\theta E_r = 4\pi r^2 E_r$$

and we obtain from Gauss' law $E_r = q/4\pi\epsilon_0 r^2$ which is Coulomb's law.

Aside Although here we have simply stated Gauss' law as fundamental it is actually a *consequence* of the Divergence theorem and Coulomb's Law (14).

Now consider a charge density within the sphere i.e. an insulating sphere with a uniform charge density ρ. Again by symmetry the electric field is radial, i.e. $E_\theta = E_\phi = 0$ everywhere. Again using a spherical closed surface of radius r to calculate the electric field E_r:

$$E_r 4\pi r^2 = \frac{\rho}{\epsilon_0}\frac{4}{3}\pi r^3$$

Inside the sphere ($r < a$):

$$E_r = \frac{\rho}{\epsilon_0}\frac{r}{3}$$

At the surface of the sphere ($r = a$):

$$E_r = \frac{\rho}{\epsilon_0}\frac{a}{3} = \frac{3Q}{4\pi\epsilon_0 a^3}\frac{a}{3} = \frac{Q}{4\pi\epsilon_0 a^2}$$

Outside the sphere ($r > a$):

$$E_r = \frac{Q}{4\pi\epsilon_0 r^2}$$

This is the same field as a point charge at the centre of the sphere!

2. 6. Electrostatic Potential

Consider Coulomb's law (14) and the result $\underline{\nabla}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\hat{\underline{r}}$. We can then write

$$\underline{E} = -\underline{\nabla}(\frac{q}{4\pi\epsilon_0 r})$$

and identify the Electrostatic Potential at $\underline{r}$ due to a point charge at $\underline{r}'$

$$\boxed{V(\underline{r}) = \frac{q}{4\pi\epsilon_0|\underline{r} - \underline{r}'|}} \tag{16}$$

Now due to superposition we can integrate Coulomb's law to get the Electric field for any charge distribution and similarly superposition holds for the potential which is then given by e.g. for a continuous charge distribution by

$$V(\underline{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\underline{r}')\mathrm{d}^3 r'}{|\underline{r} - \underline{r}'|} \tag{17}$$

Moreover, we can invoke the important theorem of section 1.7 which implies that due to the existence of the potential V, *static* electric fields generally obey

1. $\underline{\nabla} \times \underline{E} = 0$
2. $\underline{E} = -\underline{\nabla} V$
3. the line integral of the field $\int_A^B \underline{E} \cdot \underline{\mathrm{d}l} = (V_A - V_B)$ is independent of path from A to B
4. a consequence is $\oint_C \underline{E} \cdot \underline{\mathrm{d}l} = 0$ for any closed curve C

N.B this holds only for *static* fields as we shall see later.

N.B. The potential V is only defined up to a constant which may be chosen according to convenience. Often we choose $V = 0$ at $r \to \infty$.

3. gives us the **work done** to move a test charge from A to B

$$\boxed{W_{AB} = -q \int_A^B \underline{E} \cdot \underline{\mathrm{d}l} = q[V_B - V_A]} \tag{18}$$

Note that the work done is defined as the work done by moving the charge *against* the direction of force—hence the minus sign. If we take A at infinity and $V_A = 0$ the work done to move the charge from inifinty to B is the potential energy of the test charge at B

$$\boxed{U = qV_B} \tag{19}$$

Warning Beware of confusing Electrostatic potential V and potential energy U

The potential difference $V_{AB} = V_A - V_B$ is then the energy required to move a test charge between two points A and B, in units of Volts, V=JC^{-1}:

$$V_{AB} = \frac{W_{AB}}{q} \tag{20}$$

An **equipotential** is a surface connecting points in space which have the same potential. By definition $r \to \infty$ is an equipotential with $V(\infty) = 0$.
2. tells us that *The electric field is always perpendicular to an equipotential.*

EM 3 Section 3: Gauss' Law

3. 1. Conductors and Insulators

A conductor is a material in which charges can move about freely. Therefore any electric field forces the charges to rearrange themselves until a static equilibrium is reached. This in turn means that

- Inside a conductor $\underline{E}$=0 everywhere, $\rho = 0$ and any free charges must be on the surfaces.
- Inside a conductor the potential V is constant and the surfaces of a conductor are an equipotential.
- The electric field just outside a conductor must be normal to the surface and proportional to the surface charge density:

$$\underline{E} = \frac{\sigma}{\epsilon_0}\underline{\hat{n}} \tag{1}$$

In an insulator charges cannot move around, and the charge density can have any form. If $\rho(\underline{r}) \neq 0$, the potential is non-uniform, and $\underline{E} \neq 0$ inside the insulator. Insulators are often referred to as 'dielectric' materials and we shall study their properties later on.

3. 2. Gauss' law in differential form

Gauss' law reads

$$\int_A \underline{E}.\underline{\mathrm{d}S} = \frac{Q_{enc}}{\epsilon_0} = \int_V \frac{\rho(\underline{r})}{\epsilon_0}\,\mathrm{d}V$$

for any closed surface A, and enclosed volume V.

Apply **divergence theorem**

$$\begin{aligned} \int_A \underline{E}.\underline{\mathrm{d}S} &= \int_V \underline{\nabla}\cdot\underline{E}\,\mathrm{d}V \\ \Rightarrow \int_V \underline{\nabla}\cdot\underline{E}(\underline{r})\,\mathrm{d}V &= \int_V \frac{\rho(\underline{r})}{\epsilon_0}\,\mathrm{d}V \end{aligned}$$

Since this holds for **any** domain V, however small $\Rightarrow$ integrands are identical!!

$$\boxed{\underline{\nabla}\cdot\underline{E}(\underline{r}) = \frac{\rho(\underline{r})}{\epsilon_0}} \tag{2}$$

This is **Gauss's Law In Differential Form**. It is the first of the fundamental laws of electromagnetism i.e. **Maxwell I**.

NB: for static conductor this proves earlier claim that $\underline{E} = \underline{0} \Rightarrow \rho = 0$

Aside expressions for the divergence in cylindrical and spherical polar coordinates:

$$\underline{\nabla} \cdot \underline{E} = \frac{1}{\rho}\frac{\partial(\rho E_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z} \tag{3}$$

$$\underline{\nabla} \cdot \underline{E} = \frac{1}{r^2}\frac{\partial(r^2 E_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta E_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial \phi} \tag{4}$$

These are nasty and you do not need to remember them, but they simplify in the case of cylindrical or spherical symmetry e.g. for $\underline{E} = E_r \underline{e}_r$, $\underline{\nabla} \cdot \underline{E} = \frac{1}{r^2}\frac{\partial(r^2 E_r)}{\partial r}$.

3. 3. Examples of Gauss's Law

Griffiths 2.2.3 "Gauss's law affords when symmetry permits by far the quickest and easiest way of computing electric fields".

Note well the qualifier *when symmetry permits.*

Basically there are 3 kinds of symmetry which work and for which the following *gaussian surfaces* for the surface integral in Gauss' law are appropriate

1. Spherical symmetry : concentric sphere
2. Cylindrical symmetry : coaxial cylinder
3. Plane symmetry : a "pill box"

Example 1: Insulating sphere

Let us return to the example of the previous lecture i.e. an insulating sphere with a uniform charge density ρ.

Inside the sphere ($r < a$):

$$E_r = \frac{\rho}{\epsilon_0}\frac{r}{3} \qquad \underline{\nabla} \cdot \underline{E} = \frac{1}{r^2}\frac{\partial(r^2 E_r)}{\partial r} = \frac{\rho}{3\epsilon_0}\frac{1}{r^2}\frac{\partial r^3}{\partial r} = \frac{\rho}{\epsilon_0}$$

Outside the sphere ($r > a$):

$$E_r = \frac{Q}{4\pi\epsilon_0 r^2} \qquad \underline{\nabla} \cdot \underline{E} = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{r^2}\right) = 0$$

so Gauss' law holds.

Example 2: Line charge

For an infinite line charge, λ, by symmetry $E_z = E_\phi = 0$, and the closed surface is chosen to be a cylinder of length l and radius a with the line charge as its axis. **N.B.** here ρ is the radial coordinate of cylindrical polars

Figure 3: Diagram of integrating over a cylindrical surface around line charge

$$\Phi_E = E_\rho 2\pi\rho l = \frac{\lambda}{\epsilon_0} l$$

$$E_\rho = \frac{\lambda}{2\pi\epsilon_0\rho}$$

Compare this method to summing the Coulomb forces in the previous lecture!

Example 3: Surface charge

For an infinite surface charge, σ, the closed surface is chosen to be a circular "pillbox" of radius, r, and height h, with its axis normal to the surface and its centre at the surface. Note - from the symmetry of the problem the electric field parallel to the surface is zero.

Figure 4: Diagram of Gaussian pillbox around surface charge sheet

If the surface is a thin **insulating** sheet there are equal and opposite perpendicular electric fields on either side of the sheet:

$$\Phi_E = 2E_z\pi r^2 = \frac{\sigma}{\epsilon_0}\pi r^2 \qquad E_z = \frac{\sigma}{2\epsilon_0} \tag{5}$$

If instead the charge is on the surface of a large **conducting** object, the inside of the conductor has $\underline{E} = 0$, and the only contribution to the flux comes from the electric field normal to the outer surface.

$$E_z = \frac{\sigma}{\epsilon_0} \tag{6}$$

As quoted at the beginning of the lecture.

Note the factor of two between the conducting surface and the thin insulating sheet!

Remark: In both both insulating and conductor cases $\underline{\nabla} \cdot \underline{E} = \dfrac{\partial E_z}{\partial z} = 0$ for $z \neq 0$ but the electric field is *discontinuous* across the charge sheet at $z = 0$ with discontinuity σ/ϵ_0. We can write the Electric field for all z using a step function

$$\Theta(z) = \begin{cases} 1 & \text{for} \quad z > 0 \\ 0 & \text{for} \quad z < 0 \end{cases}$$

e.g. for the conducting sheet

$$E_z = \frac{\sigma}{\epsilon_0}\Theta(z)$$

Then we use the identity

$$\boxed{\delta(z) = \frac{\mathrm{d}}{\mathrm{d}z}\Theta(z)} \tag{7}$$

and find that

$$\underline{\nabla} \cdot \underline{E} = \frac{\partial E_z}{\partial z} = \frac{\sigma}{\epsilon_0}\delta(z) \tag{8}$$

Which is consistent with Equation (2) with a source of charge at $z = 0$.

3. 4. A delta function identity and point charges

Consider the vector field

$$\underline{v} = \frac{\hat{\underline{r}}}{r^2}$$

Now this is a spherically symmetric field with $v_r = 1/r^2$ so using div in spherical polars we get

$$\underline{\nabla} \cdot \underline{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = 0 \tag{9}$$

But clearly the integral over a spherical surface radius r

$$\oint \underline{v} \cdot \mathrm{d}\underline{S} = 4\pi r^2 v_r = 4\pi \tag{10}$$

So Gauss's theorem which should relate the two results appears to yield a contradiction. The source of the problems is $r = 0$ where $\underline{v}$ diverges (is singular).

The contradiction can be resolved by noting that actually

$$\boxed{\underline{\nabla} \cdot \left(\frac{\hat{\underline{r}}}{r^2}\right) = 4\pi\delta(\underline{r})} \tag{11}$$

then

$$\int_V \underline{\nabla} \cdot \underline{v}\mathrm{d}V = 4\pi \int_V \delta(\underline{r})\mathrm{d}V = 4\pi$$

Identity (11) implies that $\underline{\nabla} \cdot \underline{E} = \rho/\epsilon_0$ holds even for a point charge for which $\rho = q\delta(\underline{r})$ and $\underline{E} = \dfrac{q}{4\pi\epsilon_0}\dfrac{\hat{\underline{r}}}{r^2}$

EM 3 Section 4: Poisson's Equation

4. 1. Poisson's Equation

If we replace $\underline{E}$ with $-\underline{\nabla}V$ in the differential form of Gauss's Law we get **Poisson's Equation**:

$$\boxed{\nabla^2 V = \underline{\nabla}.\underline{\nabla}V = -\frac{\rho}{\epsilon_0}} \qquad (1)$$

where the Laplacian operator reads in Cartesians $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$

It relates the second derivatives of the potential to the local charge density.

In a region absent of free charges it reduces to **Laplace's equation**:

$$\boxed{\nabla^2 V = 0} \qquad (2)$$

Note that one solution is a uniform potential $V = V_0$, but this would only apply to the case where there are no free charges anywhere. More generally we have to solve Laplace's equation subject to certain boundary conditions and this yields non-trivial solutions.

Poisson's and Laplace's equations are among the most important equations in physics, not just EM: fluid mechanics, diffusion, heat flow etc. They can be studied using the techniques you have seen Physical Mathematics e.g. separation of variables, orthogonal polynomials etc

4. 2. Solutions of Poisson's Equation: helpful properties

If you know V everywhere you can find ρ at any point by differentiating twice.

Example

$$V = a + bx^2 y$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 2by + 0 + 0$$

$$\rho(x, y, z) = -\epsilon_0 \nabla^2 V = -2\epsilon_0 by$$

If you know ρ everywhere you can find V at any point but you have to solve Poisson's equation. *This is a harder but much more common task!* It is the central problem of electrostatics.

Helpful Property 1: Linearity

If $\nabla^2 V_1 = -\frac{\rho_1(\underline{r})}{\epsilon_0}$ and $\nabla^2 V_2 = -\frac{\rho_2(\underline{r})}{\epsilon_0}$ then

$$\nabla^2 (V_1 + V_2) = -\frac{\rho_1(\underline{r}) + \rho_2(\underline{r})}{\epsilon_0}$$

This behaviour is known as *linearity* and gives rise to the **"superposition principle"** which is used to sum potentials V arising from different charge distributions (or different pieces of the same charge distribution).

The extreme case of this is to sum the charge distribution as a set of point charges.

$$V = \sum_i V_i \qquad \rho = \sum_i \rho_i \qquad \nabla^2 V_i = -\frac{\rho_i(\underline{r})}{\epsilon_0} \tag{3}$$

Warning: must check BC's still satisfied

Superposition also applies to $\underline{E} = -\underline{\nabla} V$

often easier to superpose V (scalar) than $\underline{E}$ (vector), but not always: go case by case

Helpful Property 2: Uniqueness Theorem

If a potential obeys Poisson's equation and satisfies the known boundary conditions it is the **only** solution to a problem. This is known as the *uniqueness* theorem.

Basically if one can find a solution by whatever means —usually educated guesswork—then it is the unique solution.

Proof of Uniqueness Consider region $\mathcal{R}$ with boundary $\mathcal{B}$. Let $\rho(\underline{r})$ be specified within $\mathcal{R}$

Figure 5: Diagram of region and boundary for Uniqueness Theorem

BCs: suppose either

(i) V is specified on $\mathcal{B}$

(ii) $\underline{E} = -\underline{\nabla} V$ is specified on $\mathcal{B}$

Then **any** solution of Poisson's equation obeying the BCs is the **only** solution [up to a boring added constant in case (ii)] **NB**: $\mathcal{B}$ could be at infinity

Proof of Theorem

Suppose there are 2 different solutions, V_1, V_2. Define $\psi = V_1 - V_2$

Then $\nabla^2 \psi = 0$ in $\mathcal{R}$ (Laplace's equation)

BCs on $\mathcal{B}$: case (i) $\psi = 0$; case (ii) $\underline{\nabla}\psi = 0$

$$\text{Laplace: } \nabla^2 \psi = 0 \qquad \Rightarrow \qquad \psi \nabla^2 \psi = 0$$

$$\Rightarrow \qquad \underline{\nabla} \cdot (\psi \underline{\nabla} \psi) - (\underline{\nabla} \psi)^2 = 0$$

$$\Rightarrow \qquad \int_{\mathcal{R}} \left[\underline{\nabla} \cdot (\psi \underline{\nabla} \psi) - (\underline{\nabla} \psi)^2 \right] d\tau = 0$$

Apply divergence theorem to first term

$$\int_{\mathcal{B}} \psi \underline{\nabla} \psi . \underline{\mathrm{d}S} - \int_{\mathcal{R}} (\underline{\nabla} \psi)^2 \, d\tau = 0$$

First term now zero by BCs in either case

The remaining integrand is non-negative, so it must vanish to get zero for the integral

$$\underline{\nabla} \psi = 0 \qquad \Rightarrow \qquad \psi = V_1 - V_2 = \text{const}$$

The constant is zero by BCs in case (i)

4. 3. Simple Example: hollow conductor

Figure 6: Diagram of cavity in a conductor

Consider a cavity $\mathcal{R}$ in a conductor **Claim:** If $\rho = 0$ in cavity, then $\underline{E} = \underline{0}$ inside

Proof: Inner surface is equipotential (since it is conducting), $V = V_0$. This gives our boundary condition

Now inside the cavity $\nabla^2 V = 0$ since there is no charge.

Thus we must solve Laplace's equation subject to the condition that V is constant along the (closed) boundary.

But one solution is $V = V_0$ everywhere within $\mathcal{R}$ and this satisfies boundary condition.

Uniqueness: this is **only** solution $\Rightarrow \underline{E} = -\underline{\nabla} V = \underline{0}$ for **any** charge-free cavity within a conductor.

4. 4. The Method of Images

This is a technique for guessing (and then verifying) the solution to Poisson's equation. Due to uniqueness it is then the only solution.

The idea is to place a suitable set of "image charges" external to the physical region of the field, in such a way that they generate the required boundary conditions, without affecting Poisson's equation within the physical region (since an image charge is not in the physical region).

Point charge near a conducting plane

Consider a point charge, Q, a distance a from a flat conducting surface at a potential $V_0 = 0$.

Figure 7: Point charge near a conducting plane

The problem is to solve Poisson's equation with a point charge at $a\underline{e}_z$ and boundary condition that $V = 0$ on the boundary ($z = 0$) of the physical region $z \geq 0$.

Now the potential from the point charge at $a\underline{e}_z$ is

$$V = \frac{1}{4\pi\epsilon_0} \frac{1}{(x^2 + y^2 + (z-a)^2)^{1/2}} \tag{4}$$

The idea is to consider an 'image charge' in the unphysical region $z < 0$. In the physical region ($z > 0$) the potential due to such an image charge satisfies Laplace's equation therefore we can simple add it to (4) and still satisfy Poisson's equation.

The correct guess for the image charge is $-q$ at $-a\underline{e}_z$. This basically reflects the symmetry of the problem. To check that the boundary condition is actually satisfied we write out the potential

$$V = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{(x^2 + y^2 + (z-a)^2)^{1/2}} - \frac{1}{(x^2 + y^2 + (z+a)^2)^{1/2}} \right] \tag{5}$$

and see that it vanishes at $z = 0$ as required.

N.B. if the conducting sheet is at potential $V_0 \neq 0$ we simply add a constant V_0 to (5).

$\underline{E}$ must be normal to the conducting surface $\underline{E} = E_z \underline{e}_z$. Therefore on the surface

$$\begin{aligned} E_z &= -\left.\frac{\partial V}{\partial z}\right|_{z=0} = \frac{-q}{4\pi\epsilon_0} \left[-\frac{(z-a)}{(x^2 + y^2 + (z-a)^2)^{3/2}} + \frac{(z+a)}{(x^2 + y^2 + (z+a)^2)^{3/2}} \right]\Bigg|_{z=0} \\ &= \frac{-q}{2\pi\epsilon_0} \frac{a}{(x^2 + y^2 + a^2)^{3/2}} \end{aligned}$$

and the surface charge density is

$$\sigma = \frac{-q}{2\pi} \frac{a}{(x^2 + y^2 + a^2)^{3/2}}$$

Integrating this over the whole surface (left as exercise) shows that the surface charge is $-q$. *Note that there is an attractive force between the charge distribution on the conducting surface and the point charge above it.*

The method of images is not really a 'method' as such, more an inspired guess which works when the problem has appropriate symmetry (see tutorial problems for further examples).

EM 3 Section 5: Electric Dipoles

An electric dipole is formed by two point charges $+q$ and $-q$ connected by a vector $\underline{a}$. The **electric dipole moment** is defined as $\boxed{\underline{p} = q\underline{a}}$.

By convention the vector **a** points from the negative to the positive charge. Here we also take the origin to be at the centre and **a** to be aligned to the z axis (see diagram)

So far we are considering a *physical dipole* however it is useful to take the idealisation of an *ideal dipole* which is $\underline{a} \to 0$, $q \to \infty$ but $\underline{p}$ finite. As we shall see the ideal dipole is a useful approximation to the 'physical dipole'.

Figure 8: Diagram of electric dipole aligned along z axis

5. 1. Field of an electric dipole

We first calculate the potential and then the field:

$$V = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right) \tag{1}$$

where $r_\pm$ are the distances from the +ve(-ve) charge to the point $\underline{r}$.

Now

$$|\underline{r} \pm \underline{a}/2|^2 = (r^2 \pm \underline{a}\cdot\underline{r} + a^2/4) = r^2\left(1 \pm \frac{a}{r}\cos\theta + \frac{a^2}{4r^2}\right)$$

Now consider the "far field limit" $\boxed{r \gg a}$

$$\begin{aligned}\frac{1}{r_\pm} &= \frac{1}{r}\left(1 \mp \frac{a}{r}\cos\theta + \frac{a^2}{4r^2}\right)^{-1/2} \\ &\simeq \frac{1}{r}\left(1 \pm \frac{a}{2r}\cos\theta + O((a/r)^2)\right)\end{aligned}$$

where $O((a/r)^2)$ indicates terms proportional to $(a/r)^2$ or higher powers

Thus we obtain in the far-field limit

$$V = \frac{qa\cos\theta}{4\pi\epsilon_0 r^2} = \frac{\underline{p}\cdot\hat{\underline{r}}}{4\pi\epsilon_0 r^2} \tag{2}$$

One can check that (away from the charges) this is a solution of Laplace's equation (see tutorial)

The components of the electric field $\underline{E}= -\underline{\nabla}$V are simplest form in spherical polar coordinates:

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3} \qquad E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3} \tag{3}$$

To get a co-ordinate free form of the electric field we can use (see tutorial sheet 1)

$$\underline{E} = -\underline{\nabla} V = \frac{1}{4\pi\epsilon_0}\left(\frac{3(\underline{p}\cdot\underline{r})\underline{r}}{r^5} - \frac{\underline{p}}{r^3}\right) \tag{4}$$

The important point to note is that a dipole field is $1/r^3$*, whereas a point charge field is* $1/r^2$

Figure 9: Sketch of electric dipole field: ideal and 'physical'

5. 2. Why dipoles matter I

Many molecules have a **permanent** dipole moment **p** (e.g. H_20)
All others, and all **atoms**, acquire an **induced** dipole when placed in $\underline{E}$ field
Since atoms and molecules are (a) neutral and (b) almost pointlike, the dipole concept is crucial to understanding media see later.

5. 3. Interaction of dipole with external Electric Field

To calculate the force on a dipole in an external field $\underline{E}_{\rm ext} = -\underline{\nabla}\phi$ (note here we use ϕ for the electrostatic potenial of the external field) it is simplest to first calculate the potential energy of the dipole in this field:

$$\begin{aligned} U_{\rm dip} &= q\phi(\underline{a}/2) - q\phi(-\underline{a}/2) \\ &\simeq q\underline{a}\cdot\underline{\nabla}\phi = -\underline{p}\cdot\underline{E}_{\rm ext} \end{aligned}$$

where we have made a Taylor expansion to first order in $\underline{a}$.

Thus $U_{\rm dip}$ is minimised when the dipole is parallel to the field and maximised when antiparallel. *You need* $\Delta U = 2pE$ *to reverse the direction of the dipole!*

Moreover we can compute the force felt by the dipole through

$$\underline{F} = -\underline{\nabla} U_{\text{dip}} = \underline{\nabla}(\underline{p} \cdot \underline{E}) \tag{5}$$

There is *no net force* on a dipole in a *uniform* electric field, since the two charges of a dipole experience equal and opposite force. However, in a *non-uniform* field the force moves the dipoles along the field gradient.

In a *uniform* field there is still a *torque* which acts to align the dipole moment $\underline{p}$ along the direction of $\underline{E}$:

$$\underline{T} = (\underline{a}/2) \times (q\underline{E}) - (\underline{a}/2) \times (-q\underline{E}) = \underline{p} \times \underline{E} \tag{6}$$

The work done by the torque in rotating the dipole from an aligned position through an angle θ relative to the field is:

$$W = \int_0^\theta T d\theta = \int_0^\theta pE \sin\theta d\theta = pE(1 - \cos\theta) \tag{7}$$

To summarise

i) dipoles tend to align with an external field

ii) dipoles migrate up field gradients

5. 4. Why dipoles matter II

Let's go back to the Taylor expansion of the electrostatic potential, this time for an *arbitrary* bounded charge distribution confined to some region $\mathcal{R}$

$$\begin{aligned}
V(\underline{r}) &= \frac{1}{4\pi\epsilon_0} \int_{\mathcal{R}} \mathrm{d}V' \frac{\rho(\underline{r}')}{|\underline{r} - \underline{r}'|} \\
&= \frac{1}{4\pi\epsilon_0} \int_{\mathcal{R}} \mathrm{d}V' \frac{\rho(\underline{r}')}{(r^2 - 2\underline{r} \cdot \underline{r}' + r'^2)^{1/2}} \\
&= \frac{1}{4\pi\epsilon_0} \int_{\mathcal{R}} \mathrm{d}V' \frac{\rho(\underline{r}')}{r} \left(1 - \frac{2\underline{r} \cdot \underline{r}'}{r^2} + \frac{(r')^2}{r^2}\right)^{-1/2} \\
&\simeq \frac{1}{4\pi\epsilon_0} \int_{\mathcal{R}} \mathrm{d}V' \rho(\underline{r}') \left[\frac{1}{r} + \frac{\hat{\underline{r}} \cdot \underline{r}'}{r^2} + \frac{3(\hat{\underline{r}} \cdot \underline{r}')^2 - (r')^2}{2r^3} + \cdots\right]
\end{aligned}$$

In the last line we have used the expansion

$$(1 + x)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \cdots$$

and gathered together terms according to the power of r i.e. we have made a far-field expansion ($r \gg 1$) in powers of $1/r$.

We can tidy up the expansion to write it as

$$V(\underline{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r} + \frac{1}{4\pi\epsilon_0} \frac{\hat{\underline{r}} \cdot \underline{P}}{r^2} + \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \frac{1}{2} \sum_{i,j} \mathcal{Q}_{ij} \hat{r}_i \hat{r}_j + \cdots \tag{8}$$

where

$$
\begin{aligned}
Q &= \int_V \mathrm{d}V' \rho(\underline{r}') \\
\underline{P} &= \int_V \mathrm{d}V' \underline{r}' \rho(\underline{r}') \\
\mathcal{Q}_{ij} &= \int_V \mathrm{d}V' (3r_i' r_j' - (r')^2 \delta_{ij}) \rho(\underline{r}')
\end{aligned}
$$

Thus Q is the total net charge ; $\underline{P}$ is the dipole moment and $\dfrac{\hat{\underline{r}}^T \mathcal{Q} \hat{\underline{r}}}{r^3}$ is the quadupole moment with $\mathcal{Q}$ the quadrupole tensor (a basis dependent matrix). Each of the monopole, dipole, quadrupole fields have distinct and characteristic shapes.

The first term ('monopole term') would normally be the dominant term and this would quite reasonably represent the far-field charge distribution as that of a single lump of charge at the origin.

However, when the total charge Q vanishes (as in the case of a dipole) it is the next term ('dipole term') which is dominant. One can think of charge distributions where the second term term also vanishes (e.g. electric quadrupole where $\underline{P} = 0$ see figure) then the dominant term becomes the third term (quadrupole term) and so on.

Figure 10: 4 charges forming an Electric quadrupole

Significantly, each term in the expansion (8) is separately a solution of Laplace's equation (outside of the region V that contains the charge distribution).

Thus one can use the idea of the method of images and use an image dipole

$$
V = \frac{1}{4\pi\epsilon_0} \frac{\widehat{(\underline{r} - \underline{r}')} \cdot \underline{P}}{|\underline{r} - \underline{r}'|^2}
$$

at some point $\underline{r}'$ outside of the physical region to solve Poison's equation in the physical region, and fix certain boundary conditions on the boundary of the physical region.

An example is a conducting sphere in a uniform external field $\underline{E}_0$. The boundary condition is that $\underline{E}$ is radial at the surface of the sphere which is an equipotential. This is achieved by using an image electric dipole at the centre of the sphere - see tutorial. The result is that on the surface of the sphere there is an induced charge distribution which is positive on one side and negative on the other.

EM 3 Section 6: Electrostatic Energy and Capacitors

6. 1. Electrostatic Energy of a general charge distribution

Here we provide a proof that the electrostatic energy *density*: (energy per unit volume)

$$\boxed{u_E = \frac{dU_E}{\mathrm{d}V} = \frac{1}{2}\epsilon_0 |\underline{E}|^2} \tag{1}$$

is a completely general result for any electric field.

An assembly of $n-1$ point charges at positions $\underline{r}_j$ gives a potential at $\underline{r}_i$ (N.B. here the subscript labels the charge):

$$\phi(\underline{r}_i) = \frac{1}{4\pi\epsilon_0}\sum_{j=1}^{n-1}\frac{q_j}{|\underline{r}_i - \underline{r}_j|}$$

N.B. Here we use ϕ for potential to avoid confusion between potential and volume V.

Bringing up another charge q_n (the nth one) from infinity to point $\underline{r}_n$ requires work:

$$W_n = q_n\phi(\underline{r}_n) = \frac{q_n}{4\pi\epsilon_0}\sum_{j=1}^{n-1}\frac{q_j}{|\underline{r}_n - \underline{r}_j|}$$

i.e. for charge 1, the work is zero since there is no potential yet, for charge 2 the work is $q_2 \times$ potential due to charge 1, for charge 3 the work is $q_3 \times$ potential due to charges 1 charge 2 etc

So the total energy U_E of the charges, which is equal to the total work required to assemble all n charges, is

$$U_E = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j<i}\frac{q_i q_j}{|\underline{r}_i - \underline{r}_j|} = \frac{1}{8\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j=1(j\neq i)}^{n}\frac{q_i q_j}{|\underline{r}_i - \underline{r}_j|}$$

Make sure you understand the factor of 1/2 which appears in the second equality when we allow both sums to go over all charges.

We can write the final equality as

$$\boxed{U_E = \frac{1}{2}\sum_i q_i\phi_i} \tag{2}$$

where $\phi_i = \displaystyle\sum_{j\neq i}\frac{q_j}{4\pi\epsilon_0|\underline{r}_i - \underline{r}_j|}$ is the potential at $\underline{r}_i$ due to the *other* charges j

This can be generalized in the limit of a continuous charge distribution to an integral over the charge density ρ:

$$\boxed{U_E = \frac{1}{2}\int \rho(\underline{r})\phi(\underline{r})\mathrm{d}V} \tag{3}$$

It turns out that we can write this integral in another way. First recall

$$\underline{\nabla}\cdot\underline{E} = \frac{\rho}{\epsilon_0}\ .$$

Then (3) becomes

$$U_E = \frac{\epsilon_0}{2} \int (\underline{\nabla} \cdot \underline{E}) \phi \, \mathrm{d}V \tag{4}$$

Now use a product rule from section 1 to write

$$\begin{aligned} \phi(\underline{\nabla} \cdot \underline{E}) &= \underline{\nabla} \cdot (\phi \underline{E}) - (\underline{\nabla} \phi) \cdot \underline{E} \\ &= \underline{\nabla} \cdot (\phi \underline{E}) + |\underline{E}|^2 \end{aligned}$$

whence (4) yields two integrals. The first can be rewritten using the divergence theorem

$$\int_V \underline{\nabla} \cdot (\phi \underline{E}) \mathrm{d}V = \int_S (\phi \underline{E}) \cdot \mathrm{d}\underline{S}$$

Then taking V as *all space* and the boundary S at infinity where we have boundary conditions $\phi(\infty) = 0$ and $\underline{\nabla}\phi = 0$ which means that this integral is zero. Therefore the final result comes from the second integral:

$$U_E = \frac{\epsilon_0}{2} \int_{\text{all space}} |\underline{E}|^2 \mathrm{d}V \tag{5}$$

and from this we get (1) for the energy density.

Warning: The general result (5) was derived for a continuous charge distribution since we began from (3). When there are point charges we have to be careful with *self-energy* contributions which should be excluded from the integral (3), otherwise they lead to divergences.

Equation (3) leads us to think that Electrostatic energy lies in the charge distribution whereas from (5) we might infer that the energy is stored in the electrostic field. Which picture is correct? In fact these interpretations are tautologous.

A final thing to note is that since (5) is quadratic in the field strength we do not have superposition of energy density.

6. 2. Capacitors

A **capacitor** is formed when two neighbouring conducting bodies (any shape) have equal and opposite surface charges. Suppose we have two conductors one with charge Q and the other with charge $-Q$. Since V is constant on each conductor the potential difference between the two is $V = V_1 - V_2$. In general to actually find $V(\underline{r})$ and $\underline{E}$ can be difficult (need to solve Poisson's equation between conductors). However there will be a unique well-defined value of the capacitance defined as the ratio of the charge on each body to the potential difference between the bodies:

$$\boxed{C = \frac{Q}{V_d}} \tag{6}$$

Capacitance is measured in Farads = Coulombs/Volt.

A capacitor is basically a device which stores electrostatic energy by charging up

Figure 11: Diagram of Parallel Plate Capacitor

Parallel Plate Capacitor

Two parallel plates of area A have a separation d. They carry surface charges of $+\sigma$ and $-\sigma$, which are all on the *inner surface* of the plates because of the attractive force between the charges on the two plates. The normal to the plate is taken in $\underline{e}_z$ direction (positive up).

To obtain the Electric field we use Gauss' Law. First take a pillbox that straddles the *inner surface* of the upper plate say. Now inside the conducting plate $\underline{E} = 0$. Therefore between the two plates the field normal to the inner surface of the upper plate is

$$\underline{E}_{inside} = \left(\frac{(+\sigma)}{2\epsilon_0}\right)(-\underline{e}_z) = -\frac{\sigma}{\epsilon_0}\underline{e}_z$$

Now take a pillbox that straddles the *outer surface* of the upper plate since there is no charge on the upper surface and since inside the plate $\underline{E}$ is zero we must have that $\underline{E} = 0$ at the outer surface.

$$\underline{E}_{outside} = 0 \qquad \underline{E}_{inside} = -\frac{\sigma}{\epsilon_0}\underline{e}_z$$

Now

$$E_z = -\frac{\partial V}{\partial z} \quad \Rightarrow \quad V = \frac{\sigma z}{\epsilon_0}$$

and the potential difference between the plates is:

$$V_d = \frac{Qd}{A\epsilon_0} \tag{7}$$

$$C = \frac{Q}{V_d} = \frac{A\epsilon_0}{d} \tag{8}$$

Note that C is a purely geometric property of the plates!

6. 3. Electrostatic Energy

Let us compare the energy of the charge distribution in the capacitor using the two formulas (3,5) derived in the last section.

First use (3): The integral simplifies to a sum of two contributions from the upper plate which has charge Q and potential ϕ_1 and the lower plate which has charge $-Q$ and potential ϕ_2 $U = \frac{Q}{2}(\phi_1 - \phi_2) = \frac{QV_d}{2}$ Thus

$$U = \frac{QV_d}{2} = \frac{CV_d^2}{2} = \frac{Q_d^2}{2C} \tag{9}$$

where we have used the definition of capacitance.

On the other hand we can integrate over the electric field, which is constant between the plates, using (5).

$$U_E = \frac{\epsilon_0}{2} \int |\underline{E}|^2 \mathrm{d}V = \frac{\epsilon_0}{2} Ad \left(\frac{\sigma}{\epsilon_0}\right)^2$$

and using the definition of capacitance the result (8)

$$U_E = \frac{d}{2\epsilon_0} \frac{Q^2}{A} = \frac{Q^2}{2C}$$

6. 4. **Finite size disc capacitors*

So far we have assumed the capacitor plates are effectively infinite. In this case the electric field between the two plates was uniform *When should you worry about the finite size of capacitor plates?*

For a *finite-size* capacitor it is possible that there are edge effects where the field can bulge out of the capacitor and also non-uniformity of the field within the capacitor. To get a feeling for when such effects become important let us compute the potential and field due to a *finite-size* disc.

For a finite size disc of charge in the x–y plane, carrying a surface charge density σ, we perform a two-dimensional integral over the charge distribution to obtain the potential at a height z along the axis of the disc

$$V = \frac{1}{4\pi\epsilon_0} \int_S \frac{\sigma \mathrm{d}S}{(x^2+y^2+z^2)^{1/2}}$$

This integral is easy when we use plane polar co-ordinates for the disc $\mathrm{d}S = \rho \mathrm{d}\rho \mathrm{d}\phi$ where $\rho^2 = x^2 + y^2$ and

$$\begin{aligned} V(z) &= \frac{\sigma}{4\pi\epsilon_0} \int_0^{2\pi} \mathrm{d}\phi \int_0^{\infty} \mathrm{d}\rho \frac{\rho}{(\rho^2+z^2)^{1/2}} \\ &= \frac{\sigma}{4\pi\epsilon_0} 2\pi \left[(\rho^2+z^2)^{1/2}\right]_{\rho=0}^{R} = \frac{\sigma}{2\epsilon_0}\left[(R^2+z^2)^{1/2} - z\right] \end{aligned}$$

Now the z component of the field is

$$E_z = -\frac{\partial V}{\partial z} = -\frac{\sigma}{2\epsilon_0}\left[\frac{z}{(R^2+z^2)^{1/2}} - 1\right]$$

and we see that there is a z dependence so the field is not uniform.

The limit $R \ll z$ corresponds to a point charge:

$$\underline{E}(R \ll z) = \frac{\sigma}{2\epsilon_0}\left(1 - (1 + \frac{R^2}{z^2})^{-1/2}\right)\underline{e}_z = \frac{\sigma R^2}{4\epsilon_0 z^2} = \frac{Q}{4\pi\epsilon_0 z^2}\underline{e}_z$$

The limit $R \gg z$ gives the uniform field of an infinite plane of charge:

$$\underline{E}(R \gg z) = \frac{\sigma}{2\epsilon_0}\underline{e}_z \qquad (10)$$

Only when $R \approx z$ do you need to consider the finite size of the disc!

Edge effects for a capacitor are limited to the regions where the gap distance is comparable to the distance to the edges of the plates.

EM 3 Section 7: Magnetic force, Currents and Biot Savart Law

7. 1. Magnetic force

The magnetic field $\underline{B}$ is defined by the force on a moving charge:

$$\boxed{\underline{F} = q(\underline{E} + \underline{v} \times \underline{B})} \tag{1}$$

This is the **Lorentz Force Law**. The second term is the magnetic force. The unit of magnetic field is the Tesla (T) which is $\text{NA}^{-1}\text{m}^{-1}$. Actually this is a pretty big unit and a Gauss = 10^{-4}T.

7. 2. Current density and current elements

The first thing to note is that a moving charge by itself does not constitute a current. Instead we need a moving density of charges. For the moment we will consider **steady** currents so that at any point we have a constant density of charged particles moving past the point (clearly we will need sources of current somewhere but let's not worry about that for the moment). Also we can have zero net charge but a steady current, if the densities of positive and negative particles are the same but their velocities are different. A current consists of n charges q per unit volume moving with average velocity $\underline{v}$. These charges form a local current density:

$$\underline{J} = nq\underline{v} \tag{2}$$

The total current passing *through* a surface is obtained by integration:

$$\text{I} = \int_A \underline{J} \cdot \text{d}\underline{S} \tag{3}$$

where as usual $\text{d}\underline{S}$ points normal to the surface.

Units

The unit of current is the Ampere (A), which is a base SI unit, $1\text{A} = 1\text{Cs}^{-1}$.The unit of bulk current density $\underline{J}$ is A/m^2. We can also have surface current densities usually denoted $\underline{K}$ or $\underline{j}$ (units A/m) and line current densities usually denoted $\underline{I}$ (units A). Calling all of these 'densities' is a bit confusing since none has units of current per unit volume but that is the way it is!

What we shall see is that steady currents play the key role in magnetism as do electric charges in electrostatics, that is

Stationary charges	$\Rightarrow$	constant electric fields: electrostatics
Steady currents	$\Rightarrow$	constant magnetic fields: magnetostatics

A **current element**, denoted here $\underline{\mathrm{d}\mathcal{I}}$, has units A m and is a vectro

$$\begin{aligned}\underline{\mathrm{d}\mathcal{I}}(\underline{r}) &= \underline{J}(\underline{r})\mathrm{d}V \qquad \text{Current element in bulk}\\ \underline{\mathrm{d}\mathcal{I}}(\underline{r}) &= \underline{K}(\underline{r})\mathrm{d}S \qquad \text{Current element on surface}\\ \underline{\mathrm{d}\mathcal{I}}(\underline{r}) &= \underline{I}(\underline{r})\mathrm{d}l \qquad \text{Current element along a wire}\end{aligned}$$

Warning: you need to take care with current elements e.g. $\underline{K}\mathrm{d}S \neq K\underline{\mathrm{d}S}$ since the left hand side points in the direction of the current vector on the surface, but the right hand side points normal to the surface. On the other hand $\underline{I}\mathrm{d}l = I\underline{\mathrm{d}l}$ since a line current element always points along the wire in the direction $\underline{\mathrm{d}l}$.

Figure 12: Diagram of rotating charged disc

Example: Rotating disc An insulating disc of radius R, carrying a uniform surface charge density σ, is mechanically rotated about its axis with an angular velocity ω (in the $\underline{e}_z$ direction). As a result it has a current density on its surface:

$$\underline{K} = \sigma\underline{v} = \sigma\underline{\omega} \times \underline{r} = \sigma r\omega\underline{e}_\phi \tag{4}$$

Note that the current density increases linearly with r. Check that you understand how the direction comes from the right hand rule.

Conductivity

The quantity σ (conductivity) or ρ (resistivity) describes the intrinsic conduction properties of a bulk material in response to an electric field. The current density is:

$$\underline{J} = \sigma\underline{E} \qquad \rho = \frac{1}{\sigma} \tag{5}$$

Note that σ is a property of a particular material, and that it depends on temperature. A typical value of σ for a metal is $6 \times 10^7\ \Omega^{-1}m^{-1}$.

Remark: Actually (5) makes a crucial assumption that $\underline{J}$ and $\underline{E}$ are parallel which is not necessarily the case for some non-isotropic materials where for example current can only flow in certain directions. In such cases one needs a conductivity tensor.

The force on a steady current element is

$$\boxed{\underline{\mathrm{d}F} = \underline{\mathrm{d}\mathcal{I}} \times \underline{B}} \tag{6}$$

In particular, if the current element comes from a bulk current density $\underline{d\mathcal{I}} = \underline{J}\,dV$ we have:

$$\underline{dF} = \underline{J} \times \underline{B}\,dV \tag{7}$$

7. 3. Biot Savart Law and Calculation of Magnetic Fields

Just as a charge creates an electric field, so a current element at $\underline{r}'$ creates a magnetic field at a position $\underline{r}$:

$$\underline{dB}(\underline{r}) = \frac{\mu_0}{4\pi} \frac{\underline{d\mathcal{I}}(\underline{r}') \times (\widehat{\underline{r} - \underline{r}'})}{|\underline{r} - \underline{r}'|^2} \tag{8}$$

This is known as the **Biot-Savart Law**. The law was established expermentally; we shall take it as our starting point. It plays the same role for magnetostatics as Coulomb's law for the electric field due to a point charge in electrostatics.

The constant μ_0 is known as the *permeability* of free space:

$$\mu_0 = 4\pi \times 10^{-7}\mathrm{Hm}^{-1} \tag{9}$$

The direction of the field is perpendicular to $\underline{d\mathcal{I}}(\underline{r}')$ and the vector from the current element to the point $\underline{r}$, $\widehat{\underline{r} - \underline{r}'}$. The direction can be remembered from the usual right hand rule for vector products.

Note that **superposition** holds for magnetic fields, therefore the magnetic field can be calculated by integration over current elements:

$$\boxed{\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi} \int \frac{\underline{d\mathcal{I}}(\underline{r}') \times (\widehat{\underline{r} - \underline{r}'})}{|\underline{r} - \underline{r}'|^2}} \tag{10}$$

or by integration over a volume containing a distribution of current density:

$$\boxed{\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\underline{J}(\underline{r}') \times (\widehat{\underline{r} - \underline{r}'})}{|\underline{r} - \underline{r}'|^2} dV'} \tag{11}$$

7. 4. Magnetic Force between Currents

Substituting the Biot Savart Law for the magnetic field due to a current element into (6)

$$\boxed{\underline{dF}_{12} = \frac{\mu_0}{4\pi r_{12}^2} \underline{d\mathcal{I}}_1 \times (\underline{d\mathcal{I}}_2 \times \hat{\underline{r}}_{12})} \tag{12}$$

which corresponds to the **force between a pair of current elements**. Here $\underline{r}_{12}$ points from $\underline{d\mathcal{I}}_1$ to $\underline{d\mathcal{I}}_2$ Equation (12) is also referred to as Biot-Savart law and is the equivalent of Coulomb's law for the force between two point charges.

If the current elements are due to current densities we have

$$\underline{dF}_{12} = \frac{\mu_0}{4\pi r^2} \underline{J}_1 \times (\underline{J}_2 \times \hat{\underline{r}}) dV_1 dV_2 \tag{13}$$

where the force is attractive for parallel currents, and repulsive for antiparallel currents.

Figure 13: Diagram for calculating $\underline{B}$ from an infinite straight wire

7. 5. Example of long straight wire

We consider a long straight wire which we choose to be along the z axis so that a point $\underline{r}'$ on the wire is given by $\underline{r}' = z'\underline{e}_z$. We want to compute the field using (10). It is best to use cylindrical polars: we choose the origin along the wire so that $\underline{r}$ is $\perp$ to $\underline{e}_z$ i.e. $\underline{r} = \rho\underline{e}_\rho$ where ρ is the radial distance of the point from the wire.

Now $\underline{\mathrm{d}\mathcal{I}} = I\mathrm{d}z'\underline{e}_z$ and $\underline{\mathrm{d}r'} = \mathrm{d}z'\underline{e}_z$ so

$$\underline{\mathrm{d}r'} \times (\underline{r} - \underline{r}') = \underline{\mathrm{d}r'} \times \underline{r} = \rho\mathrm{d}z'\underline{e}_\phi$$

and we find

$$\underline{B}(\underline{r}) = \frac{\mu_0 I \rho}{4\pi}\underline{e}_\phi \int_{-\infty}^{\infty} \frac{\mathrm{d}z'}{|\rho^2 + z'^2|^{3/2}}$$

To evaluate the remaining integral we use substitution $z' = \rho\tan\theta$ so that $\mathrm{d}z' = \rho\sec^2\theta\,\mathrm{d}\theta$ and $\rho^2 + z'^2 = \rho^2\sec^2\theta$. Then we obtain

$$\underline{B}(\underline{r}) = \frac{\mu_0 I}{4\pi\rho}\underline{e}_\phi \int_{-\pi/2}^{\pi/2} \cos\theta\mathrm{d}\theta = \frac{\mu_0 I}{2\pi\rho}\underline{e}_\phi$$

We now consider the force on a current element of a second parallel wire (at distance d

Figure 14: Two parallel wires separated by distance d

from the first) coming from the magnetic field $\underline{B}_1(\underline{r})$ due to the first wire. Again we choose coordinates so that this current element lies at $\underline{r} = \rho\underline{e}_\rho$ in cylindrical polars

$$\begin{aligned}
\underline{\mathrm{d}F} &= \underline{\mathrm{d}I}_2(\underline{r}) \times \underline{B}_1(\underline{r}) = I_2\mathrm{d}z\underline{e}_z \times \frac{\mu_0}{2\pi d}I_1\underline{e}_\phi \\
&= -\frac{\mu_0 I_1 I_2}{2\pi d}\mathrm{d}z\underline{e}_\rho
\end{aligned}$$

There is an *attractive* force per unit length between two parallel infinitely long straight wires: this is the basis of the definition of the Ampère.

EM 3 Section 8: Divergence and Curl of $\underline{B}$; Gauss and Ampere's laws

8. 1. Divergence of $\underline{B}$ and Gauss' Law for Magnetic Fields

We can write the Biot-Savart Law for $\underline{B}$ due to a bulk current density using the expression for $\underline{\nabla}(1/r)$ as

$$\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi}\int_V \underline{J}(\underline{r}') \times \frac{(\widehat{\underline{r}-\underline{r}'})}{|\underline{r}-\underline{r}'|^3}\mathrm{d}V' = -\frac{\mu_0}{4\pi}\int_V \underline{J}(\underline{r}') \times \underline{\nabla}\left(\frac{1}{|\underline{r}-\underline{r}'|}\right)\mathrm{d}V' \tag{1}$$

Now since $\underline{\nabla}$ is with respect to the $\underline{r}$ coordinates and $\underline{J}(\underline{r}')$ depends on $\underline{r}'$ we find

$$\underline{\nabla}\cdot\left(\underline{J}(\underline{r}')\times\underline{\nabla}\left(\frac{1}{|\underline{r}-\underline{r}'|}\right)\right) = \underline{J}(\underline{r}')\cdot\left(\underline{\nabla}\times\underline{\nabla}\left(\frac{1}{|\underline{r}-\underline{r}'|}\right)\right) = 0$$

where the last equality follows since 'curlgrad =0'

Therefore

$$\boxed{\underline{\nabla}\cdot\underline{B} = 0} \tag{2}$$

This remarkable result is **the second fundamental law of electromag (Maxwell II)**

A magnetic field has no divergence which is a mathematical statement that there are no magnetic monopoles

This means that there are *no point sources* of magnetic field lines, instead the magnetic fields form *closed loops* round conductors where current flows.

Now the divergence theorem states that

$$\int_V \underline{\nabla}\cdot\underline{B}\,\mathrm{d}V = \oint_A \underline{B}\cdot\mathrm{d}\underline{S}$$

Thus the net magnetic field through any closed surface A must be zero

$$\boxed{\oint_A \underline{B}\cdot\mathrm{d}\underline{S} = 0} \tag{3}$$

which is sometimes referred to as Gauss' law for magnetic fields.

8. 2. Magnetic Dipoles

Since there are no magnetic monoples we should identify what is the equivalent of an electric dipole i.e. a magnetic dipole. It turns out this is a **current loop**. Consider a circular current loop radius a carrying steady current I in the clockwise direction with axis in the $\underline{e}_z$ direction

We consider the contribution to the magnetic field at $\underline{r}$ *along the axis of the loop* due to the current element $\mathrm{d}\underline{\mathcal{I}}(\underline{r}')$ at $\underline{r}'$ using the Biot-Savart law

$$\mathrm{d}\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi}\frac{\mathrm{d}\underline{\mathcal{I}}(\underline{r}')\times(\widehat{\underline{r}-\underline{r}'})}{|\underline{r}-\underline{r}'|^2} \tag{4}$$

Figure 15: Simple current loop with axis along z axis

We choose coordinates so that $\underline{r} = z\underline{e}_z$, $\underline{r}' = a\underline{e}_\rho{}'$, $\mathrm{d}\underline{\mathcal{I}} = I\mathrm{d}l\underline{e}_\phi{}'$ and $|\underline{r} - \underline{r}'| = (z^2 + a^2)^{1/2}$. Then

$$\mathrm{d}\underline{\mathcal{I}}(\underline{r}') \times \widehat{(\underline{r} - \underline{r}')} = I\mathrm{d}l(r\underline{e}_\rho{}' + a\underline{e}_z)$$

Now we see that the $\mathrm{d}\underline{B}$ is not along $\underline{e}_z$ but when we integrate around the current loop the perpendicular components cancel. Therefore we consider

$$\mathrm{d}B_z \quad = \quad \frac{\mu_0 I a \mathrm{d}l}{4\pi(z^2 + a^2)^{3/2}}$$

Note that this does not depend on the angle ϕ' around the ring therefore when we integrate over $\mathrm{d}l$ we simply get a factor $2\pi a$ and

$$B_z = \frac{\mu_0 \mathrm{I} a^2}{2(a^2 + z^2)^{3/2}} \tag{5}$$

At the centre of the loop ($z = 0$):

$$B_z = \frac{\mu_0 \mathrm{I}}{2a}$$

At a large distance from the loop ($z \gg a$):

$$B_z \simeq \frac{\mu_0 \mathrm{I} a^2}{2z^3}$$

To extend this calculation to an arbitrary position $\underline{r}$ (at all angles θ relative to the axis of the loop) is tedious, but it can be shown that the field of a loop is a magnetic dipole field i.e. in the far field limit of $r \gg a$ one finds

$$\underline{B}_{\mathrm{dip}}(\underline{r}) = \frac{\mu_0}{4\pi r^3}\left[3(\underline{m} \cdot \underline{\hat{r}})\underline{\hat{r}} - \underline{m}\right] \tag{6}$$

where the magnetic dipole moment, $\underline{m}$, is the product of the current and the vector area of the loop:

$$\underline{m} = I\pi a^2 \underline{e}_z = IA\underline{e}_z \tag{7}$$

In fact this result holds for a small current loop of *any* shape and with magnetic dipole moment, $\underline{m}$, defined as

$$\boxed{\underline{m} = I\underline{A} = I\int \mathrm{d}\underline{S}} \tag{8}$$

where $\underline{a}$ is the vector area of the loop

The (ideal) magnetic dipole field has the same form as the (ideal) electric dipole field:

$$B_r = \mu_0 \frac{2m\cos\theta}{4\pi r^3} \qquad B_\theta = \frac{\mu_0 m \sin\theta}{4\pi r^3} \tag{9}$$

However the 'physical' versions of electric and magnetic dipole look a bit different

Figure 16: Sketch of ideal and physical magnetic dipole lines *Griffiths fig 5.55*

It can be shown that: an external magnetic field creates a torque on a magnetic dipole:

$$\underline{T} = \underline{m} \times \underline{B}_{\text{ext}} \tag{10}$$

and the potential energy of the dipole in the field is:

$$U = -\underline{m} \cdot \underline{B}_{\text{ext}} \tag{11}$$

So one can think of a compass needle (magnetic moment along the needle) aligning with the Earth's magnetic field.

One could think of a magnetic dipole being composed of two monopoles (the 'Gilbert Model') and this gives the correct results for torque and energy. However this picture is basically wrong as the fundamental difference between a magnetic dipole and an electric dipole is that it is impossible to separate the N and S poles of a bar magnet for example.

8. 3. What has this got to do with everyday magnets?

You might wonder what current loops have got to do with ordinary bar or fridge magnets. The point is that in most atoms the electrons orbiting an atomic nucleus act as current loops, so atoms can have magnetic dipole moments. Moreover even an electron has 'spin' which generates a magnetic moment.

8. 4. Curl of $\underline{B}$ and Ampère's Law

Consider again the Biot-Savart law in form (1) and take the curl

$$\underline{\nabla} \times \underline{B}(\underline{r}) = -\frac{\mu_0}{4\pi} \int_V \underline{\nabla} \times \left(\underline{J}(\underline{r}') \times \underline{\nabla} \left(\frac{1}{|\underline{r} - \underline{r}'|} \right) \right) \mathrm{d}V' \tag{12}$$

where the curl is with respect $\underline{r}$ coordinates so can be taken inside the integral which is over $\underline{r}'$ coordinates. Now using a product rule from lecture 1 and remembering that $\underline{J}(\underline{r}')$ does not depend on the co-ordinates of $\underline{r}$

$$\begin{aligned}\underline{\nabla} \times \left(\underline{J}(\underline{r}') \times \underline{\nabla} \left(\frac{1}{|\underline{r} - \underline{r}'|} \right) \right) &= \nabla^2 \left(\frac{1}{|\underline{r} - \underline{r}'|} \right) \underline{J}(\underline{r}') - (\underline{J}(\underline{r}') \cdot \underline{\nabla}) \underline{\nabla} \left(\frac{1}{|\underline{r} - \underline{r}'|} \right) \\ &= -4\pi\delta(\underline{r} - \underline{r}')\underline{J}(\underline{r}') - (\underline{J}(\underline{r}') \cdot \underline{\nabla}) \underline{\nabla} \left(\frac{1}{|\underline{r} - \underline{r}'|} \right) \qquad (13)\end{aligned}$$

where we have used the now familiar result $\nabla^2 \left(\frac{1}{r} \right) = -4\pi\delta(\underline{r})$. When we insert (13) back into the integral (12), the second term can be shown to give zero since it can be written as a 'boundary term' which vanishes (see tutorial). The first term in (13) however yields

$$\boxed{\underline{\nabla} \times \underline{B} = \mu_0 \underline{J}} \qquad (14)$$

This is another fundamental law of electromagnetism i.e. Maxwell IV
The curl of a magnetic field around an axis is proportional to the component of the current density along the axis.

To obtain an integral form of Ampère's law we use Stokes' theorem:

$$\oint_C \underline{B} \cdot \underline{\mathrm{d}l} = \int_A (\underline{\nabla} \times \underline{B}) \cdot \underline{\mathrm{d}S}$$

Thus (14) becomes when we integrate over any an open surface bounded by closed loop C

$$\int_A (\underline{\nabla} \times \underline{B}) \cdot \underline{\mathrm{d}S} = \oint_C \underline{B} \cdot \underline{\mathrm{d}l} = \mu_0 \int_A \underline{J} \cdot \underline{\mathrm{d}S}$$

The integral of the magnetic field round a closed loop is related to the total current flowing across the surface enclosed by the loop:

$$\boxed{\oint_C \underline{B} \cdot \underline{\mathrm{d}l} = \mu_0 \mathrm{I} = \mu_0 \int_A \underline{J} \cdot \underline{\mathrm{d}S}} \qquad (15)$$

In a similar way to Gauss' law in electrostatics, Ampère's law is very useful for calculating magnetic fields when there is a high degree of symmetry to the problem.

Example: An infinite wire of finitie radius a carries a uniform current density, $\underline{J}$.
Outside the wire at radial distance ρ:

$$\begin{aligned}B_\phi 2\pi\rho &= \mu_0 \int_A \underline{J} \cdot \underline{\mathrm{d}S} = \mu_0 \mathrm{I} \\ B_\phi &= \frac{\mu_0 \mathrm{I}}{2\pi\rho} \qquad (16)\end{aligned}$$

The field outside the wire drops off with distance as $1/\rho$. This is a much easier derivation than integrating the Biot-Savart law

Now consider the field inside the wire:

$$\begin{aligned}B_\phi 2\pi\rho &= \mu_0 \int_A \underline{J} \cdot \underline{\mathrm{d}S} = \mu_0 J \pi \rho^2 = \mu_0 \mathrm{I} \frac{\rho^2}{a^2} \\ B_\phi &= \frac{\mu_0 \mathrm{I} \rho}{2\pi a^2} \qquad (17)\end{aligned}$$

The field inside the wire increases with radius.

EM 3 Section 9: Applications of Ampère's Law; Magnetic Vector Potential

9. 1. Applications of Ampère's Law

$$\boxed{\oint_C \underline{B} \cdot \mathrm{d}\underline{l} = \mu_0 \int_A \underline{J} \cdot \mathrm{d}\underline{S} = \mu_0 \mathrm{I}} \tag{1}$$

Like Gauss' law for electric fields Ampère's law is the most efficient way of calculating magnetic fields *when the system has some symmetry.* The symmetries which work are

- Infinite straight lines (see straight wire example from last lecture)
- Infinite planes (see next example of current sheet)
- Infinite solenoids (see tutorial 5.1)
- Toroids (see toroidal example below)

The difficult part is working out the *direction* of the magnetic field; after that Ampere's law readily gives the answer by choosing the Amperian loop appropriately

Field of an infinite slab of current (*Griffiths Example 5.8*)

An infinite sheet of conductor of thickness d, carries a uniform current density $\underline{J}$ parallel to the surface of the sheet. Let us take $\underline{e}_z$ normal to the sheet and choose $\underline{e}_x$ to be along

Figure 17: Infinite current sheets and Amperian loops (Griffiths Fig 5.33)

the direction of the current. By B-S law $\underline{B}$ field has to be perpendicular to $\underline{J}$. Now the symmetry of the infinite plane means that any component of $\underline{B}$ in the $\underline{e}_z$ direction cancels. Thus the planar symmetry implies that $\underline{B}$ is in the $\underline{e}_y$ direction i.e. $\parallel$ to plane and $\perp$ to current.

We take the integral round a rectangular loop $\parallel$ to the y–z plane loop of length l and height h enclosing the sheet. The magnetic field *outside* the slab is then:

$$2|B_y|l = \mu_0 J l d \qquad |B_y| = \frac{\mu_0 J d}{2}$$

This is a uniform magnetic field but *note that the directions of the field on the two sides of the sheet are opposite to each other!*

$$\underline{B} = -\frac{\mu_0 J d}{2} \underline{e}_y \quad \text{for} \quad z > d/2 \qquad \underline{B} = +\frac{\mu_0 J d}{2} \underline{e}_y \quad \text{for} \quad z < -d/2 \tag{2}$$

The field *inside* the conducting sheet can also be calculated by choosing a loop in the y–z plane that straddles the surface of the sheet. Then using the above result for the portion outside yields that *inside* the slab

$$B_y = -\mu_0 J z \qquad |z| < d/2$$

where $z = 0$ is at the centre of the sheet. (Exercise)

Field of a toroid (*Griffiths Example 5.10*)

A toroid consists of a set of coils of radius R, carrying a current I, and formed into a larger circle of radius a, so that they look like a doughnut. There are n coils per unit length around the larger circle. The toroidal symmetry is a little subtle: there is clearly symmetry with

Figure 18: Doughnut shaped toroid (see Griffiths Fig 5.39 for more general toroid)

respect to rotation about z axis (no dependence on ϕ) but also since the current always flows in the $\underline{e}_\rho - \underline{e}_z$ plane one can deduce from the BS law that the field must always be in the $\underline{e}_\phi$ direction i.e. it is circumferential (since other components cancel).

Griffiths Ex 5.10 gives a proof of this for a toroid of arbitrary cross-section.

Then we take our Amperian loops to be circles $\perp$ to $\underline{e}_z$. If the circle is not enclosed by the toroid, the current which cuts the circle is zero. Therefore $\underline{B} = 0$ outside the toroid

If the Amperian loop is a circle *enclosed by the toroid* of radial distance from z-axis ρ, then

$$B_\phi 2\pi\rho = \mu_0 n 2\pi a \mathrm{I}$$

Note that the rhs is constant since the same number of turns is always enclosed by such a loop. The field inside the toroid coils is:

$$B_\phi = \frac{\mu_0 n \mathrm{I} a}{\rho} \tag{3}$$

Note that this is not uniform, but only depends on ρ the radial distance from the z-axis.

9. 2. The Magnetic Vector Potential

Just as the theorem of 1.7 was the heart of Electrostatics the following theorem is the heart of magnetostatics:

The following three statements concerning a vector field $\underline{B}$ over some region in space are equivalent

1. $\underline{\nabla} \cdot \underline{B} = 0$ *the vector field is "solenoidal"*
2. $\underline{B} = \underline{\nabla} \times \underline{A}$ *the vector field may be written as the curl of a vector potential*
3. the surface integral of the field $\int_A \underline{B} \cdot \underline{\mathrm{d}S}$ is independent of the shape of the surface for a given boundary curve; a consequence is $\oint_A \underline{B} \cdot \underline{\mathrm{d}S} = 0$ for any **closed** surface A

We do not prove all the equivalences (see Griffiths 1.6) but it is clear that 2. implies 1. since 'div curl =0'

$$\underline{\nabla} \cdot \underline{B} = \underline{\nabla} \cdot (\underline{\nabla} \times \underline{A}) = 0 \tag{4}$$

Thus starting from the key property of the magnetic field is $\underline{\nabla} \cdot \underline{B} = 0$ (no monopoles), we find from 2. that we may always write the magnetic field as the curl of a vector potential

$$\boxed{\underline{B} = \underline{\nabla} \times \underline{A}} \tag{5}$$

and 3. gives the integral form of Gauss' law for magnetic fields

Using Stokes' theorem

$$\Phi_B \equiv \int_A \underline{B} \cdot \underline{\mathrm{d}S} = \oint_C \underline{A} \cdot \underline{\mathrm{d}l} \tag{6}$$

The magnetic flux through a surface is given by the integral of the magnetic vector potential around the loop enclosing that surface.

9. 3. Poisson's equation for the vector potential

Ampère's law can be written in the form:

$$\underline{\nabla} \times \underline{B} = \underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = \mu_0 \underline{J}$$

Using a vector operator identity for "curlcurl" (see lecture 1) this becomes:

$$\nabla^2 \underline{A} - \underline{\nabla}(\underline{\nabla} \cdot \underline{A}) = -\mu_0 \underline{J} \tag{7}$$

In the same way as we are free to *choose* the value of the scalar potential in electrostatics to be $V(\infty) = 0$, we are free to choose the divergence of the magnetic vector potential.

This property is known as gauge invariance.

The choice of $\underline{\nabla} \cdot \underline{A} = 0$ is known as the **Coulomb gauge**. It leads from (7) to Poisson's equation for the magnetic vector potential:

$$\boxed{\nabla^2 \underline{A} = -\mu_0 \underline{J}} \tag{8}$$

These are three equations, one for each of the components of the vector potential :

$$\nabla^2 A_x = -\mu_0 J_x \qquad \nabla^2 A_y = -\mu_0 J_y \qquad \nabla^2 A_z = -\mu_0 J_z \tag{9}$$

Assuming that $\underline{J}$ goes to zero at infinity we can read off the solution using our knowledge of the solution of Poisson's equation

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \int \frac{\underline{J}(\underline{r}')}{|\underline{r} - \underline{r}'|} \mathrm{d}V' \tag{10}$$

The equivalent of the expression the electrostatic potential from a charge can be written down for the magnetic vector potential at $\underline{r}$ due to a current element $I\underline{\mathrm{d}l}'$ or $\underline{J}\mathrm{d}V'$ at $\underline{r}'$:

$$\underline{\mathrm{d}A}(\underline{r}) = \frac{\mu_0 \mathrm{I}(\underline{r}')\underline{\mathrm{d}l}'}{4\pi|\underline{r} - \underline{r}'|} = \frac{\mu_0 \underline{J}(\underline{r}')\mathrm{d}V'}{4\pi|\underline{r} - \underline{r}'|} \tag{11}$$

Note that the direction of $\underline{\mathrm{d}A}$ *is parallel to the current element whereas* $\underline{\mathrm{d}B}$ *is perpendicular by B-S law.*

Example: vector potential of magnetic dipole (see tutorial)

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \frac{\underline{m} \times \hat{\underline{r}}}{r^2} \tag{12}$$

9. 4. Pause for thought and summary of statics

Electrostatics: Stationary charges $\frac{\partial \rho}{\partial t} = 0$ are source of electric fields

$$\underline{\nabla} \cdot \underline{E} = \frac{\rho}{\epsilon_0} \qquad \text{M1} \tag{13}$$

Coulomb's law (field due to point charge) leads to

$$\underline{\nabla} \times \underline{E} = 0 \quad \text{MIII} \quad \Rightarrow \quad \underline{E} = -\underline{\nabla} V$$

In turn the above lead to Poisson's equation for V

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

Magnetostatics: Steady current loops $\frac{\partial \underline{J}}{\partial t} = 0$ are source of magnetic fields.

$$\underline{\nabla} \cdot \underline{B} = 0 \qquad \text{MII} \tag{14}$$

Biot-Savart law (field due to cuurent element) leads to

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J} \quad \text{MIV} \quad \text{and} \quad \underline{B} = \underline{\nabla} \times \underline{A}$$

In turn in the Coulomb gauge the above lead to a vector Poisson equation for $\underline{A}$

$$\nabla^2 \underline{A} = -\mu_0 \underline{J}$$

In the following we shall see how MIII and MIV need to be modified when time-varying fields are present.

EM 3 Section 10: Electromotive force and Faraday's Law

10. 1. emf

So far we have considered steady currents and in particular steady current loops. However they actually don't exist! (at least not without a little help). To see this let's assume Ohm's law $\underline{J} = \sigma \underline{E}$ holds. Then

$$\oint_C \underline{J} \cdot \mathrm{d}\underline{r} = \sigma \oint_C \underline{E} \cdot \mathrm{d}\underline{r} = \sigma \int_S \underline{\nabla} \times \underline{E} \cdot \mathrm{d}\underline{S} = 0$$

since $\underline{\nabla} \times \underline{E} = 0$. Therefore I must be zero for the loop. The way out is to have a "battery" somewhere in the loop which causes a jump in the electric potential so that the integral from one terminal of the battery to the other

$$\mathcal{E} = \oint \underline{E} \cdot \mathrm{d}\underline{l} \tag{1}$$

where $\mathcal{E} = \Delta V$ is the potential supplied by the battery. This is known confusingly as an "electromotive force" (emf) (although it's not a force it's electrostatic potential difference). Then $\oint \underline{J} \cdot \mathrm{d}\underline{l} = IL = \sigma \Delta V$ and we get the elementary form of Ohm's law $V = IR$ with $R = L/\sigma$ and L the length of the loop.

Generally we can express the emf by considering the force on a *unit* charge $\underline{f} = \underline{f}_s + \underline{E}$ where $\underline{f}_s$ is the force due to some external source (e.g. battery) and $\underline{E}$ is the force from the electric field. Then

$$\mathcal{E} = \oint_C \underline{f} \cdot \mathrm{d}\underline{l} = \oint_C \underline{f}_s \cdot \mathrm{d}\underline{l}$$

10. 2. Induced emf

Consider a rectangular loop of conducting wire l moving with velocity $\underline{v}$ perpendicular to a uniform magnetic field (into the page). The charges in the segment of the loop (ab)

Figure 19: Rectangular conducting loop moving in magnetic field (*Griffiths Fig.7.10*)

perpendicular both to $\underline{B}$ and to $\underline{v}$ experiences a force in the ab direction

$$\underline{f}_{mag} = q\underline{v} \times \underline{B} \qquad \mathcal{E} = \oint \underline{f}_{mag} \cdot \mathrm{d}\underline{l} = vBh \tag{2}$$

where h is the length from a to b.

Aside: you may worry that a magnetic force $\underline{f}_{mag} = q\underline{v} \times \underline{B}$ should do no work! —this is because if the charge q moves $\underline{\mathrm{d}l} = \underline{v}\mathrm{d}t$ then $\mathrm{d}W_{mag} = \underline{f}_{mag} \cdot \underline{\mathrm{d}l} = q(\underline{v} \times \underline{B}) \cdot \underline{v}\mathrm{d}t = 0$. So actually it is the person pulling the loop that is doing the work to produce a current i.e. when the current flows in ab it creates a magnetic force $\mathrm{d}\underline{f}_{mag} = \underline{\mathrm{d}\mathcal{I}} \times \underline{B}$ to the left on an element of the wire and this must be balanced by a pulling force to the right.

Now consider the magnetic flux through the loop

$$\Phi_B = \int \underline{B} \cdot \underline{\mathrm{d}S} = BA = Bhx$$

where A is the area and the flux intergal is simply current$\times$ area here becuase $\underline{\mathrm{d}S} \parallel \underline{B}$ in this example.

$$\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = Bh\frac{\mathrm{d}x}{\mathrm{d}t} = -Bhv$$

Therefore comparing with (2) we see

$$\boxed{\mathcal{E} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}} \tag{3}$$

This result actually holds for general loops, $\underline{B}$ and $\underline{v}$ (for proof see Griffiths fig. 7.13). It is known as the flux rule or *Faraday's law of induction*. $\mathcal{E}$ is known as an **induced electromotive force (emf)**.

10. 3. Faraday's Law

As we have seen in (3) the induced emf can be understood in terms of the time variation of the magnetic flux through the current loop. Note that the magnetic flux $\phi_B = \int \underline{B} \cdot \underline{\mathrm{d}S}$ can be changed by varying any of $|\underline{B}|$, $\int \underline{\mathrm{d}S}$ or the angle between $\underline{B}$ and $\underline{\mathrm{d}S}$.

Figure 20: Summary of Faraday's experiments (*Griffiths Fig. 7.20*)

In a famous seris of experiments Faraday found that an emf can be induced by:

1. Pulling a current loop through a magnetic field.
2. Moving the magnet and area containing a field to the left
3. Changing the strength of the field

N.B. In case 2. we clearly see from relativity that the two scenarios must yield the same result i.e. in both cases the loop moves relative to the magnet with the same velocity. But actually this has enormous consequences for the physics: in case 2 the loop is stationary therefore, for the charges to feel a force, there **must** be an electric field present i.e. we deduce that the changing magnetic field *induces* an electric field.

10. 4. Differential form of Faraday's Law

Let us take Faraday's law (3) and use our definition of emf (1) to find

$$\oint \underline{E} \cdot \mathrm{d}\underline{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \int \underline{B} \cdot \mathrm{d}\underline{S} = -\int \frac{\partial \underline{B}}{\partial t} \cdot \mathrm{d}\underline{S}$$

where in the last equality we have assumed that only the magnetic field is changing

Now use Stokes' theorem

$$\oint_C \underline{E} \cdot \mathrm{d}\underline{l} = \int_A (\underline{\nabla} \times \underline{E}) \cdot \mathrm{d}\underline{S} = -\int_A \frac{\partial \underline{B}}{\partial t} \cdot \mathrm{d}\underline{S}$$

which holds for an arbitrary surface A implying

$$\boxed{\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}} \tag{4}$$

This is the full third fundamental law of electromagnetism MIII.
The curl of an electric field around an axis is proportional to the time variation of the magnetic field along the axis.

10. 5. Lenz's Law

Determing the sign of the flux in Faraday's law often proves troublesome. But there is a simple rule known as Lenz's Law that gives the right answer.

The induced emf always acts to oppose the change that causes it or **Nature abhors a change in flux**!

- In the case of the moving current loop at the start of this lecture, there is a force on the current in the wire $f = IlB$ which acts to decelerate the wire.
- Additional work has to be done to move a current loop into or out of a magnetic field, or through a non-uniform field. This is to overcome the induced emf.
- For a rotating current loop (see next lecture), there is a torque $\underline{m} \times \underline{B}$ due to the magnetic moment of the loop. This always acts to slow down the rotation.
- A time-varying magnetic field produces *eddy currents* in conducting loops.
  The dipole fields of these loops act to reduce the time-variation.

10. 6. Faraday's Law in terms of Magnetic Vector Potential

The two equations:

$$\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \qquad \underline{B} = \underline{\nabla} \times \underline{A}$$

can be combined to give:

$$\underline{\nabla} \times \underline{E} = -\frac{\partial}{\partial t}(\underline{\nabla} \times \underline{A})$$

A solution to this equation is clearly:

$$\underline{E} = -\frac{\partial \underline{A}}{\partial t} \tag{5}$$

The electric field due to induction can be expressed as the time-derivative of the magnetic vector potential.

The general solution is obtained by adding in a static electric field, which is expressed as the gradient of the scalar potential:

$$\boxed{\underline{E} = -\frac{\partial \underline{A}}{\partial t} - \underline{\nabla} V} \tag{6}$$

This does not change the form of Faraday's law because "curlgrad = 0 ".

Note that the equation $\underline{E} = -\underline{\nabla} V$ only applies to electrostatic situations when there are no time-varying fields.

Thus we can think of two kinds of electric field: those coming from a static charge distribution and which may be written as $\underline{E} = -\underline{\nabla} V$; those coming from a changing magnetic field and which may be written $\underline{E} = -\frac{\partial \underline{A}}{\partial t}$.

EM 3 Section 11: Inductance

11. 1. Examples of Induction

As we saw last lecture an emf can be induced by changing the area of a current loop in a magnetic field or moving a current loop into or out of a magnetic field.

Here we consider some common examples of rotation of a current loop about its axis in a uniform magnetic field.

AC generator

A generator has a coil of area A rotating about its diameter in a uniform magnetic field with angular velocity ω: In this case it is only the angle between the field and the loop that is

Figure 21: AC generator

varying:

$$\Phi_B = AB\cos\omega t \tag{1}$$

$$\mathcal{E} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = -AB\omega\sin\omega t \tag{2}$$

This system generates an alternating current (AC) with frequency ω. The current is $\pi/2$ out of phase with the rotation, so the peak current is obtained when the flux is zero, i.e. when the loop is parallel to the magnetic field. There is zero current when the loop is perpendicular to the field.

Rotating disc of charge

An insulating disc with a uniform surface charge rotates around its axis. There is a uniform magnetic field parallel to the axis of the disc.

The force on an element of charge, q, on the disc at radius, r, is:

$$\underline{\mathrm{d}F} = qvB\underline{e}_r = qr\omega B\underline{e}_r \tag{3}$$

where $\underline{e}_r$ is the radial basis vector on the disc. This magnetic force is equivalent to a *radial* electric field:

$$\underline{E}' = \underline{F}/q = r\omega B\underline{e}_r \tag{4}$$

and there is an induced emf between the centre and outer radius of the disc:

$$\mathcal{E} = \int \underline{E}' \cdot \mathrm{d}\underline{r} = \frac{\omega B a^2}{2} \tag{5}$$

This emf acts outwards to try and move the charge to the outside of the disc. *If the disc were a conductor this would actually happen.*

So what happens to the flux rule for this type of problem? Basically it is not clear if there is any current loop to consider a flux through. Thus, as it stands, the flux rule $\mathcal{E} = -\mathrm{d}\Phi_B/\mathrm{d}t$ only works when there is a fixed current loop.

11. 2. Mutual Inductance

Consider two current loops I_1,I_2 at rest. The current I_1 will lead to a magnetic field $\underline{B}_1$ which will lead to a magnetic flux through loop 2

$$\boxed{\Phi_2 = \int \underline{B}_1 \cdot \mathrm{d}\underline{S}_2 \equiv M_{21} I_1} \tag{6}$$

M_{21} is the **mutual inductance** of the two loops; it relates the flux through loop 2 to the current in loop 1.

Now let us use the vector potential and Stokes' theorem to obtain an explicit form for M_{12}

$$\begin{aligned} \Phi_2 &= \int (\underline{\nabla} \times \underline{A}_1) \cdot \mathrm{d}\underline{S}_2 = \oint_2 \underline{A}_1 \cdot \mathrm{d}\underline{l}_2 \\ &= \frac{\mu_0 I_1}{4\pi} \oint_1 \oint_2 \frac{\mathrm{d}\underline{l}_1 \cdot \mathrm{d}\underline{l}_2}{|\underline{r}_1 - \underline{r}_2|} \end{aligned}$$

where we have used the formula for the vector potential from section 9 equation (10). Thus

$$\mathrm{M}_{12} = \frac{\mu_0}{4\pi} \oint_1 \oint_2 \frac{\mathrm{d}\underline{l}_1 \cdot \mathrm{d}\underline{l}_2}{|\underline{r}_1 - \underline{r}_2|} \tag{7}$$

where the integrals are taken round both current loops. This is known as the Neumann formula but it is not very useful for most practical applications. What it does reveal is that

$$\mathrm{M}_{12} = \mathrm{M}_{21} = \mathrm{M} \tag{8}$$

which is a remarkable result i.e the flux through 1 when there is current I in 2 is the same as the flux through 2 when there is current I in 1 whatever the geometry of the loops! The relative geometry of the two conductors enters through M which is a purely geometric quantity (a double integral around the loops)

Now let us introduce time-dependence and vary the currentI_1 in 1. The changing flux through 2 then gives rise to an emf

$$\mathcal{E} = -\frac{\mathrm{d}\Phi_2}{\mathrm{d}t} = -M\frac{\mathrm{d}I_1}{\mathrm{d}t}$$

By Lenz's law this emf *opposes* the change in current.

11. 3. Self-Inductance

The above discussion similarly applies to the source loop itself i.e. a changing current in a loop induces a "back emf" which opposes the change in current.

The self-inductance of the loop, L, is defined as the ratio of the induced emf to the current change:

$$\mathcal{E} = -\mathrm{L}\frac{\mathrm{dI}}{\mathrm{d}t} \tag{9}$$

It can also be written as:

$$\boxed{\Phi_B = LI} \tag{10}$$

The unit of inductance is the Henry (H), which is 1 Vs/A.

Inductance of a Solenoid

For a long solenoid (length l, radius a, with n loops per unit length) there is a uniform magnetic field along the axis of the solenoid:

$$B_z = \mu_0 n\mathrm{I} \tag{11}$$

This result can be shown using Ampère's Law (see tutorial 5.1).

The flux through *all* nl loops is:

$$\Phi_B = \int_A \underline{B} \cdot \mathrm{d}\underline{S} = \mu_0 n\mathrm{I}\pi a^2 nl$$

and the self-inductance of the solenoid is:

$$\mathrm{L} = \mu_0 n^2 \pi a^2 l = \mu_0 n^2 V \tag{12}$$

11. 4. Energy Stored in Inductors

The work done to create a current in a loop *against* the induced emf is related to the self-inductance L:

$$\frac{\mathrm{d}U_M}{\mathrm{d}t} = -\mathcal{E}\mathrm{I} = \mathrm{LI}\frac{\mathrm{dI}}{\mathrm{d}t}$$

Integrating this gives:

$$\boxed{U_M = \frac{1}{2}\mathrm{LI}^2} \tag{13}$$

For two coils with a mutual inductance:

$$U_M = \frac{1}{2}\mathrm{L}_1{\mathrm{I}_1}^2 + \frac{1}{2}\mathrm{L}_2{\mathrm{I}_2}^2 + M_{12}\mathrm{I}_1\mathrm{I}_2$$

Example of solenoid For a long solenoid the self inductance and magnetic field are:

$$L = \mu_0 n^2 \pi a^2 l \qquad \underline{B} = \mu_0 n\mathrm{I}$$

The energy stored in the solenoid is:

$$U_M = \frac{1}{2}\mu_0 n^2 \mathrm{I}^2 \pi a^2 l = \frac{1}{2\mu_0}|\underline{B}|^2 \pi a^2 l$$

This can be written in terms of the energy density associated with the magnetic field:

$$\frac{\mathrm{d}U_M}{\mathrm{d}V} = \frac{|\underline{B}|^2}{2\mu_0}$$

Note that this treatment of the energy density of a magnetic field in an inductor is very similar to the treatment of the energy density of an electric field in a capacitor.

We can write the result (13) in a form that uses the magnetic vector potential and the current density. As before the flux is given by

$$\Phi_B = \int (\underline{\nabla} \times \underline{A}) \cdot \mathrm{d}\underline{S} = \oint \underline{A} \cdot \mathrm{d}\underline{l}$$

Thus using the definition of L

$$LI = \oint \underline{A} \cdot \mathrm{d}\underline{l}$$

and we find for a current loop that

$$U_M = \frac{I}{2} \oint \underline{A} \cdot \mathrm{d}\underline{l} = \frac{1}{2} \oint \underline{A} \cdot \underline{I} \mathrm{d}l$$

The generalisation to volume currents is

$$\boxed{U_M = \frac{1}{2} \int \underline{A} \cdot \underline{J}\, \mathrm{d}V} \qquad (14)$$

We can develop (14) further by using Ampère's law and a product rule from lecture 1

$$\begin{aligned} \mu_0 \underline{A} \cdot \underline{J} &= \underline{A} \cdot (\underline{\nabla} \times \underline{B}) \\ &= \underline{B} \cdot (\underline{\nabla} \times \underline{A}) - \underline{\nabla} \cdot (\underline{A} \times \underline{B}) \\ &= \underline{B} \cdot \underline{B} - \underline{\nabla} \cdot (\underline{A} \times \underline{B}) \end{aligned}$$

Consequently

$$\begin{aligned} U_M &= \frac{1}{2\mu_0} \left[\int B^2 \mathrm{d}V - \int \underline{\nabla} \cdot (\underline{A} \times \underline{B})\, \mathrm{d}V \right] \\ &= \frac{1}{2\mu_0} \left[\int B^2 \mathrm{d}V - \oint_S (\underline{A} \times \underline{B}) \cdot \mathrm{d}\underline{S} \right] \end{aligned}$$

The second intergal is a boundary term which vanishes when we take the volume over all space, therefore

$$\boxed{U_M = \frac{1}{2\mu_0} \int_{\mathrm{allspace}} |\underline{B}|^2 \mathrm{d}V} \qquad (15)$$

In a similar way to the electrostatic energy U_E, we can think of the magnetic energy being stored either in the (localised) current distribution (14) or throughout all space in the magnetic field (15).

EM 3 Section 12: The Displacement Current

In this lecture we complete the discussion of the fundamental laws of electromagnetism, and introduce electromagnetic waves for the first time.

12. 1. Continuity equation

Consider a *conserved quantity* for example electric charge—experimentally it is known that electric charge is always conserved.

We consider a volume V and the rate of change of the total charge Q in that volume. In the case where there is no creation or spontaneous loss of charge inside the volume we have

$$-\frac{\partial Q}{\partial t} = \oint_A \underline{J} \cdot \mathrm{d}\underline{S} \tag{1}$$

where the right hand side is a flux integral which expresses the total current *out* of the volume, therefore the left hand side has a negative sign.

Writing the left hand side as a volume integral over charge density ρ and the right hand side as a volume integral by virtue of the divergence theorem gives

$$-\frac{\partial}{\partial t}\int_V \rho\,\mathrm{d}V = \int_V \underline{\nabla} \cdot \underline{J}\,\mathrm{d}V$$

Since this must hold for an arbitrary volume V we deduce the differential form:

$$\boxed{\frac{\partial \rho}{\partial t} = -\underline{\nabla} \cdot \underline{J}} \tag{2}$$

The divergence of the current density at any point is proportional to the rate of change of the charge density at that point.

This is **continuity equation** which is a statement of local conservation (here for charge). In fact it holds for any conserved quantity (mass, energy, electric charge, momentum, and even probability) and is one of the most general and useful equations in physics.

12. 2. The Displacement Current

Let us return to the differential form of Ampère's law

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J} \tag{3}$$

and take the divergence of both sides:

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{B} = \mu_0 \underline{\nabla} \cdot \underline{J}$$

Now since the divergence of a curl is always zero we find

$$\underline{\nabla} \cdot \underline{J} = 0$$

This result is inconsistent with the continuity equation since generally (unless the charge distribution is static)

$$\frac{\partial \rho}{\partial t} = -\underline{\nabla} \cdot \underline{J} \neq 0$$

To satisfy the continuity equation generally we need to modify Ampère's law (MIV) by the addition of a **displacement current** term to go along with $\underline{J}$ i.e. we want to have when we take the divergence of the modified MIV

$$\underline{\nabla} \cdot (\underline{\nabla} \times \underline{B}) = \mu_0 \left(\underline{\nabla} \cdot \underline{J} + \frac{\partial \rho}{\partial t} \right) = 0 \tag{4}$$

Using Gauss' law MI we can replace ρ with the divergence of the electric field:

$$\underline{\nabla} \cdot (\underline{\nabla} \times \underline{B}) = \mu_0 \left(\underline{\nabla} \cdot \underline{J} + \epsilon_0 \frac{\partial}{\partial t} (\underline{\nabla} \cdot \underline{E}) \right)$$

The order of the time derivative and the divergence of the electric field can be reversed, and the divergence operation removed from all terms to leave:

$$\boxed{\underline{\nabla} \times \underline{B} = \mu_0 \left(\underline{J} + \epsilon_0 \frac{\partial \underline{E}}{\partial t} \right)} \tag{5}$$

This is the Ampère-Maxwell law (**MIV**) which holds for both static and time-varying charge distributions and fields.
Maxwell's stroke of genius was to include the displacement current term—often called the Maxwell correction—albeit for different reasons than we have given here! In any case we conclude that
The effect of a time-varying electric field is to produce an additional contribution to the curl of the magnetic field.

Is the displacement actually a current? Answer is not really (see next subsection) but it does, of course, have the dimensions of a current.

12. 3. Capacitor Paradox and Resolution

Consider the circuit in the figure which illustrates a parallel plate capacitor charging up

Figure 22: Capacitor paradox *(Griffiths fig 7.42)*

and current $I(t)$ flowing in the wire. If we want to compute $\underline{B}$ by taking an Amperian loop in the form of a circle around the wire (outside of the capacitor) then the surface S that we should take to compute

$$\oint \underline{B} \cdot \underline{\mathrm{d}l} = \mu_0 \int_S \underline{J} \cdot \underline{\mathrm{d}S}$$

does not appear to be well-defined e.g. taking $S = S_1$ as the surface of the disc in the plane of the loop gives $\int_{S_1} \underline{J} \cdot \underline{\mathrm{d}S} = I$; but taking $S = S_2$ as an extended surface which goes through the gap between the plates and which does not cross the wire gives $\int_{S_2} \underline{J} \cdot \underline{\mathrm{d}S} = 0$ since there is no current flowing between the plates. But really Ampère's law should hold independent of the surface bounded by the fixed loop.

If, on the other hand, we consider MIV with the Maxwell correction we *replace* the old Ampère's law in integral form by the new version

$$\int_S (\underline{\nabla} \times \underline{B}) \cdot \underline{\mathrm{d}S} = \oint \underline{B} \cdot \underline{\mathrm{d}l} = \mu_0 \int_S \left(\underline{J} + \epsilon_0 \frac{\partial \underline{E}}{\partial t} \right) \cdot \underline{\mathrm{d}S}$$

Now we know (at least quasistatically) that between the plates of the capacitor, $\underline{E}$ is normal to the plates and $|\underline{E}| = \dfrac{Q}{\epsilon_0 A}$. Therefore $\epsilon_0 \dfrac{\partial \underline{E}}{\partial t}$ is a vector with magnitude $\dot{E} = \dfrac{I}{A}$. Thus inside the plates $\epsilon_0 \displaystyle\int_{S_2} \frac{\partial \underline{E}}{\partial t} \cdot \underline{\mathrm{d}S} = I$ and gives the same contribution as does $\displaystyle\int_{S_1} \underline{J} \cdot \underline{\mathrm{d}S}$ outside the plates—see tutorial sheet 6. Thus the capacitor paradox is resolved. Also we see that the diplacement current is not a real current as no current flows between the capacitor plates.

One final thing to notice about the displacement current term is that, due to the factor $\epsilon_0 \simeq 9 \times 10^{-12} \mathrm{C}^2/\mathrm{NM}^2$, it is typically *much* smaller than the current term. Thus when there is a current flowing the current term dominates the displacement current term.

12. 4. Maxwell's Equations

The laws of electromagnetism are summarised in four differential equations (MI-IV) known as Maxwell's equations:

$$\underline{\nabla} \cdot \underline{E} = \frac{\rho}{\epsilon_0} \tag{6}$$

$$\underline{\nabla} \cdot \underline{B} = 0 \tag{7}$$

$$\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \tag{8}$$

$$\underline{\nabla} \times \underline{B} = \mu_0 \left(\underline{J} + \epsilon_0 \frac{\partial \underline{E}}{\partial t} \right) \tag{9}$$

MII and MII are Gauss' Laws for electric and magnetic fields
MIII is Faraday's law of induction
MIV is Ampère-Maxwell law including the displacement current

In the *electrostatic limit* Poisson's equation is obtained from MI & MIII:

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \qquad \text{when} \qquad \frac{\partial \underline{B}}{\partial t} = 0$$

In the *magnetostatic limit* Poisson's equations for the magnetic vector potential are obtained from MII & MIV:

$$\nabla^2 \underline{A} = -\mu_0 \underline{J} \qquad \text{when} \qquad \frac{\partial \underline{E}}{\partial t} = 0$$

The continuity equation is obtained from MI & MIV:

$$\underline{\nabla} \cdot \underline{J} = -\frac{\partial \rho}{\partial t} \tag{10}$$

12. 5. Solution of Maxwell's Equations in Vacuo

In a vacuum there are no charges present:

$$\rho = 0 \qquad \underline{J} = 0 \tag{11}$$

We take the curl of MIII

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = -\frac{\partial(\underline{\nabla} \times \underline{B})}{\partial t}$$

which inserting MIV yields

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = -\epsilon_0 \mu_0 \frac{\partial^2 \underline{E}}{\partial t^2}$$

Similarly taking the curl of MIV leads to

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{B}) = -\epsilon_0 \mu_0 \frac{\partial^2 \underline{B}}{\partial t^2}$$

Now we make use of the vector identity (to be memorised) for a vector field $\underline{F}$:

$$\boxed{\underline{\nabla} \times (\underline{\nabla} \times \underline{F}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{F}) - \nabla^2 \underline{F}} \tag{12}$$

In the absence of charges MI becomes $\underline{\nabla} \cdot \underline{E} = 0$ and from MII $\underline{\nabla} \cdot \underline{B} = 0$, we are left with two **wave equations**:

$$\nabla^2 \underline{E} = \epsilon_0 \mu_0 \frac{\partial^2 \underline{E}}{\partial t^2} \tag{13}$$

$$\nabla^2 \underline{B} = \epsilon_0 \mu_0 \frac{\partial^2 \underline{B}}{\partial t^2} \tag{14}$$

Thus we have *decoupled* the four (first order) Maxwell's equations for $\underline{B}$ and $\underline{E}$ in the vacuum, at the price of now having second order equations. But we know that the solution of these second order wave equations (to be revised next lecture) will be electromagnetic waves. The velocity of the electromagnetic waves is the speed of light:

$$c^2 = \frac{1}{\epsilon_0 \mu_0} = (3 \times 10^8 \mathrm{ms}^{-1})^2 \tag{15}$$

Maxwell's equations predict that light, radio waves, X-rays etc. are all types of waves associated with oscillating electric and magnetic fields in a vacuum.

N.B. There are no charges present in a vacuum, and the waves propagate without the presence of matter!

EM 3 Section 13: Description of Electromagnetic Waves

13. 1. Recap of wave equations

Let us recall (see Mathematics for Physics 4 and Physics 2A) the wave equation in 1d (i.e. one spatial dimension x and one time dimension t) for a scalar field u

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}} \tag{1}$$

Now, as can readily be checked by substitution into (1) the general solution is *any* function f of the form

$$u(x,t) = f(kx - \omega t) \tag{2}$$

where the wave velocity c is given by

$$c = \frac{w}{k} \tag{3}$$

A convenient solution of special interest is

$$f = A \exp i(kx - \omega t) = A\cos(kx - \omega t) + iA\sin(kx - \omega t) \tag{4}$$

These are sinusoidal waves and A is the constant amplitude (which may be complex) N.B. the real (cosine) and imaginary (sine) parts are independent solutions. Moreover it is a **monochromatic wave** since there is a single angular frequency ω. These are the basis of Fourier methods where we build up waves of arbitrary shape by superposition of sines and cosines

If for physical reasons we want to get a real solution from (4) we simply take the real part

$$\begin{aligned} u(x,t) &= \mathrm{Re}\left[A \exp i(kx - \omega t)\right] \\ &= \mathrm{Re}A\,\cos(kx - \omega t) - \mathrm{Im}A\,\sin(kx - \omega t) \end{aligned} \tag{5}$$

Important things to remember are : k is the wavenumber; the angular frequency is $\omega = 2\pi\nu$ where ν is the frequency; the wavelength is $\lambda = 2\pi/k$; the whole wave proceeds to the right with speed c, but at any fixed x the wave oscillates with period $T = 2\pi/\omega = 2\pi/kc$.

The 1d equation (1) generalises easily to 3d

$$\boxed{\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}} \tag{6}$$

where the second derivative w.r.t. x has been replaced by the Laplacian operator.

The solution (2) generalises to

$$u(\underline{r},t) = f(\underline{k}\cdot\underline{r} - \omega t) \tag{7}$$

where the **wavevector** $\underline{k} = (k_x, k_y, k_z)$ and the velocity is again

$$c = \frac{w}{k} \tag{8}$$

where $k = |\underline{k}|$. We can also write (7) as

$$u(\underline{r}, t) = g(\underline{\hat{n}} \cdot \underline{r} - ct) \tag{9}$$

where $\underline{\hat{n}}$ is the unit vector in the direction of $\underline{k}$.

13. 2. Plane Waves

The generalisation of the 1d sinusoidal solution (4) is to the 3d **plane wave** solution

$$u(\underline{r}, t) = A \exp i(\underline{k} \cdot \underline{r} - \omega t) \tag{10}$$

The is called a plane wave because it takes the same (complex) value whenever

$$\underline{k} \cdot \underline{r} = \omega t + \text{constant} \tag{11}$$

which at any fixed t is the equation of a *plane* with normal in the $\underline{k}$ direction.
To see that (2) is a solution to (12) note that

$$\begin{aligned} \underline{\nabla} \exp i\underline{k} \cdot \underline{r} &= \left[\underline{e}_x \frac{\partial}{\partial x} + \underline{e}_y \frac{\partial}{\partial y} + \underline{e}_z \frac{\partial}{\partial x} \right] \exp i(k_x x + k_y y + k_z z) \\ &= i\underline{k} \exp i\underline{k} \cdot \underline{r} \end{aligned}$$

and

$$\nabla^2 \exp i\underline{k} \cdot \underline{r} = \underline{\nabla} \cdot \underline{\nabla} \exp i\underline{k} \cdot \underline{r} \;=\; i\underline{\nabla} \cdot (\underline{k} \exp i\underline{k} \cdot \underline{r}) = i\underline{k} \cdot \underline{\nabla} \exp i\underline{k} \cdot \underline{r} = -k^2 \exp i\underline{k} \cdot \underline{r}$$

where we used the product identity $\underline{\nabla} \cdot (\underline{k} f) = f \underline{\nabla} \cdot \underline{k} + \underline{k} \cdot \underline{\nabla} f = \underline{k} \cdot \underline{\nabla} f$ since $\underline{k}$ is constant.
Also

$$\frac{\partial^2 \exp i(\underline{k} \cdot \underline{r} - \omega t)}{\partial t^2} = -\omega^2 \exp i(\underline{k} \cdot \underline{r} - \omega t)$$

Finally we can generalise to the 3d wave equation for a *vector field* $\underline{F}$

$$\boxed{\nabla^2 \underline{F} = \frac{1}{c^2} \frac{\partial^2 \underline{F}}{\partial t^2}} \tag{12}$$

for which a plane wave solution is

$$\underline{F} = \underline{F}_0 \exp i(\underline{k} \cdot \underline{r} - \omega t) \tag{13}$$

where $\underline{F}_0$ is a constant (complex) vector. The key things to remember with this plane wave solution are

$$\underline{\nabla} \cdot \underline{F} = i\underline{k} \cdot \underline{F} \tag{14}$$

$$\underline{\nabla} \times \underline{F} = i\underline{k} \times \underline{F} \tag{15}$$

$$\nabla^2 \underline{F} = -k^2 \underline{F} \tag{16}$$

13. 3. Electromagnetic Plane Waves

Previously we saw that *in vacuo* Maxwell's equations with $\rho = 0$, $\underline{J} = 0$ read

$$\underline{\nabla} \cdot \underline{E} = 0 \tag{17}$$

$$\underline{\nabla} \cdot \underline{B} = 0 \tag{18}$$

$$\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \tag{19}$$

$$\underline{\nabla} \times \underline{B} = \mu_0 \epsilon_0 \frac{\partial \underline{E}}{\partial t} \tag{20}$$

and reduce to the decoupled wave equations

$$\nabla^2 \underline{E} = \epsilon_0 \mu_0 \frac{\partial^2 \underline{E}}{\partial t^2} \tag{21}$$

$$\nabla^2 \underline{B} = \epsilon_0 \mu_0 \frac{\partial^2 \underline{B}}{\partial t^2} \tag{22}$$

Clearly we have plane solutions

$$\underline{E} = \underline{E}_0 \exp i(\underline{k} \cdot \underline{r} - \omega t) \quad \underline{B} = \underline{B}_0 \exp i(\underline{k} \cdot \underline{r} - \omega t) \tag{23}$$

moving at the speed of light $c = \omega/k = \dfrac{1}{\sqrt{\epsilon_0 \mu_0}}$. However Maxwell's equations imply more constraints on our plane wave solutions. First MI, MII imply

$$i\underline{k} \cdot \underline{E}_0 = 0 \qquad i\underline{k} \cdot \underline{B}_0 = 0$$

i.e. $\underline{E}_0$ and $\underline{B}_0$ and hence $\underline{E}$ and $\underline{B}$ are *perpendicular* to the direction of propagation $\underline{k}$. That is, the wave is **transverse**.

It is usually convenient to take the direction of propagation $\underline{k}$ in the $\underline{e}_z$ direction;

$$\underline{k} = k\underline{e}_z \tag{24}$$

therefore $\underline{E}_0$ and $\underline{B}_0$ lie in the x–y plane. Substituting (3) in MIII we find

$$i\underline{k} \times \underline{E}_0 \exp i(\underline{k} \cdot \underline{r} - \omega t) \quad = i\omega \underline{B}_0 \exp i(\underline{k} \cdot \underline{r} - \omega t)$$

or more compactly

$$\underline{B}_0 = \frac{k}{\omega}(\underline{e}_z \times \underline{E}_0) \tag{25}$$

Now since $\underline{e}_z$ and $\underline{E}_0$ are orthogonal we can take magnitudes

$$|\underline{B}_0| = \frac{k}{\omega}|\underline{E}_0| \tag{26}$$

Now we should choose the directions of $\underline{B}_0$ and $\underline{E}_0$. (12) tells us that **The magnetic field is perpendicular to the electric field, and both are perpendicular to the direction of propagation of the wave**.

There are two comon **polarisation states** which can be defined in various ways:

- Linearly (or plane) polarized - direction of $\underline{E}$ is always in x or y direction.
- Circularly polarized - direction of $\underline{E}$ rotates clockwise or anticlockwise around the z axis in the x–y plane.

An unpolarised electromagnetic wave has a random direction for $\underline{E}$ *as a function of z.*

13. 4. Linear (Plane) Polarisation

Let us first consider the case where $\underline{B}_0$ and $\underline{E}_0$ are *real*. Then, since they lie in the x–y plane it is conventional to take $\underline{E}_0 = E_0\underline{e}_x$ and $\underline{B}_0 = B_0\underline{e}_y$. This is referred to as linear polarisation in the x direction i.e. the electric field is always in the x direction and magnetic field is always in the y direction and $\underline{k}$ is in the z direction as usual. Polarisation in the y direction would have $\underline{E}_0 = E_0\underline{e}_y$, $\underline{B}_0 = -B_0\underline{e}_x$. More generally we can take $\underline{E}_0 = E_0\underline{\hat{n}}$

Figure 23: Plane polarisation in x direction *(Griffiths fig 9.10)*

$$\underline{E}_0 \cdot \underline{e}_x = E_0\underline{\hat{n}} \cdot \underline{e}_x = E_0 \cos\theta$$

where $\underline{\hat{n}}$ is the polarisation vector and θ is the polarisation angle.

13. 5. Circular Polarisation

Now consider taking $\underline{E}_0$ as a *complex* vector

$$\underline{E}_0 = \frac{E_0}{\sqrt{2}}(\underline{e}_x \pm i\underline{e}_y)\mathrm{e}^{i\phi} \tag{27}$$

Then we find that the real part of $\underline{E}$ is given by

$$\mathrm{Re}\,\underline{E} = \frac{E_0}{\sqrt{2}}\left[\underline{e}_x \cos(\underline{k}\cdot\underline{r} - \omega t + \phi) \mp \underline{e}_y \sin(\underline{k}\cdot\underline{r} - \omega t + \phi)\right] \tag{28}$$

The *minus* sign in (27) implies that the polarisation vector rotates anticlockwise about the $\underline{e}_z$: i.e. at time $\omega t = \underline{k}\cdot\underline{r} + \phi$, Re$\underline{E}$ is in the $\underline{e}_x$ direction but as time increases the polarisation vector rotates towards $-\underline{e}_y$. This also referred to left circular polarisation or positive helicity

Likewise the *plus* sign in (27) implies that the polarisation vector rotates clockwise about the $\underline{e}_z$ This is referred to as right circular polarisation or negative helicity.

EM 3 Section 14: Electromagnetic Energy and the Poynting Vector

*14. 1. Poynting's Theorem (*Griffiths 8.1.2*)*

Recall we saw that the total energy stored in electromagnetic fields is:

$$U = U_M + U_E = \frac{1}{2}\int_{allspace}\left(\frac{1}{\mu_0}B^2 + \epsilon_0 E^2\right)\mathrm{d}V \qquad (1)$$

Let us now derive this more generally. Consider some distribution of charges and currents. In small time dt a charge will move $\underline{v}\mathrm{d}t$ and, according to the Lorentz force law, the work done on the charge will be

$$\mathrm{d}U = \underline{F}\cdot\underline{\mathrm{d}l} = q(\underline{E} + \underline{v}\times\underline{B})\cdot\underline{v}\mathrm{d}t = q\underline{E}\cdot\underline{v}\mathrm{d}t$$

where as usual the magnetic forces do no work. Now let $q = \rho\mathrm{d}V$ (usual definition of charge density) and $\rho\underline{v} = \underline{J}$ (usual definition of current). Then dividing through by dt and integrating over a volume V containing the charges, we find that the rate at which work is done (i.e. the power delivered to the system) is

$$\boxed{\frac{\mathrm{d}U}{\mathrm{d}t} = \int_V \underline{E}\cdot\underline{J}\,\mathrm{d}V} \qquad (2)$$

Thus $\underline{E}\cdot\underline{J}$ is the power delivered per unit volume. Now use MIV to express

$$\underline{E}\cdot\underline{J} = \frac{1}{\mu_0}\underline{E}\cdot(\underline{\nabla}\times\underline{B}) - \epsilon_0\underline{E}\cdot\frac{\partial\underline{E}}{\partial t}$$

Furthermore we can use a product rule from lecture 1 to write

$$\begin{aligned}\underline{E}\cdot(\underline{\nabla}\times\underline{B}) &= \underline{B}\cdot(\underline{\nabla}\times\underline{E}) - \underline{\nabla}\cdot(\underline{E}\times\underline{B})\\ &= -\underline{B}\cdot\frac{\partial\underline{B}}{\partial t} - \underline{\nabla}\cdot(\underline{E}\times\underline{B})\end{aligned}$$

where we used MIII in the last line. Putting it all together, and noting

$$\underline{B}\cdot\frac{\partial\underline{B}}{\partial t} = \frac{1}{2}\frac{\partial B^2}{\partial t} \qquad \underline{E}\cdot\frac{\partial\underline{E}}{\partial t} = \frac{1}{2}\frac{\partial E^2}{\partial t}\ ,$$

yields

$$\underline{E}\cdot\underline{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\underline{\nabla}\cdot(\underline{E}\times\underline{B})$$

Finally we can integrate over the volume V containing the currents and charges and use the divergence theorem on the second term to obtain from (9)

$$\boxed{\frac{\mathrm{d}U}{\mathrm{d}t} = -\frac{\partial}{\partial t}\int_V \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)\mathrm{d}V - \frac{1}{\mu_0}\oint_S(\underline{E}\times\underline{B})\cdot\underline{\mathrm{d}S}} \qquad (3)$$

Let us now examine each term in **Poynting's Theorem** (10): the left hand side is the power delivered to the volume i.e. the rate of *gain* in energy of the particles; the first term on the right hand side is the rate of *loss* of electromagnetic energy stored in fields *within* the volume; the second term is the rate of energy transport *out* of the volume i.e. across the surface S.

Thus Poynting's theorem reads: energy *lost* by fields = energy *gained* by particles+ energy flow out of volume. Hence we can identify the vector

$$\boxed{\underline{S} = \frac{1}{\mu_0}\underline{E} \times \underline{B}} \tag{4}$$

as the **energy flux density** (energy per unit area per unit time) and it is known as the **Poynting vector** (it 'Poynts' in the direction of energy transport).

Also we can write Poynting's theorem as a continuity equation for the total energy $U = U_{em} + U_{mec}$. The left hand side of (10) is the rate of change of *mechanical energy* thus

$$\frac{\mathrm{d}(U_{em} + U_{mec})}{\mathrm{d}t} = -\oint_S \underline{S} \cdot \mathrm{d}\underline{A}$$

(to avoid a nasty clash of notation with $\underline{S}$ as Poynting vector we use $\mathrm{d}\underline{A}$ rather than $\mathrm{d}\underline{S}$ as vector element of area). As usual, expressing energy as a volume over energy *densities* u_{em},u_{mec} and using the divergence theorem on the right hand side we arrive at

$$\boxed{\frac{\partial}{\partial t}(u_{em} + u_{mec}) = -\underline{\nabla} \cdot \underline{S}} \tag{5}$$

which is the continuity equation for energy density. Thus the Poynting vector represents the flow of energy in the same way that the current $\underline{J}$ represents the flow of charge.

*14. 2. Energy of Electromagnetic Waves (*Griffiths 9.2.3*)*

As we saw last lecture a monochromatic plane wave in vacuo propagating in the $\underline{e}_z$ direction is described by the fields:

$$\underline{E} = \underline{e}_x E_0 \cos(kz - \omega t) \qquad \underline{B} = \underline{e}_y B_0 \cos(kz - \omega t) \tag{6}$$

where

$$B_0 = \frac{E_0}{c}$$

The total energy stored in the fields associated with the wave is:

$$U = U_E + U_M \quad = \quad \frac{1}{2}\int_V \left(\frac{B^2}{\mu_0} + \epsilon_0 E^2\right) \mathrm{d}V$$

Now since $|\underline{B}| = \frac{|\underline{E}|}{c}$ and $c = 1/\sqrt{\mu_0 \epsilon_0}$ we see that the electric and magnetic contributions to the total energy are equal and the electromagnetic energy *density* is (for a linearly polarised wave)

$$u_{EM} = \epsilon_0 E^2 = \epsilon_0 E_0^2 \cos^2(kz - \omega t)$$

The Poynting vector becomes for monochromatic waves

$$\underline{S} = \frac{1}{\mu_0}(\underline{E} \times \underline{B}) = c\epsilon_0 E_0^2 \cos^2(kz - \omega t)\underline{e}_z = u_{EM} c\underline{e}_z$$

That $\underline{S}$ is just the energy density multiplied by the velocity of the wave $c\underline{e}_z$ as it should be. Generally

$$\underline{S} = u_{EM} c \hat{\underline{k}}$$

N.B To compute the Poynting vector is is simplest to use a real form for the fields $\underline{B}$ and $\underline{E}$ rather than a complex exponential representation.

The time average of the energy density is is defined as the average over one period T of the wave

$$\begin{aligned} \langle u_{EM} \rangle &= \frac{\epsilon_0 E_0^2}{T} \int_0^T \cos^2(kz - \omega t) \mathrm{d}t \\ &= \frac{\epsilon_0 E_0^2}{T} \frac{T}{2} = \frac{1}{2} \epsilon_0 E_0^2 = \frac{1}{2} \frac{B_0^2}{\mu_0} \end{aligned}$$

The energy density of an electromagnetic wave is proportional to the square of the amplitude of the electric (or magnetic) field.

14. 3. Example of discharging capacitor

Consider a discharging circular parallel plate capacitor (plates area A) in a circuit with a

Figure 24: Discharging capacitor in a circuit with a resistor

resistor R. Ohm's law gives

$$V_d = \frac{Q}{C} = IR$$

or

$$I = -\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{Q}{RC} \quad \Rightarrow \quad Q = Q_0 \mathrm{e}^{-t/RC} \quad I = -\frac{Q_0}{RC} \mathrm{e}^{-t/RC}$$

Now assume 'quasistatic' approximation that we can treat the fields as though they were static:

$$\underline{E} = -\frac{Q}{A\epsilon_0} \hat{\underline{n}} = -\frac{Q}{A\epsilon_0} \mathrm{e}^{-t/RC}$$

We take the normal to the plates (direction of $\underline{E}$) is $\hat{\underline{n}}$. Now we can compute $\underline{B}$ through Ampère-Maxwell noting that the cylindrical symmetry implies that $\underline{B}$ is circumferential. The Amperian loop is a circle radius r between the capacitor plates where $\underline{J} = 0$

$$\oint \underline{B} \cdot \mathrm{d}\underline{l} = \mu_0 \int_S \left(\underline{J} + \epsilon_0 \frac{\partial \underline{E}}{\partial t} \right) \cdot \mathrm{d}\underline{S} = -\mu_0 \pi r^2 \epsilon_0 \frac{\partial}{\partial t} \left(\frac{Q}{A\epsilon_0} \mathrm{e}^{-t/RC} \right) = \frac{\mu_0 I(t)}{2A} r$$

so

$$\underline{B} = \frac{\mu_0 I(t) r}{2A} \underline{e}_\phi$$

The Poynting vector is given by

$$\underline{S} = \frac{1}{\mu_0} \underline{E} \times \underline{B} = -\frac{Q}{A\epsilon_0} \mathrm{e}^{-t/RC} \mu_0 I_0 \frac{r}{2A} \mathrm{e}^{-t/RC} \underline{e}_z \times \underline{e}_\phi = \frac{I_0^2 CR}{2A^2 \epsilon_0} r \mathrm{e}^{-2t/RC} \underline{e}_r$$

Thus the Poynting vector points *radially out of the capacitor* and this is the direction of energy flow.

14. 4. Momentum of electromagnetic radiation

Let us reinterpret the Poynting vector from a quantum perspective. Due to wave-particle duality, radiation can be thought of as photons travelling with speed c with energy

$$\varepsilon = \hbar\omega = h\nu$$

The momentum of a single photon

$$\underline{p} = \hbar \underline{k} = \frac{\varepsilon}{c} \hat{\underline{k}}$$

For n photons per unit volume travelling at speed c we can interpret the average Poynting vector as average energy density $n\varepsilon$ multiplied by velocity vector $c\hat{\underline{k}}$

$$\langle \underline{S} \rangle = n\varepsilon c \hat{\underline{k}} = \langle u_{EM} \rangle c \hat{\underline{k}}$$

Again thinking of the energy transport as effected by photons, we must have an accompanying **momentum flux** $\widetilde{\underline{P}}$

$\widetilde{\underline{P}}$ is defined as the momentum carried across a plane normal to propagation, per unit area per unit time

For each photon $p = \varepsilon/c$ (along $\hat{\underline{k}}$) so

$$\tilde{\underline{P}} = \underline{S}/c$$

If light strikes the absorber (normal incidence) momentum is absorbed, this creates a force per unit area equal to the incoming (normal) momentum flux

This causes radiation pressure

$$p_{rad} = \widetilde{\underline{P}} \cdot \hat{\underline{n}} = S/c \quad \Rightarrow \quad p_{rad} = \langle u_{EM} \rangle$$

If light is reflected not absorbed so twice the momentum is imparted, p_{rad} doubles but so does $\langle u_{EM} \rangle$, and this result still holds.

To understand radiation pressure classically let's go back to the example of an x polarised wave propagating in $\underline{e}_z$ direction: the electric field moves charges, on the surface the radiation strikes, in the x direction; then the Lorentz force $q\underline{v} \times \underline{B}$ (with $\underline{v}$ in the x direction and $\underline{B}$ in the y direction) is in the $\underline{e}_z$ direction and creates the pressure.

Above is for a **collimated** light beam (i.e. single direction) The other extreme is "diffuse radiation" = light bouncing around in all directions; this gives instead

$$p_{rad} = \langle u_{EM} \rangle / 3$$

(factor 1/3 as in kinetic theory of gases)

EM 3 Section 15: Dielectric Materials

15. 1. Overview

So far we have developed Maxwell's equations and they offer a complete and general description of electrodynamics. However the input we have to make is to define the charge and current densities ρ and $\underline{J}$ with *microscopic precision*. In the real world (i.e. not in vacuo) this would be a huge task as materials are made up of atoms/molecules which all contain charge distributions and currents (through electronic orbits). This as the atomic level of description.

Instead we want to develop a *macroscopic description* of materials in terms of smoothly varying quantities: these turn out to be the density ρ_f and current $\underline{J}_f$ of **free charges**. The **bound charges** which are bound up in the atomic structure are dealt with by defining new fields $\underline{D}$ the **Electric Displacement Field** and $\underline{H}$ the **Auxiliary (magnetic) Field** Then we end up with a complementary *macroscopic form of Maxwell's equations.* Although it may seem annoying to have to learn a second set of Maxwell's equations, they are in some ways simpler than the microscopic ones.

15. 2. Dielectric Materials

Roughly speaking we can classify materials as conductors or dielectrics (insulators). A perfect conductor will have an 'unlimited' supply of free charges whereas at the other extreme a perfect dielectric will have no free charges and instead all charges are bound up in atoms/molecules.

Figure 25: Polarization of Dipoles in a Dielectric

Let us consider the effect of an electric field on a dielectric. The field will induce a dipole moment in two ways

- the charge distribution of some atoms/molecules is distorted
- already polar molecules (e.g. H_2O) will tend to align with the external field (rotation)

These effects *polarize* the material and result in an induced dipole moment for each atom

$$\langle \underline{p}_{atom} \rangle = \alpha \underline{E} \tag{1}$$

where α is the atomic polarizability. We take an average in (1) as an atom's dipole moment will not be constant due to thermal fluctuations. All these atomic dipole moments give rise to the dipole moment per unit volume $\underline{P}$ or **Polarization**

$$\boxed{\underline{P} = n\langle \underline{p}_{atom} \rangle} \tag{2}$$

Here n is the number of atoms per unit volume and $\langle \underline{p}_{atom} \rangle$ is the average atomic dipole moment induced by the field

Then we can write the dipole moment for some some volume $\mathrm{d}V$ as

$$\boxed{\mathrm{d}\underline{p} = \underline{P}\,\mathrm{d}V} \tag{3}$$

Let us now consider the field *due* to the polarized molecules. Recall that for a single dipole at $\underline{r}'$ the potential at $\underline{r}$ is

$$V(\underline{r}) = \frac{1}{4\pi\epsilon_0}\frac{(\underline{r} - \underline{r}') \cdot \underline{p}}{|\underline{r} - \underline{r}'|^3}$$

This generalises by superposition to the potential due to the Polarization field $\underline{P}(\underline{r}')$

$$V(\underline{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{(\underline{r} - \underline{r}') \cdot \underline{P}(\underline{r}')}{|\underline{r} - \underline{r}'|^3}\mathrm{d}V'$$

We now note a usual identity but this time for the gradient wrt the primed coordinates

$$\underline{\nabla}'\left(\frac{1}{|\underline{r} - \underline{r}'|}\right) = \frac{(\underline{r} - \underline{r}')}{|\underline{r} - \underline{r}'|^3}$$

Then we perform 'integration by parts' using the divergence theorem

$$\begin{aligned} V(\underline{r}) &= \frac{1}{4\pi\epsilon_0}\int_V \underline{P}(\underline{r}') \cdot \underline{\nabla}'\left(\frac{1}{|\underline{r} - \underline{r}'|}\right)\mathrm{d}V' \\ &= \frac{1}{4\pi\epsilon_0}\left[\int_V \underline{\nabla}' \cdot \left(\frac{\underline{P}}{|\underline{r} - \underline{r}'|}\right)\mathrm{d}V' - \int_V \frac{1}{|\underline{r} - \underline{r}'|}\underline{\nabla}' \cdot \underline{P}\mathrm{d}V'\right] \\ &= \frac{1}{4\pi\epsilon_0}\oint_S \frac{\underline{P} \cdot \mathrm{d}\underline{S}}{|\underline{r} - \underline{r}'|} - \frac{1}{4\pi\epsilon_0}\int_V \frac{1}{|\underline{r} - \underline{r}'|}\left(\underline{\nabla}' \cdot \underline{P}\right)\mathrm{d}V' \end{aligned}$$

Now the first term on the right hand side is equivalent to the potential due to a surface charge distribution on S i.e. $\underline{P} \cdot \mathrm{d}\underline{S} \to \sigma \mathrm{d}S$ or

$$\boxed{\sigma_b = \underline{P} \cdot \underline{\hat{n}}} \tag{4}$$

The second term on the lhs is equivalent to the potential due to a volume charge distribution ρ_b which is given by

$$\boxed{\rho_b = -\underline{\nabla} \cdot \underline{P}} \tag{5}$$

The subscript b refers to the fact the charges are bound (to the atoms)

15. 3. Electric displacement vector and Gauss' law in media

We are now in a position to develop Gauss' law in the case of media. The key idea is to divide up the charge distribution into bound and free charges

$$\rho = \rho_b + \rho_f$$

Then Gauss's law (MI) becomes

$$\underline{\nabla} \cdot \underline{E} = \frac{\rho_f}{\epsilon_0} + \frac{\rho_b}{\epsilon_0} = \frac{\rho_f}{\epsilon_0} - \frac{\underline{\nabla} \cdot \underline{P}}{\epsilon_0}$$

or

$$\underline{\nabla} \cdot (\epsilon_0 \underline{E} + \underline{P}) = \rho_f \tag{6}$$

Now let us define the **Electric displacement** as

$$\boxed{\underline{D} \equiv \epsilon_0 \underline{E} + \underline{P}} \tag{7}$$

Gauss' law in media then becomes

$$\boxed{\underline{\nabla} \cdot \underline{D} = \rho_f} \tag{8}$$

15. 4. Linear Homogeneous Media

So far, so good, but at the expense of the introduction of a new field $\underline{D}$ in addition to $\underline{E}$. However things become simpler when we consider an ideal type of medium which is *linear, isotropic and homogeneous* (LIH).

Isotropic means there is no preferred direction which implies through symmetry that $\underline{P}$ is $\parallel$ to $\underline{E}$. *Linear* means that the applied $\underline{E}$ field results in a generally small polarization of molecules through distortion and rotation, and we expect a *linear* response to the field

$$\underline{P} = \chi_E \epsilon_0 \underline{E} \tag{9}$$

χ_E (chi) is the susceptibility—large χ_E means a large response to the applied field and the medium is easier to polarize.

Homogeneous means the medium has the same properties at all points in space so that χ_E has no spatial dependence.

Using (9) results in

$$\begin{aligned} \underline{D} &= \epsilon_0 \underline{E} + \underline{P} = \epsilon_0 (1 + \chi_E) \underline{E} \\ &\equiv \epsilon_0 \epsilon_r \underline{E} \end{aligned} \tag{10}$$

ϵ_r is the *relative permittivity* (or dielectric constant) of the medium and is a dimensionless constant $= 1$ for vacuum; for most insulators $\epsilon_r = 1.05 - 1.3$. Some crystals have high ϵ_r, e.g. mica: $\underline{e}_r = 7$. For dipolar fluids, e.g. deionized water: $\epsilon_r = 80$.

The important point is that for LIH we have a linear *constitutive relation* (10) between $\underline{E}$ and $\underline{D}$.

15. 5. Example: Dielectrics in Capacitors

The space between the two plates of a capacitor can be filled with an insulating material rather than with a vacuum. There are induced polarization (bound) charges on the surfaces next to the plates. These change the capacitance in a way that depends on the geometry of the insulator and the plates. For a *parallel plate capacitor*: the electric field is simply

Figure 26: Parallel plate capacitor with dielectric

the superposition of the field from the free charges on the plates and bound charges at the surface of the dielectric

$$\underline{E} = \underline{E}_0 + \underline{E}_P = \frac{1}{\epsilon_0}(\sigma_f - \sigma_b)\underline{\hat{n}} \tag{11}$$

where $\underline{\hat{n}}$ is normal to the plates The electric field $\underline{E}$ as a function of the free charge density on the plates σ_f is *reduced* by the polarization of the dielectric between the plates. N.B. the total free charge on a plate is still $Q = A\sigma_f$. Also the electric displacement turns out to simply be

$$\underline{D} = \sigma_f \underline{\hat{n}}$$

this can be checked by the modified version of Gauss's Law which gives

$$\oint_S \underline{D} \cdot \mathrm{d}\underline{S} = \int_V \rho_f \, \mathrm{d}V = (Q)_{enc} \tag{12}$$

Taking a Gaussian pillbox area a straddling a plate one finds that

$$a|\underline{D}| = a\sigma_f$$

The parallel plate capacitance is given in terms of the potential difference V_d, which remains

$$V_d = -\int_1^2 \underline{E} \cdot \mathrm{d}\underline{l}$$

When we integrate along the normal from plate 1 to plate 2

$$V_d = Ed = \frac{Dd}{\epsilon_0 \epsilon_r}$$

and

$$C = \frac{Q}{Ed} = \frac{A\sigma_f}{Ed} = \frac{AD}{Ed} = \frac{A\epsilon_r\epsilon_0}{d} = \epsilon_r C_0 \tag{13}$$

where C_0 is the capacitance without the dielectric present. For any geometry of capacitor there is an *increase* in the capacitance due to the presence of a dielectric between the plates. *Note that it is not necessarily by just a factor ϵ_r!*

EM 3 Section 16: Magnetic Media

16. 1. Magnetic Materials

Generally more complicated than the dielectrics that we reviewed last lecture.

When an external magnetic field is applied to a material it produces a **magnetization** of the atoms of the material. There are several different types of magnetization:

- Diamagnetism - the orbital angular momentum of the atomic electrons is increased slightly due to electromagnetic induction.
  This magnetization is *opposite* to the external magnetic field.
- Paramagnetism - if the atoms of a material have intrinsic magnetic moments, they align with the applied field, due to $U = -\underline{m} \cdot \underline{B}$.
  This magnetization is *parallel* to the external magnetic field.
- Ferromagnetism - in a few materials the intrinsic magnetic moments of the atoms $\underline{m}_{atom}$ *spontaneously align* due to mutual interactions of a quantum nature called 'exchange interactions'. They form *domains* with moments $\underline{m}_{atom}$ all in the same direction. This magnetization can form *permanent magnets*.

16. 2. The Magnetization Vector

In analogy with the polarisation vector for dielectrics the magnetization vector, $\underline{M}$, is the key macroscopic field for magnetic media.

The infinitesimal magnet (equivalent to small current loop) in volume dV is given by the magnetic dipole moment per unit volume:

$$\mathrm{d}\underline{m} = \underline{M}\mathrm{d}V \tag{1}$$

The units of magnetization $\underline{M}$ are Am^{-1} and $\underline{m}$ in Am^2.

Figure 27: Magnetization loops

An array of small magnetic dipoles can be thought of as producing macroscopic current loops on the surface of the material. These currents circulate round the direction of $\underline{M}$, with a magnetization current density $\underline{J}_M$ (see figure).

Similarly, spatial variation of the magnetisation can be expected to produce a bulk magnetisation current.

To quantify these effects let us calculate the field of a magnetised object. Recall that the magnetic vector potential at $\underline{r}$ of a magnetic dipole at $\underline{r}'$ is

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\frac{\underline{m} \times (\underline{r} - \underline{r}')}{|\underline{r} - \underline{r}'|^3} \tag{2}$$

This generalises, when we replace $\underline{m}$ by $\underline{M}\mathrm{d}V'$ and integrate the magnetisation over some volume V, to

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\underline{M} \times (\underline{r} - \underline{r}')}{|\underline{r} - \underline{r}'|^3}\,\mathrm{d}V' \tag{3}$$

Now we recall that

$$\frac{(\underline{r} - \underline{r}')}{|\underline{r} - \underline{r}'|^3} = \underline{\nabla}' \frac{1}{|\underline{r} - \underline{r}'|} \tag{4}$$

and use the product rule

$$\underline{\nabla}' \times \left(\frac{\underline{M}(\underline{r}')}{|\underline{r} - \underline{r}'|}\right) = \frac{1}{|\underline{r} - \underline{r}'|}\underline{\nabla}' \times \underline{M}(\underline{r}') + \underline{\nabla}'\left(\frac{1}{|\underline{r} - \underline{r}'|}\right) \times \underline{M}(\underline{r}')$$

to obtain

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\int_V \left[\frac{1}{|\underline{r} - \underline{r}'|}\underline{\nabla}' \times \underline{M}(\underline{r}') - \underline{\nabla}' \times \left(\frac{\underline{M}}{|\underline{r} - \underline{r}'|}\right)\right]\mathrm{d}V'$$

We can rewrite the second intergal as a surface integral (see tutorial sheet 9) to obtain

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\int_V \frac{1}{|\underline{r} - \underline{r}'|}\underline{\nabla}' \times \underline{M}(\underline{r}')\mathrm{d}V' + \frac{\mu_0}{4\pi}\oint_S \frac{1}{|\underline{r} - \underline{r}'|}\underline{M}(\underline{r}') \times \underline{\mathrm{d}S}' \tag{5}$$

Now, the first term on the right hand side is equivalent to the potential due to a *volume* current in V

$$\boxed{\underline{J}_M = \underline{\nabla} \times \underline{M}} \tag{6}$$

and the second term is equivalent to the potential due to a *surface* current on S (normal $\hat{\underline{n}}$)

$$\boxed{\underline{j}_M = \underline{M} \times \hat{\underline{n}}} \tag{7}$$

We use the subscript M to indicate that these are *magnetisation* currents resulting from microscopic current loops. Do they really exist? not really but they are effectively present.

Example: Bar magnet "A cylindrical bar magnet has uniform magnetisation M along its axis. To what current distribution is this equivalent?"

Now $\underline{M}$ is uniform so $\underline{\nabla} \times \underline{M} = 0$ and no bulk $\underline{J}_M$

Surface current density $\underline{j}_{mag} = \underline{M} \times \hat{\underline{n}} = M\underline{e}_z \times \underline{e}_\rho = M\underline{e}_\phi$ has magnitude M and is 'solenoidal', i.e. resembling a solenoid with current flowing circumferentially

Example: Toroidal magnet "A long cylindrical bar magnet of uniform $\underline{M}$ is bent into a loop. What is the equivalent current distribution?"

Curl in cylindrical polars (ρ, ϕ, z) reads:

$$\begin{aligned} \underline{\nabla} \times \underline{K} \quad &= \quad \left[\frac{1}{\rho} \frac{\partial K_z}{\partial \phi} - \frac{\partial K_\phi}{\partial z} \right] \underline{e}_\rho + \left[\frac{\partial K_\rho}{\partial z} - \frac{\partial K_z}{\partial \rho} \right] \underline{e}_\phi \\ &\quad + \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho K_\phi) - \frac{\partial K_\rho}{\partial \phi} \right] \underline{e}_z \end{aligned}$$

Direction of $\underline{M}$ now varies with position

$$\underline{M} = M \underline{e}_\phi$$

In the curl formula, the only survivor is

$$\underline{\nabla} \times \underline{M} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho M) \underline{e}_z = \frac{M}{\rho} \underline{e}_z$$

Alongside the solenoidal (circumferential around the toroid) $j_{mag} = M$ on surface, we now have bulk magnetisation current N.B. $\underline{j}_{mag} = \underline{M} \times \underline{\hat{n}}$ has constant magnitude: larger net

Figure 28: Magnetization currents in bar magnet and toroidal magnet

current on outer than inner surface. $\underline{J}_M$ makes up the difference

16. 3. Modification to Ampere's Law

The Ampère-Maxwell law still holds for **full** current density $\underline{J}$

$$\underline{\nabla} \times \underline{B} = \mu_0 \left(\underline{J} + \epsilon_0 \frac{\partial \underline{E}}{\partial t} \right)$$

The key idea is to divide this into three contributions $\underline{J} = \underline{J}_f + \underline{J}_M + \underline{J}_P$

$\underline{J}_f$, current of free charges i.e. the *conduction* current

$\underline{J}_M = \underline{\nabla} \times \underline{M}$, *magnetisation* current we have just met

$\underline{J}_P$ = *polarisation* current — this new term comes from electric dipoles moving around

To find $\underline{J}_P$ we use the (definition) $\rho_P = -\underline{\nabla} \cdot \underline{P}$ and the continuity equation

$$\dot{\rho}_P = -\underline{\nabla} \cdot \underline{J}_P$$

from whicn we deduce

$$\boxed{\underline{J}_P = \frac{\partial \underline{P}}{\partial t}} \tag{8}$$

We would like Ampère-Maxwell in terms of $\underline{J}_f$ only:

$$\begin{aligned}
\underline{\nabla} \times \underline{B} &= \mu_0 \left(\underline{J}_f + \underline{J}_M + \underline{J}_P + \epsilon_0 \frac{\partial \underline{E}}{\partial t} \right) \\
&= \mu_0 \left(\underline{J}_f + \underline{\nabla} \times \underline{M} + \frac{\partial \underline{P}}{\partial t} + \epsilon_0 \frac{\partial \underline{E}}{\partial t} \right) \\
&= \mu_0 \left(\underline{J}_f + \underline{\nabla} \times \underline{M} + \frac{\partial \underline{D}}{\partial t} \right)
\end{aligned}$$

where we have used the definition of $\underline{D}$. Now shift $\underline{\nabla} \times \underline{M}$ onto left, divide by μ_0:

$$\underline{\nabla} \times \left(\frac{\underline{B}}{\mu_0} - \underline{M} \right) = \underline{J}_f + \frac{\partial \underline{D}}{\partial t}$$

We define

$$\boxed{\underline{H} = \frac{\underline{B}}{\mu_0} - \underline{M}} \tag{9}$$

then

$$\boxed{\underline{\nabla} \times \underline{H} = \underline{J}_f + \frac{\partial \underline{D}}{\partial t}} \tag{10}$$

which is **Ampère-Maxwell law in media**. The Integral form of Ampère-Maxwell reads

$$\oint_C \underline{H} \cdot \underline{\mathrm{d}l} = \int_S (\underline{J}_f + \partial \underline{D}/\partial t) \cdot \underline{\mathrm{d}S}$$

where C is a closed circuit bounding S

We run into difficulties in terminology for $\underline{B}$, $\underline{H}$. It is actually simplest and easiest to call them 'magnetic field B' (units Tesla) and 'magnetic field H' in units of Am^{-1}. But be warned in some text $\underline{B}$ is the 'magnetic field' and $\underline{H}$ is the 'auxiliary field'; in others $\underline{B}$ is the 'magnetic flux density' and $\underline{H}$ is the 'magnetic field strength' (which is really confusing!)

We now basically have Maxwell's equations in media since MII and MIII do not need to be modified as they contain no $\underline{J}$ or ρ. See next lecture for summary

The **magnetic susceptibility** χ_M describes the relationship between magnetization and applied field, by relating $\underline{M}$ to $\underline{H}$. We will assume again an *LIH medium* (linear, isotropic, homogeneous). Then the relation may be written

$$\underline{M} = \chi_M \underline{H} \tag{11}$$

Warning—some books, e.g. Grant & Phillips, use $\chi_B \underline{B} = \mu_0 \underline{M}$ *which can be very confusing!*

The equivalent of the dielectric constant is known as the **relative permeability** of a material, μ_r:

$$\boxed{\underline{B} = \mu_r \mu_0 \underline{H}} \tag{12}$$

where $(\mu_r - 1) = \chi_M$. The limit of no magnetization is $\chi_M = 0$ and $\mu_r = 1$.

In contrast to dielectrics, the magnetic susceptibility χ_M can be either positive or negative, and $\mu_r < 1$ or $\mu_r > 1$.

EM 3 Section 17: Summary of EM in media; boundary conditions on fields

17. 1. Effect of Magnetic Materials on Inductance

First we have to finish off our description of magnetism with a look at how inductance is affected by magnetisation currents

Example: conducting core in solenoid "A long solenoid of n turns per unit length, length L and cross sectional area $\mathcal{A}$ is filled with ferrite, in which $\underline{M}$ obeys $\underline{M} = \chi_m \underline{H}$ where $\chi_m = 900$. Find the self inductance $\mathcal{L}$."

Recall the definition $\mathcal{L} = \Phi_B / I$ this stems from Faraday's law MIII, and is therefore **unchanged** by media. Ampère's law in the static situation $\partial \underline{D}/\partial t = 0$ becomes

$$\begin{aligned} \underline{\nabla} \times \underline{H} &= \underline{J}_f + \underline{0} \\ \Rightarrow \oint \underline{H} \cdot \underline{\mathrm{d}l} &= \int \underline{J}_f \cdot \underline{\mathrm{d}S} = nLI \end{aligned}$$

in integral form where I is the usual conduction current. Now note the symmetry: $\underline{H}$ is axial within the solenoid and vanishes outside for large L. Taking a loop as shown in figure,

Figure 29: Solenoid with conducting core: Amperian loop

$H = nI$, so $\underline{M}$ is axial; magnitude $M = \chi_m nI$

Then B also must be axial:

$$\begin{aligned} B &= \mu_0 (H + M) = (\chi_m + 1)\mu_0 nI \\ \Rightarrow \Phi_B &= n\mathcal{A}LB = (\chi_m + 1)\mu_0 n^2 \mathcal{A}LI \\ \Rightarrow \mathcal{L} &= \Phi_B / I = (\chi_m + 1)\mu_0 n^2 \mathcal{A}L \end{aligned}$$

Thus $\mathcal{L}$ is 901 times larger than in vacuum (vacuum case: $\chi_m = 0$). For a ferromagnetic material there is a *very large increase* in self inductance.

On the other hand for diamagnetic/paramagnetic materials there is a small decrease/increase in the self-inductance.

For ferromagnetic materials the energy stored in an inductor increases by a large factor $\mu_r \approx 10^3 - 10^6$: See section 17.3 for energy stored in fields

17. 2. Electromagnetism with media: summary

Maxwell's equations in general form read

$$\underline{\nabla} \cdot \underline{D} = \rho_f \qquad (1)$$

$$\begin{aligned}\underline{\nabla} \times \underline{E} &= -\frac{\partial \underline{B}}{\partial t} &(2)\\ \underline{\nabla} \cdot \underline{B} &= 0 &(3)\\ \underline{\nabla} \times \underline{H} &= \underline{J}_f + \frac{\partial \underline{D}}{\partial t} &(4)\end{aligned}$$

Definitions of $\underline{D}, \underline{H}$ are

$$\underline{D} = \epsilon_0 \underline{E} + \underline{P} \qquad \underline{B} = \mu_0(\underline{H} + \underline{M}) \tag{5}$$

Relations for LIH Media

$$\begin{aligned}\underline{P} &= \chi_E \epsilon_0 \underline{E} & \underline{M} &= \chi_m \underline{H} &(6)\\ \underline{D} &= \epsilon_0 \epsilon_r \underline{E} \equiv \epsilon \underline{E} & \underline{B} &= \mu_0 \mu_r \underline{H} \equiv \mu \underline{H} &(7)\\ \epsilon_r &= 1 + \chi_E & \mu_r &= 1 + \chi_m &(8)\end{aligned}$$

17. 3. Energy densities and Poynting Vector

Recall that $\underline{E} \cdot \underline{J}_f$ is the power delivered per unit volume so the *energy density* u obeys

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \underline{E} \cdot \underline{J}_f \tag{9}$$

Now use modified MIV to express

$$\underline{E} \cdot \underline{J}_f = \underline{E} \cdot (\underline{\nabla} \times \underline{H}) - \underline{E} \cdot \frac{\partial \underline{D}}{\partial t}$$

Furthermore we can use a product rule from lecture 1 to write

$$\begin{aligned}\underline{E} \cdot \underline{J} &= \underline{H} \cdot (\underline{\nabla} \times \underline{E}) - \underline{\nabla} \cdot (\underline{E} \times \underline{H}) - \underline{E} \cdot \frac{\partial \underline{D}}{\partial t}\\ &= -\underline{H} \cdot \frac{\partial \underline{B}}{\partial t} - \underline{\nabla} \cdot (\underline{E} \times \underline{H}) - \underline{E} \cdot \frac{\partial \underline{D}}{\partial t}\\ &= -\frac{\partial}{\partial t}\left(\frac{1}{2}\underline{E} \cdot \underline{D} + \frac{1}{2}\underline{B} \cdot \underline{H}\right) - \underline{\nabla} \cdot (\underline{E} \times \underline{H})\end{aligned}$$

provided that $\underline{E} \cdot \underline{\dot{D}} = \underline{\dot{E}} \cdot \underline{D}$ and $\underline{B} \cdot \underline{\dot{H}} = \underline{\dot{B}} \cdot \underline{H}$ which is true for linear static media. Then integrating over a volume V of the medium and using the divergence theorem on the second term as usual, we obtain from (9) for the total energy

$$\boxed{\frac{\mathrm{d}U}{\mathrm{d}t} = -\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\underline{E} \cdot \underline{D} + \frac{1}{2}\underline{B} \cdot \underline{H}\right) \mathrm{d}V - \oint_S (\underline{E} \times \underline{H}) \cdot \underline{\mathrm{d}S}} \tag{10}$$

From the first term we identify the electric and magnetic energy densities as

$$u_M = \frac{1}{2}\underline{B} \cdot \underline{H} \qquad u_E = \frac{1}{2}\underline{E} \cdot \underline{D} \tag{11}$$

and from the second term we identify the Poynting vector as

$$\underline{S} = \underline{E} \times \underline{H} \tag{12}$$

17. 4. Boundary Matching Problems

There are often have sharp interfaces between media. These boundaries acquire nonzero values of σ_P surface polarization charge and $\underline{j}_{mag}$ surface magnetisation current

In keeping with use of MI-MIV in general form, we want to avoid considering these, and think about **free** charges and currents only ...

1. First condition (from $\underline{\nabla} \cdot \underline{D} = \rho_f$): Divergence theorem:

Figure 30: Gaussian surface for deriving continuity conditions on normal components (similar to Griffiths Fig 2.36)

$$\underline{\nabla} \cdot \underline{D} = \rho_f \qquad \Rightarrow \qquad \oint \underline{D} \cdot \mathrm{d}\underline{S} = Q_{f,enclosed}$$

Apply to small pillbox or "patch", vector area $\mathrm{d}\underline{S} = \hat{\underline{n}}\mathrm{d}S$

$$(\underline{D}_2 - \underline{D}_1) \cdot \hat{\underline{n}} \, \mathrm{d}S = \sigma_f \, \mathrm{d}S$$

surface density of FREE charges only. In the absence of free surface charges D_{normal} is continuous. We can also write this as

$$(\underline{D}_2 - \underline{D}_1) \cdot \hat{\underline{n}} = \sigma_f$$

2. Second condition (from $\underline{\nabla} \cdot \underline{B} = 0$):

$$\underline{\nabla} \cdot \underline{B} = 0 \qquad \Rightarrow \qquad \int \underline{B} \cdot \mathrm{d}\underline{S} = 0$$

Apply to small Gaussian pill box (or "patch")

$$\int \underline{B} \cdot \mathrm{d}\underline{S} = (\underline{B}_2 - \underline{B}_1) \cdot \hat{\underline{n}} \, \mathrm{d}S = 0$$

Therefore B_{normal} is continuous. This is completely general.

3. Third condition (from $\underline{\nabla} \times \underline{E} = -\partial \underline{B}/\partial t$): $\hat{\underline{t}}$ = unit tangent satisfies $\hat{\underline{t}} \cdot \hat{\underline{n}} = 0$; we take a rectangular loop straddling the interface length ℓ height h

$$\oint \underline{E} \cdot \mathrm{d}\underline{l} = (\underline{E}_1 - \underline{E}_2) \cdot \hat{\underline{t}} \, l \qquad = -\frac{\partial}{\partial t} \Phi_B$$

Unless $\underline{B}$ is infinite, the magnetic flux cutting the loop $\Phi_B \to 0$ as $h \to 0$

$$\Rightarrow (\underline{E}_1 - \underline{E}_2) \cdot \hat{\underline{t}} = 0$$

Figure 31: Amperian loop for deriving continuity conditions on tangential components (similar to Griffiths Fig 2.37)

but $\hat{\underline{t}}$ is arbitrary within plane of the surface: $\underline{E}_{tangential}$ is continuous is completely general as it stands. **N.B.** this is **two** conditions in 3D

4. Fourth condition ($\underline{\nabla} \times \underline{H} = \underline{J}_f + \partial \underline{D}/\partial t$):

$\underline{j}_f$ = free surface current / unit area

$$\oint \underline{H} \cdot \mathrm{d}\underline{l} = \underline{j}_f \cdot \hat{\underline{s}}\, \ell + \frac{\partial \underline{D}}{\partial t} \cdot \hat{\underline{s}}\, \ell\, h$$

where $\hat{\underline{s}} = \hat{\underline{t}} \times \hat{\underline{n}}$ = unit vector $\perp$ to Ampèrian loop

Now take $h \to 0$: last term vanishes

$$\oint \underline{H} \cdot \mathrm{d}\underline{l} = (\underline{H}_1 - \underline{H}_2) \cdot \hat{\underline{t}}\, \ell = \underline{j}_f \cdot \hat{\underline{s}}\, \ell$$

In the absence of free surface currents $\underline{H}_{tangential}$ is continuous

The general form is rarely needed and may be written in several equivalent ways:

$$\begin{aligned} (\underline{H}_1 - \underline{H}_2) \cdot \hat{\underline{t}} &= \underline{j}_f \cdot \hat{\underline{s}} \\ (\underline{H}_2^{tang} - \underline{H}_1^{tang}) &= \underline{j}_f \times \hat{\underline{n}} \\ (\underline{H}_2 - \underline{H}_1) \times \hat{\underline{n}} &= -\underline{j}_f \end{aligned}$$

Summary of the continuity conditions

1.	D_n	continuous	if $\sigma_f = 0$
2.	B_n	continuous	always
3.	$\underline{E}_t$	continuous	always
4.	$\underline{H}_t$	continuous	if $\underline{j}_f = \underline{0}$

These are key results and you should know the derivations.

Problems with nonzero σ_f or $\underline{j}_f$ are uncommon but for these:

$(\underline{D}_2 - \underline{D}_1) \cdot \hat{\underline{n}} = \sigma_f$ replaces 1

$(\underline{H}_2^{tang} - \underline{H}_1^{tang}) = \underline{j}_f \times \hat{\underline{n}}$ replaces 4

EM 3 Section 18: Examples of continuity conditions; waves in media

18. 1. Continuity conditions: examples

Example: Inclined dielectric slab

"The electric field $\underline{E}^o$ outside a large dielectric slab of relative permittivity ϵ_r is uniform and at angle θ to the *normal* to the slab. What is the electric field $\underline{E}^i$ inside the slab?".

Figure 32: Similar to Griffiths Fig 4.34

Outside: $\underline{D}^o = \epsilon_0 \underline{E}^o$ inside: $\underline{D}^i = \epsilon_r \epsilon_0 \underline{E}^i$. Let ψ be the angle between the normal to the plane and $\underline{E}^i$

Take the normal to slab in $\underline{e}_z$ direction and tangent in $\underline{e}_x$ direction and write (x, z) components of fields as

$$\begin{aligned}\underline{E}^o &= (E^o \sin\theta, E^o \cos\theta) \qquad \underline{D}^o = \epsilon_0 (E^o \sin\theta, E^o \cos\theta)\\ \underline{E}^i &= (E^i \sin\psi, E^i \cos\psi) \qquad \underline{D}^i = \epsilon_0 \epsilon_r (E^i \sin\psi, E^i \cos\psi)\end{aligned}$$

Now impose b.c.s:

1. $D_n = D_z$ continuous:

$$D^i_z = D^o_z \quad \Rightarrow \quad \epsilon_0 E^o \cos\theta = \epsilon_0 \epsilon_r E^i \cos\psi$$

$$\Rightarrow E^i = \frac{E^o}{\epsilon_r} \frac{\cos\theta}{\cos\psi}$$

2. $E_t = E_x$ continuous:

$$E^i_x = E^o_x \quad \Rightarrow \quad E^o \sin\theta = E^i \sin\psi$$

$$\Rightarrow E^i = E^o \frac{\sin\theta}{\sin\psi}$$

Result:

$$\begin{aligned}\frac{1}{\epsilon_r} \frac{\cos\theta}{\cos\psi} &= \frac{\sin\theta}{\sin\psi}\\ \Rightarrow \quad \psi &= \tan^{-1}(\epsilon_r \tan\theta)\end{aligned}$$

Checks: For $\theta = \pi/2$: $E^i = E^o$, $\psi = \pi/2$ ($\underline{E}$ is purely tangential, continuous)

For $\theta = 0$: $E^i = E^o/\epsilon_r$, $\psi = 0$ ($\underline{D}$ is purely normal, continuous)

Remarks $\underline{D} \parallel \underline{E}$ everywhere; but the angle of *both* is altered within slab. $\underline{E}^i$ is the superposition of uniform $\underline{E}^o$ with that of the polarization charges on surface of slab. Unless $\theta = 0$, as in a parallel plate capacitor, $\underline{D}/\epsilon_0$ is not "the $\underline{E}$ field you would have had" without the slab which would be $\underline{D}^i = \epsilon_0 \epsilon_r \underline{E}^o$

Example: Spherical cavity in dielectric

"A large block of dielectric of relative permittivity $\epsilon_r > 1$ contains a spherical cavity. The $\underline{E}$ field far away from the cavity is uniform, with magnitude E_0. What are $\underline{E}, \underline{D}$ within the cavity?"

Figure 33: Spherical cavity in dielectric - Griffiths Example 4.7

Use spherical polars with origin at the centre of the sphere and take z axis $\parallel \underline{E}_0$. Therefore there is symmetry w.r.t. ϕ.

At the surface of the spherical cavity $\sigma_p = \underline{P} \cdot \hat{\underline{n}}$ where $\hat{\underline{n}}$ is outwards normal of material (inwards normal of sphere). Therefore the field inside is **enhanced** by $\sigma_p(\theta)$.

The charge around the cavity $\sigma_p(\theta)$ forms an effective *dipole*. Outside the field lines are **distorted** locally by $\sigma_p(\theta)$

Try a **uniform** field in z direction within cavity:

$$V(r < a) = -E_{in} z = -E_{in} r \cos\theta \ .$$

Try the uniform field $\underline{E}_0$ plus a **dipole** form outside

$$V(r > a) = -E_0 r \cos\theta + \frac{A\cos\theta}{r^2}$$

where A is a constant to be fixed.

Recall that these two expressions satisfy Laplace's equation away from the boundary where there are no charges. We now just need to satisfy the boundary conditions on the fields.

First recall that $\underline{E} = -\underline{\nabla} V$ and in spherical polars

$$\underline{\nabla} V = \underline{e}_r \frac{\partial V}{\partial r} + \underline{e}_\theta \frac{1}{r}\frac{\partial V}{\partial \theta} + \underline{e}_\phi \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}$$

At the boundary: E_t continuous requires E_θ continuous at $r = a$:

$$-E_0 a \sin\theta + \frac{A\sin\theta}{a^2} = -E_{in} a \sin\theta$$

D_n continuous requires $D_r = -\epsilon_r \partial V/\partial r$ continuous at $r = a$:

$$\epsilon_r \left(E_0 \cos\theta + \frac{A\cos\theta}{a^3} \right) = E_{in} \cos\theta$$

Combine these

$$E_{in} = \epsilon_r \left(E_0 + \frac{2A}{a^3} \right) = E_0 - \frac{A}{a^3}$$

eliminate A/a^3:

$$E_{in} = E_0 \frac{3\epsilon_r}{1 + 2\epsilon_r}$$

with $\underline{E}_{in} = E_{in}\underline{e}_z$. Then $\underline{D}_{in} = \epsilon_r \epsilon_0 \underline{E}_{in} = \epsilon_0 \underline{E}_{in}$ (since the cavity has $\epsilon_r = 1$).

Check: $E_{in} > E_0$ if $\epsilon_r > 1$, field inside enhanced.

Uniqueness $\Rightarrow$ problem solved!

This example may be used to derive an approximate formula for atomic polarizability the Clausius Mosotti equation - see tutorial 10.

18. 2. Waves in media

As a first look at waves in media let's consider a *non-conducting* medium with $\rho_f = 0$, $\underline{J}_f = 0$. Let us write the permittivity $\epsilon = \epsilon_0 \epsilon_r$ and the permeability $\mu = \mu_0 \mu_r$.

As before when we considered waves in vacuo in lecture 13 we can reduce Maxwell's equation to two decoupled wave equations

$$\nabla^2 \underline{E} = \epsilon\mu \frac{\partial^2 \underline{E}}{\partial t^2} \qquad (1)$$

$$\nabla^2 \underline{B} = \epsilon\mu \frac{\partial^2 \underline{B}}{\partial t^2} \qquad (2)$$

These are precisely the same as in lecture 13 but with ϵ_0 replaced by ϵ and μ_0 replaced by μ.

Clearly we have plane solutions

$$\underline{E} = \underline{E}_0 \exp i(\underline{k}\cdot\underline{r} - \omega t) \quad \underline{B} = \underline{B}_0 \exp i(\underline{k}\cdot\underline{r} - \omega t) \qquad (3)$$

where $k^2 - \mu\epsilon\omega^2 = 0$. Thus the wave speed is

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}$$

and recalling $c = \dfrac{1}{\sqrt{\mu_0\epsilon_0}}$

$$\boxed{\frac{v^2}{c^2} = \frac{1}{\mu_r \epsilon_r} \equiv n^2} \qquad (4)$$

where n is called the refractive index of the medium

As in lecture 13 MI,MII imply

$$i\underline{k} \cdot \underline{E}_0 = 0 \qquad i\underline{k} \cdot \underline{B}_0 = 0$$

i.e. $\underline{E}$ and $\underline{B}$ are *perpendicular* to the direction of propagation $\underline{k}$ and the wave is **transverse**.

This may seem like a trivial generalisation of waves in vacuo but the physics is remarkable— we have managed to deal with all the atoms, atomic dipoles, polarisation etc by wrapping them up into ϵ and μ and the net result is simply to change the velocity of the wave.

18. 3. Waves in conductors

In conductors there is free charge and currents flow in response to an electric field. As we shall see this has a serious effect on the propagation of an EM wave in a conductor.

Let us start with MIV and use the linear relations

$$\underline{B} = \mu \underline{H} \qquad \underline{D} = \epsilon \underline{E}$$

and Ohm's law $\underline{J} = \sigma \underline{E}$

$$\begin{aligned} \underline{\nabla} \times \underline{H} &= \frac{\partial \underline{D}}{\partial t} + \underline{J}_f \\ \to \underline{\nabla} \times \underline{B} &= \mu\epsilon \frac{\partial \underline{E}}{\partial t} + \mu\sigma \underline{E} \end{aligned}$$

Now as usual MIII yields

$$\frac{\partial}{\partial t}(\underline{\nabla} \times \underline{B}) = -\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = \nabla^2 \underline{E} - \underline{\nabla}(\underline{\nabla} \cdot \underline{E})$$

The microscopic M1 reads $\underline{\nabla} \cdot \underline{E} = \rho/\epsilon$. Let us assume a *uniform* charge density so that $\underline{\nabla}\rho = 0$.

Then finally we obtain

$$\boxed{\nabla^2 \underline{E} = \mu\epsilon \frac{\partial^2 \underline{E}}{\partial t^2} + \mu\sigma \frac{\partial \underline{E}}{\partial t}} \tag{5}$$

we note an additional term on the rhs whose origin is the free current in Maxwell IV. How will this term affect the wave?

A similar calculation (**Exercise**) yields

$$\boxed{\nabla^2 \underline{B} = \mu\epsilon \frac{\partial^2 \underline{B}}{\partial t^2} + \mu\sigma \frac{\partial \underline{B}}{\partial t}} \tag{6}$$

Let us proceed blindly and bravely by making an ansatz of plane wave moving in the z direction $\underline{E} = \underline{E}_0 \exp i(\tilde{k}z - \omega t)$. When we sub this into (1) we obtain

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega$$

Clearly something has to become complex to solve this!

EM 3 Section 19: Waves in Conductors: Skin Effect

19. 1. Recap: Waves in conductors

Last time we derived the equation

$$\boxed{\nabla^2 \underline{E} = \mu\epsilon \frac{\partial^2 \underline{E}}{\partial t^2} + \mu\sigma \frac{\partial \underline{E}}{\partial t}} \tag{1}$$

where σ is the conductivity. Substituting a plane wave ansatz

$$\underline{E} = \underline{\tilde{E}}_0 \exp i(\tilde{k}z - \omega t) \tag{2}$$

yields

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega \tag{3}$$

To solve this we have to take a complex wavenumber from (2)

$$\tilde{k} = k + i\kappa \tag{4}$$

Equating the real and imaginary parts in (3) yields

$$k^2 - \kappa^2 = \mu\epsilon\omega^2 \tag{5}$$

$$2k\kappa = \mu\sigma\omega \tag{6}$$

The second equation can be solved for $\kappa = \dfrac{\mu\sigma\omega}{2k}$ then eliminating κ from (5) yields

$$k^4 - \left(\frac{\mu\sigma\omega}{2}\right)^2 = \mu\epsilon\omega^2 k^2$$

This is a quadratic in k^2 with solution

$$\begin{aligned} k^2 &= \frac{1}{2}\mu\epsilon\omega^2 + \frac{1}{2}\left((\mu\epsilon\omega^2)^2 + (\mu\sigma\omega)^2\right)^{1/2} \\ &= \frac{\mu\epsilon\omega^2}{2}\left[\left(1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2\right)^{1/2} + 1\right] \end{aligned} \tag{7}$$

(we have taken the positive square root so that the solution for k^2 is positive). Then we can use (5) to obtain

$$\kappa^2 = \frac{\mu\epsilon\omega^2}{2}\left[\left(1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2\right)^{1/2} - 1\right] \tag{8}$$

Now the complex wavenumber (4) implies

$$\underline{E} = \underline{\tilde{E}}_0 \mathrm{e}^{-\kappa z} \mathrm{e}^{i(kz - \omega t)} \tag{9}$$

The first exponential decays with z and causes *attenuation* of the wave. The characterisitic distance over which the wave decays is known as the *skin depth* and is given by

$$\boxed{\delta = \frac{1}{\kappa}} \tag{10}$$

Thus the skin depth is the typical distance a wave penetrates into a conductor.

In the result (8) the ratio $\frac{\sigma}{\epsilon\omega}$ is significant. $1/\omega$ has the dimensions of time as does ϵ/σ. Thus this quantity is a ratio of two timescales.

19. 2. Good and poor conductors

In order to understand the timescale ϵ/σ let us return to the continuity equation for free charge

$$\frac{\partial \rho_f}{\partial t} = -\underline{\nabla} \cdot \underline{J}_f \tag{11}$$

Using Ohm's law and Gauss's law (plus linear media property)

$$\underline{\nabla} \cdot \underline{J}_f = \sigma \underline{\nabla} \cdot \underline{E} = \frac{\sigma}{\epsilon} \underline{\nabla} \cdot \underline{D} = \frac{\sigma}{\epsilon} \rho_f$$

So finally

$$\frac{\partial \rho_f}{\partial t} = -\frac{\sigma}{\epsilon} \rho_f$$

which has solution

$$\rho_f(t) = \rho_f(0) \mathrm{e}^{-(\sigma/\epsilon)t}$$

So the free charge density decays on a timescale $\tau = \dfrac{\epsilon}{\sigma}$ which is the *relaxation time*. If this is small then any free excess charge is quickly rearranged away and the medium is a good conductor. A perfect conductor would have this timescale tending to zero i.e. $\sigma \to \infty$.

On the other hand if τ is large, free charge hangs around for a long time and the medium is a poor conductor.

Returning to the quantity

$$\frac{\sigma}{\epsilon\omega} = \frac{1}{2\pi} \frac{T}{\tau}$$

we see that is (roughly) the ratio of the oscillation period of the wave to the charge relaxation time in the conductor. If, for a given frequency ω, this ratio is large the medium is a good conductor, whereas if the ratio is small the medium is a poor conductor for that frequency.

In the tutorial you are invited to work out the different limits. One finds from (8) that the skin depth

$$\begin{aligned} \delta &\simeq \left(\frac{2}{\mu\omega\sigma} \right)^{1/2} \quad \text{for} \quad \sigma \gg \epsilon\omega \\ \delta &\simeq \left(\frac{4\epsilon}{\mu\sigma^2} \right)^{1/2} \quad \text{for} \quad \sigma \ll \epsilon\omega \end{aligned}$$

Thus the skin depth is much smaller for a good conductor. Also note that for a poor conductor the behaviour does not depend on frequency.

Typical metals are good conductors up to about 1 MHz

$\delta \simeq$ 1cm at 50 Hz (mains frequency)

$\delta \simeq$ 10 μm at 50 MHz

Consequences / Applications of Skin effect

- shielding of sensitive electronics (metal casework)
- power lines and cable design: conductors > 1cm thick are wasted since the current resides only in the skin layer around the outside and there is a 'dead zone' in the centre
- submarines can't use radio
- mobile phones don't work inside metal boxes (? paint concert halls with metal paint?)
- microwave oven doors: metal mesh stops radiation escaping, holes $\ll \lambda$ are OK

19. 3. Phase lag of magnetic field

MI and MII imply further constraints on our wave. As usual

$$i\underline{\tilde{k}} \cdot \underline{\tilde{E}}_0 = 0 \qquad i\underline{\tilde{k}} \cdot \underline{\tilde{B}}_0 = 0$$

Take the direction of propagation $\underline{\tilde{k}}$ in the $\underline{e}_z$ direction and $\underline{\tilde{E}}_0$ in the $\underline{e}_x$ direction. Substituting in MIII

$$\begin{aligned} i\underline{\tilde{k}} \times \underline{\tilde{E}}_0 &= i\omega\underline{\tilde{B}}_0 \\ \Rightarrow \quad \underline{\tilde{B}}_0 &= \frac{\tilde{k}\tilde{E}_0}{\omega}\underline{e}_y \end{aligned} \tag{12}$$

However $\tilde{k}$ is complex so $\tilde{E}_0$ and $\tilde{B}_0$ will also be complex. Let us write

$$\tilde{k} = R\mathrm{e}^{i\phi}$$

Then using (7,8)

$$\begin{aligned} R &= \left(k^2 + \kappa^2\right)^{1/2} = (\mu\epsilon\omega^2)^{1/2}\left(1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2\right)^{1/4} \\ \phi &= \tan^{-1}\left(\frac{\kappa}{k}\right) = \tan^{-1}\left[\frac{\left(1+\left(\frac{\sigma}{\epsilon\omega}\right)^2\right)^{1/2} - 1}{\left(1+\left(\frac{\sigma}{\epsilon\omega}\right)^2\right)^{1/2} + 1}\right]^{1/2} \end{aligned}$$

For a good conductor

$$\phi \to \tan^{-1}[1] = \pi/4$$

and

$$\tilde{k} \simeq (\mu\omega\sigma)^{1/2}\mathrm{e}^{i\pi/4} \tag{13}$$

The vectors $\underline{\tilde{E}}_0$, $\underline{\tilde{B}}_0$ are also complex. Let us write

$$\tilde{E}_0 = E_0\mathrm{e}^{i\delta_E} \qquad \tilde{B}_0 = B_0\mathrm{e}^{i\delta_B} \tag{14}$$

Putting these in (12) yields

$$
\begin{aligned}
B_0 \mathrm{e}^{i\delta_B} &= \frac{R\mathrm{e}^{i\phi}}{\omega} E_0 \mathrm{e}^{i\delta_E} && (15)\\
\Rightarrow \delta_B - \delta_E &= \phi && (16)
\end{aligned}
$$

Condition (16) means that the magnetic field lags behind the electric field by angle ϕ.

Finally taking the real part to get real fields we have

$$
\begin{aligned}
\underline{E} &= E_0 \mathrm{e}^{-\kappa z} \cos(kz - \omega t + \delta_E) \underline{e}_x && (17)\\
\underline{B} &= B_0 \mathrm{e}^{-\kappa z} \cos(kz - \omega t + \delta_E + \phi) \underline{e}_y && (18)
\end{aligned}
$$

Figure 34: Electric and magnetic fields and the skin depth *(Griffiths fig 9.18)*

19. 4. Intrinsic Impedance

As we have seen

$$\underline{E} = \underline{e}_x \, \tilde{E}_0 e^{i(kz-\omega t)} \qquad ; \qquad \underline{B} = \underline{e}_y \, \tilde{B}_0 e^{i(kz-\omega t)}$$

where $\tilde{E}_0$ and $\tilde{B}_0$ are complex

Whereas in vacuum $\underline{E}$ and $\underline{H} = \underline{B}/\mu_0$ are *in phase*, here there are not. The complex number

$$\boxed{Z \equiv \frac{\tilde{E}_0}{\tilde{H}_0}} \qquad (19)$$

is the **Intrinsic Impedence** of the medium. One can think of it as the generalised resistance (when Z is real it reduces to the resistance). Dimensions are Ω (Ohms): check units $E = V/m; H = A/m \quad \Rightarrow E/H = V/A = \Omega$

In a vacuum

$$\frac{E_0}{H_0} = \frac{E_0 \mu_0}{B_0} = c\mu_0 \equiv Z_{vac} = 377\Omega$$

This is real since $\underline{E}, \underline{H}$ are in phase

In a dielectric

$$\frac{E_0}{H_0} = \frac{E_0 \mu}{B_0} = \sqrt{\mu_r \epsilon_r}\, Z_{vac}$$

As we have seen in a good conductor we have $\tilde{k} \approx \sqrt{i\mu\omega\sigma}$ (13)

$$Z = \frac{\tilde{E}_0}{\tilde{H}_0} = \frac{\tilde{E}_0 \mu}{\tilde{B}_0} = \frac{\omega\mu}{\tilde{k}} \simeq \left(\frac{\mu\omega}{\sigma}\right)^{1/2} \mathrm{e}^{-i\pi/4}$$

which is complex.

EM 3 Section 20: Reflection at boundaries: normal incidence

20. 1. Reminder on plane waves and amplitudes

Consider a plane polarized wave propagating in the $\underline{e}_z$ direction

$$\underline{E} = \underline{E}_0 \mathrm{e}^{i(kz-\omega t)} \qquad \underline{E} = \underline{E}_0 \mathrm{e}^{i(kz-\omega t)}$$

As we have seen Maxwell III implies

$$ik\underline{e}_z \times \underline{E}_0 = i\omega B_0 \underline{e}_y$$

Usually we take $\underline{E}_0 = E_0 \underline{e}_x$ and

$$\underline{B}_0 = \frac{kE_0}{\omega} \underline{e}_y \, .$$

Now E_0,B_0 can, in principle, be complex, as they were for waves in conductor. Previously we indicated this by a tilde e.g. $\tilde{E}_0$. but to lighten notation we won't do that here and instead just refer to E_0 as the complex amplitude; the (real) amplitude is then the modulus $|E_0|$ i.e. $E_0 = |E_0|\mathrm{e}^{i\delta_E}$. Recall that the complex impedance is given by the ratio of complex amplitudes

$$Z = \frac{E_0}{H_0} = \frac{\mu E_0}{B_0}$$

as we have seen complex Z allows a *phase shift* between $\underline{E}$ and $\underline{H}$

20. 2. Waves at interfaces

Now consider a plane polarized wave propagating in the $\underline{e}_z$ direction *normal incidence* to an interface and call this $\underline{E}_{inc}$. Generally medium 1 has complex impedance $Z = Z_1$ and medium 2 has complex impedance $Z = Z_2$. We take coordinates: $\underline{e}_x$ along $\underline{E}_{inc}$; $\underline{e}_y$ along $\underline{H}_{inc}$; $\underline{e}_z$ along $\underline{k}_1$ (forming a right handed triad).

We place the boundary at $z = 0$ so that the x–y plane is the interface between the two media

Figure 35: Wave at interface between two media *similiar to Griffiths fig. 9.13*

20. 3. Interfaces between two dielectric media

It is simplest to start by considering two dielectric media where we have seen that

$$Z_i = v_i \mu_i$$

is real and there is no phase lag between $\underline{E}$ and $\underline{H}$

$$\underline{E}_{inc} = E_I\, \underline{e}_x\, e^{i(k_1 z - \omega t)}$$

$$\underline{H}_{inc} = \frac{E_I}{\mu_i v_1}\, \underline{e}_y\, e^{i(k_1 z - \omega t)}$$

Also we can take the amplitude E_I to be real. Likewise for transmitted and reflected waves (see diagram):

$$\underline{E}_{trans} = E_T\, \underline{e}_x\, e^{i(k_2 z - \omega t)}$$

$$\underline{H}_{trans} = \frac{E_T}{\mu_2 v_2}\, \underline{e}_y\, e^{i(k_2 z - \omega t)}$$

$$\underline{E}_{ref} = E_R\, \underline{e}_x\, e^{i(-k_2 z - \omega t)}$$

$$\underline{H}_{ref} = -\frac{E_R}{\mu_1 v_1}\, \underline{e}_y\, e^{i(-k_2 z - \omega t)}$$

N.B. The reflected wave propagates in $-$ve z direction hence *sign switch* in the exponential (so that wave speed is $v = -\omega/k$) and *sign switch* in $\underline{H}_{ref}$ (so that $-\underline{e}_z$, $\underline{E}$, $\underline{H}$ form a right-handed triad).

Now invoke continuity conditions (see sections 17 and 18): $\underline{e}_x$ and $\underline{e}_y$ are both *tangential* to interface and tangential components of $\underline{E}$ and $\underline{H}$ are **continuous**. Note that we assume that there *no surface currents or charges* which is usually the case. Then the continuity conditions become

$\underline{E}_{tan} = E_x$ is continuous

$$\Rightarrow E_I + E_R = E_T$$

$\underline{H}_{tan} = H_y$ is continuous

$$\Rightarrow \frac{E_I}{\mu_1 v_1} - \frac{E_R}{\mu_1 v_1} = \frac{E_T}{\mu_2 v_2}$$

Solve for E_T and E_R, knowing E_I: add the equations to find

$$\frac{2E_I}{\mu_1 v_1} = \left[\frac{1}{\mu_1 v_1} + \frac{1}{\mu_2 v_2}\right] E_T$$

Also recall that

$$v_i = \frac{1}{\sqrt{\mu_i \epsilon_i}} = \frac{c}{n_i}$$

then the **Amplitude transmission coefficient**

$$t \equiv \frac{E_T}{E_I} = \frac{2}{1+\beta}$$

and the **Amplitude reflection coefficient**

$$r \equiv \frac{E_R}{E_I} = \frac{1-\beta}{1+\beta}$$

where β is defined as

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2}$$

Now if the permeabilities $\mu_i = \mu_0$ (non-magnetic media) we find

$$\begin{aligned} r &= \frac{v_2 - v_1}{v_1 + v_2} = \frac{n_1 - n_2}{n_1 + n_2} \\ t &= \frac{2v_2}{v_1 + v_2} = \frac{2n_1}{n_1 + n_2} \end{aligned}$$

So the reflected wave is *in phase* if $v_2 > v_2$ but *out of phase* if $v_2 < v_1$. If $v_2 = v_1$ (two media the same) there is no reflected wave as expected.

Energy flow

The Poynting vector is given as usual by

$$\underline{S} = \underline{E} \times \underline{H} = \frac{1}{\mu} \underline{E} \times \underline{B}$$

so the energy flux per unit volume averaged over one period or **intensity** of the wave is given by

$$\langle \underline{S} \rangle = \frac{1}{\mu} \langle \underline{E} \times \underline{B} \rangle = \frac{1}{\mu v} \frac{E_0^2}{2} = \frac{\epsilon v}{2} E_0^2$$

So R the ratio of reflected to incident intensity and T the ratio of transmitted to incident intensity are given by

$$R = r^2 = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2 \qquad\qquad T = \frac{\epsilon_2 v_2}{\epsilon_1 v_1} t^2 = \frac{4 n_1 n_2}{(n_1 + n_2)^2}$$

N.B. since $R + T = 1$ we recover energy conservation

20. 4. General waves at interface: normal incidence

Basically we now repeat the above calculation but for complex impedance so that there may be phase lag between $\underline{E}$ and $\underline{H}$

$$\begin{aligned} \underline{E}_{inc} &= E_I \, \underline{e}_x \, e^{i(k_1 z - \omega t)} \\ \underline{H}_{inc} &= \frac{E_I}{Z_1} \, \underline{e}_y \, e^{i(k_1 z - \omega t)} \\ \underline{E}_{trans} &= E_T \, \underline{e}_x \, e^{i(k_2 z - \omega t)} \\ \underline{H}_{trans} &= \frac{E_T}{Z_2} \, \underline{e}_y \, e^{i(k_2 z - \omega t)} \\ \underline{E}_{ref} &= E_R \, \underline{e}_x \, e^{i(-k_2 z - \omega t)} \\ \underline{H}_{ref} &= -\frac{E_R}{Z_1} \, \underline{e}_y \, e^{i(-k_2 z - \omega t)} \end{aligned}$$

We afain assume that there *no surface currents or charges* and the continuity conditions reduce to $\underline{E}_{tan} = E_x$ continuous and $\underline{H}_{tan} = H_y$ continuous

$$\begin{aligned} E_I + E_R &= E_T \\ \frac{E_I}{Z_1} - \frac{E_R}{Z_2} &= \frac{E_T}{Z_2} \end{aligned}$$

Solve for E_T and E_R, knowing E_I as before

$$t \equiv \frac{E_T}{E_I} = \frac{2Z_2}{Z_2 + Z_1}$$

$$r \equiv \frac{E_R}{E_I} = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

N.B. These are now *complex* quantities

20. 5. Reflection at Conducting Surface: why metals are shiny

The x–y plane is a boundary between vacuum (medium 1) and a conductor (medium 2).

$$Z_1 = Z_{vac} = 377\Omega$$

$$Z_2 = \sqrt{\frac{-i\mu\omega}{\sigma}} = \frac{i-1}{\sigma\delta}$$

where $\delta = \sqrt{2/\mu\sigma\omega}$ is skin depth

Z_2 is complex and ω-dependent. But typical magnitude is tiny... e.g. Cu at 10^{10} Hz:

$$|Z_2| = 0.036\Omega = 10^{-4} Z_{vac}$$

and at 10^{15} Hz (visible light frequency)

$$|Z_2| = 3.6\Omega = 0.01 Z_{vac}$$

Amplitude reflection (note phase reversal)

$$r = \frac{Z_2 - Z_1}{Z_2 + Z_1} \simeq -1$$

to within (complex) terms of order 1 percent

Near perfect reflection (with phase reversal) is exhibited by good conductor— this explains why metals are shiny.

Physical origin is the skin effect; transmitted wave decays like $e^{-z/\delta}$, almost all the energy you put in comes back out

Energy Flow

With complex impedances we need to bit more careful with the Poynting vector. Generally we use the *time-averaged* Poynting vector which is given by

$$\langle \underline{S} \rangle = \underline{\hat{k}} \, \frac{1}{2} \Re \left(\frac{1}{Z} \right) |E_0|^2$$

and the **intensity** is given by its magnitude

$$|\langle \underline{S} \rangle| = \frac{1}{2} \Re \left(\frac{1}{Z} \right) |E_0|^2$$

EM 3 Section 21: Reflection at boundaries: oblique incidence

Last lecture we analysed the case of waves impinging on an interface at normal incidence. Here we consider a general angle of incidence

21. 1. General Angle of Incidence

As before we take an interface between two media to be the x–y plane at $z = 0$: medium 1

Figure 36: Wave at interface between two media *Griffiths fig. 9.14*

is $z < 0$; medium 2 is $z > 0$.

We can take the incident wave vector $\underline{k}_I$ to be in the x–z plane which is then the **plane of incidence**; y out of page

$$\begin{aligned} \underline{E}_{inc} &= \underline{E}_I \, e^{i(\underline{k}_I \cdot \underline{r} - \omega t)} \\ \underline{E}_{ref} &= \underline{E}_R \, e^{i(\underline{k}_R \cdot \underline{r} - \omega t)} \\ \underline{E}_{trans} &= \underline{E}_T \, e^{i(\underline{k}_T \cdot \underline{r} - \omega t)} \end{aligned}$$

We also have the corresponding magnetic field vectors e.g.

$$\underline{H}_{inc} = \frac{1}{\mu_1 v_1} \hat{\underline{k}} \times \underline{E}_{inc}$$

Now we have to fit the boundary conditions at the interface. First of all we note that all the boundary conditions will be of the form

$$(\quad)\mathrm{e}^{i(\underline{k}_I \cdot \underline{r} - \omega t)} + (\quad)\mathrm{e}^{i(\underline{k}_R \cdot \underline{r} - \omega t)} = (\quad)\mathrm{e}^{i(\underline{k}_T \cdot \underline{r} - \omega t)}$$

So for the boundary conditions to hold for all points on the interface x–y plane we must have the exponential factors (i.e. the phases) equal

$$\Rightarrow \underline{k}_I \cdot \underline{r} = \underline{k}_R \cdot \underline{r} = \underline{k}_T \cdot \underline{r} = \phi = \quad \text{constant} \qquad (1)$$

and straightaway we see that $\underline{k}_I$, $\underline{k}_R$, $\underline{k}_T$, all lie in the same plane—the plane of incidence. i.e. none of them have a component in the y direction

Then (1) becomes

$$k_I \sin\theta_I x = k_R \sin\theta_R x = k_T \sin\theta_T x$$

But this must hold for all x and also we know from $k = \omega/v$ that

$$k_I = k_R = \frac{v_2}{v_1} k_T = \frac{n_1}{n_2} k_T$$

which together imply

$$\begin{aligned} \theta_I &= \theta_R && \textbf{angle of incidence equals angle of reflection} \\ n_1 \sin\theta_I &= n_2 \sin\theta_T && \textbf{Snell's Law} \end{aligned}$$

We now have the job of satisfying the boundary conditions (see sections 17,18) which become

$$\epsilon_1(\underline{E}_I + \underline{E}_R)_z = \epsilon_2(\underline{E}_T)_z \tag{2}$$
$$(\underline{B}_I + \underline{B}_R)_z = (\underline{B}_T)_z \tag{3}$$
$$(\underline{E}_I + \underline{E}_R)_{x,y} = \epsilon_2(\underline{E}_T)_{x,y} \tag{4}$$
$$(\underline{H}_I + \underline{H}_R)_{x,y} = (\underline{H}_T)_{x,y} \tag{5}$$

The final two equations are both pairs of equation for the two transverse x,y components.

Polarisation Effects

The reflection and transmission coefficients r, t depend on the polarisation state of the incident beam. There are two basic polarisation states

A. $\underline{E}_I$ in plane of incidence ($\underline{E}_I$ has no y component and $\underline{H}_I$ along $\underline{e}_y$)

B. $\underline{H}_I$ in plane of incidence ($\underline{H}_I$ has no y component and $\underline{E}_I$ along $\underline{e}_y$)

Other polarization states can be decomposed into A+B by superposition. We will only work out **CASE A:** $\underline{E}$ in plane of incidence

Figure 37: Wave at interface between two media *Griffiths fig. 9.15*

Clearly (3) is automatically satisfied as $\underline{B}$ has no z component. It turns out (as you can check) that (5) does not give any additional information to (2) and (4)

Thus noting sign of E_z: $-$ for I,T but $+$ for R

$$\begin{aligned} \underline{E}_{inc} &= E_I \ (\underline{e}_x \cos\theta_I - \underline{e}_z \sin\theta_I)\, \mathrm{e}^{i(\phi-\omega t)} \\ \underline{E}_{ref} &= E_R \ (\underline{e}_x \cos\theta_I + \underline{e}_z \sin\theta_I)\, \mathrm{e}^{i(\phi-\omega t)} \\ \underline{E}_{trans} &= E_T \ (\underline{e}_x \cos\theta_T - \underline{e}_z \sin\theta_T)\, \mathrm{e}^{i(\phi-\omega t)} \end{aligned}$$

Then condition (2) $\Rightarrow$ $\quad \epsilon_1(-E_I + E_R)\sin\theta_I = -\epsilon_2 E_T \sin\theta_T$

and condition (4)$\Rightarrow$ $\quad (E_I + E_R)\cos\theta_I = E_T \cos\theta_T$

2 equations in 2 unknowns, solve for E_T, E_R: We define as before

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \left(\frac{\mu_1 \epsilon_2}{\epsilon_1 \mu_2}\right)^{1/2}$$

and also

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I}$$

then we find

$$\boxed{r \equiv \frac{E_R}{E_I} = \frac{\alpha - \beta}{\alpha + \beta} \qquad t \equiv \frac{E_T}{E_I} = \frac{2}{\alpha + \beta}} \tag{6}$$

which can also be written as

$$r = \frac{Z_2 \cos\theta_T - Z_1 \cos\theta_I}{Z_2 \cos\theta_T + Z_1 \cos\theta_I} \qquad t = \frac{2 Z_2 \cos\theta_I}{Z_2 \cos\theta_T + Z_1 \cos\theta_I} \tag{7}$$

where Z_i is the usual impedance. For nonmagnetic dielectrics $\mu_1 = \mu_2 = \mu_0$, $Z_1 = Z_{vac}/n_1$, $Z_2 = Z_{vac}/n_2 \Rightarrow$

$$r = \frac{n_1 \cos\theta_T - n_2 \cos\theta_I}{n_1 \cos\theta_T + n_2 \cos\theta_I}$$

use Snell's law to eliminate n's:

$$\boxed{r = \frac{\sin 2\theta_T - \sin 2\theta_I}{\sin 2\theta_T + \sin 2\theta_I}} \tag{8}$$

Fresnel Formula for case A ($\underline{E}$ in plane of incidence)

21. 2. Brewster's Angle

An interesting consequence of Fresnel equations (6) is that $r = 0$ when $\alpha = \beta$. This occurs at special angle of incidence know as **Brewster's angle** $\theta_I = \theta_B$

$$\left(1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_B\right)^{1/2} = \beta \cos\theta_B \tag{9}$$

$$\Rightarrow 1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_B = \beta^2 (1 - \sin^2\theta_B) \tag{10}$$

and finally this gives

$$\boxed{\sin^2\theta_B = \frac{1 - \beta^2}{\left(\frac{n_1}{n_2}\right)^2 - \beta^2}} \tag{11}$$

In the typical case $\mu_1 = \mu_2$ we have $\beta = n_2/n_1$ and one can show that

$$\boxed{\tan\theta_B = \frac{n_2}{n_1}} \tag{12}$$

($\theta_B \simeq 50°$ for water/air)

N.B. Brewster's angle only exists for case A: in case B there is no such effect.

Brewster angle microscopy: Shine 'case-A light' on clean surface at $\theta_I = \theta_B$: **no** reflected ray

Now adsorb thin layer of another material: reflected ray caused **solely** by film $\Rightarrow$ sensitive probe of film structure

Polarization by reflection: Unpolarised light source = random superposition of waves with E in plane of incidence (case A) and transverse to it (case B)

Near to Brewster's angle reflected ray is **almost all polarized**

One can eliminate reflected ray (glare) with polaroid filter which cuts out one plane of polarised light; basis of polaroid sunspecs etc.

21. 3. Total Internal Reflection

Choose $n_1 > n_2$ (e.g. wave leaving dielectric into vacuum) then $\theta_T > \theta_I$; $\theta_T = 90^\circ$ at $\theta_I = \theta_C$. Snell: $\sin\theta_C = n_2/n_1$. For $\theta_I > \theta_C$: we have **Total Internal Reflection**

Figure 38: Total internal reflection *Griffiths fig 9.28*

Evanescent Waves

To see what is happening for $\theta_I > \theta_C$, we persevere with the maths and note that if

$$\begin{aligned} \sin\theta_T &= \frac{n_2}{n_1}\sin\theta_I > 1 \\ \text{then} \qquad \cos\theta_T &= (1-\sin^2\theta_T)^{1/2} = i\left(\left(\frac{n_2}{n_1}\sin\theta_I\right)^2 - 1\right)^{1/2} \end{aligned}$$

clearly we can't interpret θ_T as an angle any more but the maths is valid

One can show that

$$\underline{E}_{evanescent} = \underline{E}_T e^{i(\underline{k}_T\cdot\underline{r}-\omega t)} = \underline{E}_T e^{i(kx-\omega t)}\, e^{-z/\alpha}$$

where

$$k = \frac{\omega n_1}{c}\sin\theta_I \qquad \alpha^{-1} = \frac{\omega}{c}\sqrt{n_1^2\sin^2\theta_I - n_2^2}$$

with $\alpha \simeq$ the wavelength ($\sim 0.5\mu$m). So we have **attenuation** in the z direction

The transmission coefficient $t \neq 0$ but no energy is carried into medium 2.
Instead there is a travelling wave directed along the interface, which decays in the z direction (into medium 2):

The decay is **not** adsorption **or** the skin effect but can be though of as **Light tunnelling**: light can tunnel across a thin layer of medium 2 via the evanescent wave.