Junior Honours

Electromagnetism **Problem Sheet 3**

More Electrostatics: Images; Dipoles

HAND-IN DEADLINE: you must bring your solution to Q3.4 to the teaching office on Friday week 3 before noon You are strongly advised to work through the preceding questions, obtaining help from tutors where necessary, **before** attempting the hand-in question. The questions that are an integral part of this course. The code beside each question has the following significance:

- **K**: key question – explores core material
- **R**: review question – an invitation to consolidate
- **C**: challenge question – going beyond the basic framework of the course
- **S**: standard question – general fitness training!

3.1 **An image problem [R,K]** (first parts done in lectures)

(i) Sketch the field lines and equipotentials around a charge Q placed at height b above a point $\underline{0}$ on a plane horizontal conducting slab.

(ii) Write down the resulting field $\underline{E}(\underline{r})$ at a general point $\underline{r}$ above the slab, in terms of vectors $\underline{r}_1, \underline{r}_2$ connecting the charge, and its image, to $\underline{r}$.

(iii) Show that, on the surface at the slab at distance s from $\underline{0}$, the field has magnitude $E(s) = \frac{Qb}{2\pi\epsilon_0}(s^2 + b^2)^{-3/2}$. Hence find the surface charge density $\sigma(s)$.

(iv) Show by integration that the total induced surface charge is $-Q$. Use superposition to show that for an arbitrary arrangement of charges Q_i near the slab, the total induced surface charge is $-\sum_i Q_i$.

3.2 **Narcissus effect [S]**

What is the force on a point charge Q a distance b from a conducting slab?

3.3 **Electric dipole [K]**

(i) Sketch the electric field lines around a point (ideal) dipole. From the expression

$$V(\underline{r}) = \frac{\underline{p} \cdot \hat{\underline{r}}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

confirm that when $\theta = \pi/2$ the electric field is antiparallel to $\underline{p}$, and find its magnitude. [In spherical polars $\underline{\nabla} V = \underline{e}_r \frac{\partial V}{\partial r} + \underline{e}_\theta \frac{1}{r}\frac{\partial V}{\partial \theta} + \underline{e}_\phi \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}$.]

(ii) Explain why the energy of a dipole in an external field is $U_{dip} = -\underline{p}.\underline{E}_{ext}$. Hence establish that, if a second dipole of strength p' is placed antiparallel to the first, displaced by a vector $\underline{r}$ that is perpendicular to both moments, the mutual electrostatic energy of the system is $U = \frac{-pp'}{4\pi\epsilon_0 r^3}$.

(iii) What is the force between the two dipoles? Is there a torque?

3.4♣ **Thundercloud [S]**

A thunder cloud can be crudely modelled by two point charges $-Q$ and $+Q$ at heights h and d above the earth's surface with $d > h$. The Earth's surface may be considered as a conducting plane at $z = 0$.

(i) Make a sketch of this charge configuration (Do this in cross-section, on a plane perpendicular to the Earth and take the origin to be vertically below the 'cloud'.)

(ii) Write down Poisson's equation for the given charge distribution and the boundary conditions for this problem.

(iii) Construct the solution for the potential using image charges and show that Poisson's equation and the boundary conditions are satisfied.

(iv) State the direction of $\underline{E}$ at the Earth's surface and compute the field component in this direction.

(v) Using your expression for $\underline{E}$, write down the induced charge density on the Earth's surface and compute the total induced charge on the Earth's surface.

(vi) Show that on the surface $z = 0$, $\underline{E} = 0$ at a radial distance ρ_0 from the origin where ρ_0 satisfies

$$\frac{(\rho_0^2 + d^2)^3}{d^2} = \frac{(\rho_0^2 + h^2)^3}{h^2}$$

(vii) Sketch the electric field lines (do this in cross-section, on a plane perpendicular to the Earth). Indicate, in particular, the direction of the fields: directly below the cloud; at radial distance ρ_0; and far away from the cloud. Giving consideration to $\underline{E}$, where's the safest place on the surface to be?

3.5 **Image in crystal ball [S]**

A point charge q is placed at distance b from an earthed conducting sphere of radius a. Find the electrostatic potential outside the conducting sphere and force between the sphere and the charge.

[Hint: consider an image charge of size $-q'$ a distance b' from the centre of the sphere, and tune q' and b' until $V = 0$ on the surface. You will need to recall the cosine rule in order to obtain the distance from the image to a general point on the sphere.]

3.6 **Cornered [S/C]**

(i) **[S]** Consider a point charge q in the x–y plane at (a, b) where a,$b > 0$. There are two conducting walls (where $V = 0$) at $x = 0$, $y > 0$ and $y = 0$, $x > 0$.

Show that the solution of Poisson's equation the positive quadrant x,$y > 0$ that satisfies these boundary conditions is given by the physical charge plus three image charges: $-q$ at $(-a, b)$; $-q$ at $(a, -b)$; $+q$ at $(-a, -b)$.

(ii) **[C]** Now consider a point charge q in 3 -dimensions at (a, b, c) where a,b,$c > 0$. There are three conducting walls at $x = 0$, y,$z > 0$, $y = 0$, x,$z > 0$ and $z = 0$, x,$y > 0$ where $V = 0$.

Write down the charge and positions of the image charges required to give the solution to Poisson's equation in the positive octant. [Hint: there are seven of them]

3.7 **Electric Quadrupole [C]** Write down the potential at a point $\underline{r}$ due to an electric dipole of moment $\underline{p}$ at the origin. Hence, or otherwise, calculate the potential at a point P with spherical polar co-ordinates (r, θ, ϕ) due to charges $-q$, $2q$ and $-q$ at points $z = -a$, $z = 0$ and $z = +a$ respectively, where $a \ll r$. Determine the radial and transverse components E_r and E_θ of the electric fields $\underline{E}$ due to this charge distribution, and show that $E_r = 0$ when $\cos^2\theta = 1/3$, while $E_\theta = 0$ when $\sin 2\theta = 0$. Sketch the electric field lines.