Junior Honours

Electromagnetism **Problem Sheet 1**

Revision of Vector Calculus

1.1 If $\underline{r} = (x, y, z)$, $r = |\underline{r}| \neq 0$, $\hat{\underline{r}} = \underline{r}/r$, while $\underline{m}$ is constant, evaluate the following:

(i) $\underline{\nabla} r$; $\underline{\nabla} r^2$; $\underline{\nabla}(1/r)$; $\underline{\nabla}(\underline{m} \cdot \underline{r})$; $\underline{\nabla}(\underline{m} \cdot \underline{r}/r^3)$;

(ii) $\underline{\nabla} \cdot \underline{r}$; $\underline{\nabla} \cdot \hat{\underline{r}}$; $\underline{\nabla} \cdot (\underline{r}/r^3)$; $\underline{\nabla} \cdot (\underline{m} \times \underline{r})$; $\underline{\nabla} \cdot (\underline{m} \times \underline{r}/r^3)$;

(iii) $\underline{\nabla} \times \underline{r}$; $\underline{\nabla} \times (\hat{\underline{r}})$; $\underline{\nabla} \times (\underline{m} \times \underline{r})$; $\underline{\nabla} \times (\underline{m} \times \underline{r}/r^3)$;

[Hint: write $r = (x^2 + y^2 + z^2)^{1/2}$ to get started. Use product rules to speed things up!]

1.2 Show that if $\delta(x)$ is the δ-function, then

(i) $\delta(kx) = \frac{1}{|k|}\delta(x)$;

(ii) $x\frac{d}{dx}\delta(x) = -\delta(x)$;

[Remember: the δ-function is *defined* such that $\delta(x) = 0$ when $x \neq 0$, but $\int_{-a}^{a} dx\, \delta(x) = 1$ for any $a > 0$.]

1.3 In question 1.1 part iii) you showed that $\underline{\nabla} \cdot \left(\frac{\underline{r}}{r^3}\right) = 0$ for $r \neq 0$.

By applying the Divergence theorem to the field $\underline{v} = \frac{\underline{r}}{r^3}$ show that

$$\underline{\nabla} \cdot \left(\frac{\underline{r}}{r^3}\right) = 4\pi\delta(\underline{r})$$

Hint: choose a sphere of radius r as the surface.

1.4 a) Evaluate, using spherical polar co-ordinates and the spherical symmetry, the volume integral

$$I_1 = \int_V \mathrm{d}V \, \frac{e^{-r}}{r^2}$$

over all space.

(b) Evaluate

$$I_2 = \int_V \mathrm{d}V \, e^{-r} \underline{\nabla} \cdot (\hat{\underline{r}}/r^2).$$

[Hint: use the result of 1.3 and the δ-fn]

(c) Check your results by showing that $I_2 = I_1$ using the identity

$$f \, \underline{\nabla} \cdot \underline{v} = \underline{\nabla} \cdot (f\underline{v}) - (\underline{\nabla} f) \cdot \underline{v}$$

and the Divergence Thm

1.5 Compute the first three terms of the Taylor expansion of

(i) $1/(x + a)$, $a \ll x$,

(ii) $e^{i\underline{k}\cdot\underline{r}}$ $r \ll 1/k$,

(iii) $1/|\underline{r} + \underline{a}|$, $|\underline{a}| \ll \underline{r}$.

[Hint: write $|\underline{r} + \underline{a}| = (r^2 + 2ar\cos\theta + a^2)^{1/2}$.]

Cont./over

Revision of elementary Electrostatics (see P2B)

1.6 Three equal charges Q are placed at the corners of an equilateral triangle of side a.

(i) Find the resultant force on each charge and describe the relative motion of the charges.

(ii) Choose the position and size of an additional charge q such that the forces on all the charges are zero.

(iii) is this a stable equilibrium?

1.7 A point charge Q is placed at the centre of a cube. What is the *electric flux* through each face of the cube? How do these fluxes change if the charge is placed at the corner of the cube? [Hint: Use Gauss' Law and symmetry]