Junior Honours

Electromagnetism **Problem Sheet 7**

Electromagnetic waves in vacuo; Poynting vector

The questions that follow on this and succeeding sheets are an integral part of this course. The code beside each question has the following significance:

- **K**: key question – explores core material
- **R**: review question – an invitation to consolidate
- **C**: challenge question – going beyond the basic framework of the course
- **S**: standard question – general fitness training!

7.1 **Beginning to see the light [K]**

Assume the following $\underline{E}$ and $\underline{B}$ fields for a propagating EM wave in a region of empty space, with $\underline{E}_0$ and $\underline{B}_0$ constant vectors:

$$\underline{E}(\underline{r},t) = \underline{E}_0 \exp i(\underline{k}\cdot\underline{r} - \omega t)$$

$$\underline{B}(\underline{r},t) = \underline{B}_0 \exp i(\underline{k}\cdot\underline{r} - \omega t)$$

Verify the **Differentiation rules**: If $\underline{f}(\underline{r},t) = \underline{f}_0 \exp i(\underline{k}\cdot\underline{r} - \omega t)$ then $\underline{\nabla}\cdot\underline{f} = i\underline{k}\cdot\underline{f}$; $\underline{\nabla}\times\underline{f} = i\underline{k}\times\underline{f}$; $\nabla^2\underline{f} = -k^2\underline{f}$; $\partial\underline{f}/\partial t = -i\omega\underline{f}$.

Using these rules show that these field configurations obey all four Maxwell equations (for a region of vacuum with no charges or currents present), if and only if:

(a) $\underline{k}\cdot\underline{E}_0 = 0$ [Hint: Gauss for $\underline{E}$]

(b) $\underline{k}\times\underline{E}_0 = \omega\underline{B}_0$ [etc.]

(c) $\underline{k}\cdot\underline{B}_0 = 0$

(d) $\underline{k}\times\underline{B}_0 = -\epsilon_0\mu_0\omega\underline{E}_0$

7.2 **EM waves in vacuum: laying down the law [K]**

Use the results (a–d) in question (1) to deduce that for the EM wave in that question:

(i) Fields $\underline{E}$ and $\underline{B}$ **must** be transverse to the direction of propagation.

(ii) If the x axis is chosen along the electric field vector $\underline{E}_0$ and the z axis chosen along the propagation direction, then $\underline{B}_0$ **must** point along y, and **must** have magnitude E_0k/ω.

(iii) The relationship $\omega^2 = c^2k^2$, where $c^2 = (\epsilon_0\mu_0)^{-1}$, **must** be obeyed.

7.3 **A continuity gaffe [S]**

A student claims that in some region of space (with $\underline{r} = (x,y,z)$) there is simultaneously a time dependent charge density $\rho(\underline{r},t) = axe^{-bxt}$, and a current density $\underline{J}(\underline{r},t) = \underline{e}_x abx^3e^{-bxt}/3$, where a and b are constants. Show that this claim is false.

7.4 **Continuity problem [S]** Show that the charge distribution within a conductor of conductivity σ decays exponentially in time, with decay constant $\tau = \epsilon_0/\sigma$.

A uniform conducting sphere of radius R has at $t = 0$ a uniform charge density ρ_0 throughout its interior. Find ρ, $\underline{J}$ and $\underline{E}$ within the sphere as functions of time. Find also the electric field outside the sphere and show that the rate of Ohmic heat generation in the sphere $\dfrac{\mathrm{d}W}{\mathrm{d}t} = \displaystyle\int_V \underline{E}\cdot\underline{J}\,\mathrm{d}V$ is equal to the rate of dissipation of electrostatic field energy.

[Hint: Use spherical symmetry and the formula for divergence in spherical polars]

7.5 **Polarisation [K]**

A transverse electromagnetic wave in vacuo propagates in the z-direction. Consider the cases that the wave is (a) linearly (or plane) polarised, $\underline{E} = \underline{E}_0 \sin(kz - \omega t)$, and (b) circularly polarised, $\underline{E} = E_0 \left(\cos(kz - \omega t)\underline{e}_x + \sin(kz - \omega t)\underline{e}_y \right)$. In each case find (i) the magnetic field, $\underline{B}(\underline{r}, t)$, and (ii) the Poynting vector $\underline{S}(\underline{r}, t)$.

7.6 **Poynting in the right direction [S]**

A long straight wire of radius a, carries a uniform current density; the total current is I. The potential difference across a length l of the wire is $V = El$.

(i) Use Ampère's law to write down the $\underline{B}$ field everywhere.

(ii) If the current density is uniform, $\underline{E}$ has the same value at all points within the wire. Use $\oint \underline{E} \cdot \mathrm{d}\underline{l} = 0$ (which holds for stationary currents and charge densities so that $\underline{E}$ is purely electrostatic and there is no magnetic vector potential) to show that the $\underline{E}$ field just outside the wire must also have this value.

(iii) Find the Poynting vector (magnitude and direction) at all points within the wire, and also just outside it.

(iv) Show that enough energy is flowing radially into this stretch of wire from the fields outside, exactly to balance the energy lost to resistance within the wire.

[For discussion] This contradicts an idea many people have, that the energy required to balance resistive losses flows along the wire itself.