Junior Honours

Electromagnetism **Problem Sheet 5**

More Magnetostatics: Ampère's law; Magnetic vector potential
HAND-IN DEADLINE: you must bring your solution to Q5.3 to the teaching office on Friday 15th February by noon

You are strongly advised to work through the preceding questions, obtaining help from tutors where necessary, **before** attempting the hand-in question.
The questions The code beside each question has the following significance:

- **K**: key question – explores core material
- **R**: review question – an invitation to consolidate
- **C**: challenge question – going beyond the basic framework of the course
- **S**: standard question – general fitness training!

5.1 **Solenoid [K]** Use Ampère's Law and Gauss' Law for magnetic fields to show the following results for a solenoid of infinite length with n turns per unit length: (a) $B_\phi = 0$ everywhere, (b) $B_r = 0$ everywhere, (c) $B_z = 0$ outside and (d) $B_z = n\mu_0 I$ inside the solenoid.

5.2 **Parallel lines [S]** Two very long thin wires carrying equal and opposite currents of I are placed parallel to the x-axis at $y = 0$ and $z = \pm a$. Calculate the magnetic field $\underline{B}$ in the $y - z$ plane. Show that, at $z = 0$, its gradient $\frac{\partial B}{\partial y}$ is greatest when $y = \pm a/\sqrt{3}$.

5.3♣ **Helmholtz coils [S]** Helmholtz coils can be modelled as a pair of current loops oriented so that they are parallel to each other with a common axis in the z direction. The two loops have the same current I, and the same radius R, and their centres are at $z = -d$ and $z = +d$, so the distance between the two loops is $2d$.

(a) Write down the magnetic field at position z along the axis of the coils (you may use without proof a result derived in lectures for a single coil). What is the magnetic field at the midpoint between the coils, i.e. at $z = 0$? What is the magnetic field at $z \gg R$ Sketch the magnetic field lines for all $\underline{r}$ [5]

(b) Compute the magnetic field gradient $\dfrac{\partial B_z}{\partial z}$ and second derivative $\dfrac{\partial^2 B_z}{\partial z^2}$. What are the values at the midpoint $z = 0$. At $z = 0$, show that $\dfrac{\partial^2 B_z}{\partial z^2} = 0$ when $d = R/2$. [6]

(c) Suggest why Helmholtz coils normally have $d = R/2$. [1]

(d) Find the force (magnitude and direction) between the two loops.
[Hint: Find the magnetic field from the first loop on the axis and use $\underline{F} = \underline{\nabla}(\underline{m} \cdot \underline{B})$]
where $\underline{m}$ is the magnetic dipole moment [3]

(e) Now consider reversing the direction of the current in one of the two coils.

How would the above result for the force change? Determine the far field ($z \gg R$, $z \gg d$) expression for the field along the axis. Make a rough sketch of what you think the field will look like for all $\underline{r}$ [5]

5.4 **Spinning wheel [S]**

A wire with uniform charge density λ per unit length is bent into a ring of radius a and rotates with angular velocity ω about an axis through its centre and perpendicular to the plane of the ring. A a second co-axial ring of the same radius, carrying charge density λ', and also rotating with angular velocity ω, placed at a distance $L \ll a$ from the first.

Compute both electrical and magnetic forces between the two rings. [Hint: in limit $L \ll a$ one can treat the rings as straight lines]

Show that the total force is zero if $\omega = c/a$ where $c = 1/\sqrt{\mu_0 \epsilon_0}$. Is this possible to achieve in practice?

5.5 **Vector potential of wire [K]**

A long straight wire carries a uniform current density $\underline{J}$ inside it

(i) What is the magentic field $\underline{B}$ inside the wire?

(ii) Show that the magnitude of the magnetic vector potential inside the wire is:

$$|\underline{A}| = -\frac{\mu_0 J r^2}{4}$$

What is the direction of $\underline{A}$? [Hint: use the expression for curl in cylindrical polars]

(iii) Confirm that $\underline{A}$ satisfies Poisson's equations $\nabla^2 \underline{A} = -\mu_0 \underline{J}$.

5.6 **Magnetic dipole vector potential [C]**

Use the formula for the vector potential of a current loop C

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \oint_C \frac{I \underline{dr'}}{|\underline{r} - \underline{r'}|}$$

to show that the vector potential for a current loop is given in the far field limit by

$$\underline{A}_{\rm dip}(\underline{r}) = \frac{\mu_0}{4\pi r^2} \oint_C \hat{\underline{r}} \cdot \underline{r'}\, \underline{dr'}$$

By setting $\underline{v} = \varphi\, \underline{c}$, $\underline{c}$ any constant vector, in Stoke's Thm, show that

$$\oint_C \varphi\, \underline{dr'} = -\int_S \underline{\nabla}\varphi \times \underline{dS}.$$

Hence show that

$$\oint_C (\underline{b} \cdot \underline{r'}) \underline{dr'} = \underline{a} \times \underline{b}.$$

where $\underline{a}$ is the vector area and $\underline{b}$ is a constant vector and therefore

$$\underline{A}_{\rm dip}(\underline{r}) = \frac{\mu_0}{4\pi} \frac{\underline{m} \times \hat{\underline{r}}}{r^2}$$

where $\underline{m}$ is the magnetic dipole of the loop.

Then derive the magnetic dipole field using a result of Q1.1.*iii*