Lecture 14: More on Vector Operators *(RHB 8.7.2, 8.7.3)*

In this lecture we combine the vector operator $\underline{\nabla}$ ('Del') with a vector field to define two new operations 'Div' and 'Curl'. Then we define the Laplacian.

14. 1. Divergence

We define the **divergence** of vector field $\underline{A}$ as:-

$$\boxed{\operatorname{div}\underline{A}(\underline{r}) \;\equiv\; \underline{\nabla}\cdot\underline{A}(\underline{r})}$$

This is pronounced 'div A'

In Cartesian coordinates

$$\boxed{\begin{aligned}\underline{\nabla}\cdot\underline{A}(\underline{r}) = \frac{\partial}{\partial x_i}\,A_i(\underline{r}) \quad &= \quad \frac{\partial A_1(\underline{r})}{\partial x_1} + \frac{\partial A_2(\underline{r})}{\partial x_2} + \frac{\partial A_3(\underline{r})}{\partial x_3} \\ \text{or} \quad &\quad \frac{\partial A_x(\underline{r})}{\partial x} + \frac{\partial A_y(\underline{r})}{\partial y} + \frac{\partial A_z(\underline{r})}{\partial z} \quad \text{in } x, y, z \text{ notation}\end{aligned}}$$

It is easy to show that $\underline{\nabla}\cdot\underline{A}(\underline{r})$ is a scalar field: Under a change of basis $\underline{e}_i \to \underline{e}_i{}' = \lambda_{ij}\,\underline{e}_j$

$$\begin{aligned}\underline{\nabla}'\cdot\underline{A}'(x'_1,\,x'_2,\,x'_3) &= \frac{\partial}{\partial x'_i}\,A'_i(x'_1,\,x'_2,\,x'_3) \\ &= \frac{\partial x_j}{\partial x'_i}\,\frac{\partial}{\partial x_j}\,\big(\lambda_{ik}A_k(x_1,\,x_2,\,x_3)\big) \\ &= \lambda_{ij}\,\lambda_{ik}\,\frac{\partial}{\partial x_j}\,A_k(x_1,\,x_2,\,x_3) \\ &= \delta_{jk}\,\frac{\partial}{\partial x_j}\,A_k(x_1,\,x_2,\,x_3) \\ &= \frac{\partial}{\partial x_j}\,A_j(x_1,\,x_2,\,x_3) = \underline{\nabla}\cdot\underline{A}(\underline{r})\end{aligned}$$

Hence $\underline{\nabla}\cdot\underline{A}$ is invariant under a change of basis, therefore it is a **scalar field**.

Example: $\underline{A}(\underline{r}) = \underline{r} \quad \Rightarrow \quad \boxed{\underline{\nabla}\cdot\underline{r} \;=\; 3}$ a *very* useful & important result!

$$\underline{\nabla}\cdot\underline{r} = \frac{\partial x_1}{\partial x_1} + \frac{\partial x_2}{\partial x_2} + \frac{\partial x_3}{\partial x_3} = 1 + 1 + 1 = 3$$

In suffix notation

$$\underline{\nabla}\cdot\underline{r} = \frac{\partial x_i}{\partial x_i} = \delta_{ii} = 3.$$

Example: In this example we use 'xyz' notation: $x_1 = x$, $x_2 = y$, $x_3 = z$. Consider $\underline{A} = x^2 z\,\underline{e}_1 - 2y^3 z^2\,\underline{e}_2 + xy^2 z\,\underline{e}_3$

$$\underline{\nabla} \cdot \underline{A} \;=\; \frac{\partial}{\partial x}\,(x^2 z) \;-\; \frac{\partial}{\partial y}\,(2y^3 z^2) \;+\; \frac{\partial}{\partial z}\,(xy^2 z)$$
$$\;=\; 2xz - 6y^2 z^2 + xy^2$$

Thus at the point (1,1,1) $\underline{\nabla} \cdot \underline{A} = 2 - 6 + 1 = -3$.

14. 2. Curl

We define the the curl of a vector field curl $\underline{A}$ as

$$\boxed{\operatorname{curl} \underline{A}(\underline{r}) \;\equiv\; \underline{\nabla} \times \underline{A}(\underline{r})}$$

Note that curl $\underline{A}$ is a *vector* field

In Cartesian coordinates

$$\underline{\nabla} \times \underline{A} \;=\; \underline{e}_i \,\left(\underline{\nabla} \times \underline{A}\right)_i$$
$$\;=\; \underline{e}_i \,\epsilon_{ijk}\, \frac{\partial}{\partial x_j}\, A_k$$

ie, the ith component of $\underline{\nabla} \times \underline{A}$ is

$$\boxed{\left(\underline{\nabla} \times \underline{A}\right)_i \;=\; \epsilon_{ijk}\, \frac{\partial}{\partial x_j}\, A_k}$$

More explicitly

$$\left(\underline{\nabla} \times \underline{A}\right)_1 \;=\; \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \quad \text{etc},$$

Instead of the above equation for curl that uses ϵ_{ijk}, one can use a determinant form (c.f. the expression of the vector product)

$$\boxed{\underline{\nabla} \times \underline{A} \;=\; \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ A_1 & A_2 & A_3 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} \underline{e}_x & \underline{e}_y & \underline{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}.}$$

Example: $\underline{A}(\underline{r}) = \underline{r} \quad \Rightarrow \quad \boxed{\underline{\nabla} \times \underline{r} \;=\; 0}$ another *very* useful & important result!

$$\underline{\nabla} \times \underline{r} \;=\; \underline{e}_i \,\epsilon_{ijk}\, \frac{\partial}{\partial x_j}\, x_k$$
$$\;=\; \underline{e}_i \,\epsilon_{ijk}\, \delta_{jk} = \underline{e}_i \,\epsilon_{ijj} = 0$$

or, using the determinant formula, $\underline{\nabla} \times \underline{r} \;=\; \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ x_1 & x_2 & x_3 \end{vmatrix} \;\equiv\; 0$

Example: Compute the curl of $\underline{V} = x^2 y \underline{e}_1 + y^2 x \underline{e}_2 + xyz \underline{e}_3$:

$$\underline{\nabla} \times \underline{V} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 y & y^2 x & xyz \end{vmatrix} = \underline{e}_1(xz - 0) - \underline{e}_2(yz - 0) + \underline{e}_3(y^2 - x^2)$$

14. 3. Physical Interpretation of 'div' and 'curl'

Complete interpretions of the divergence and curl of a vector field are best left until after we have studied the Divergence Theorem and Stokes' Theorem respectively. However, we can gain some intuitive understanding by looking at simple examples where div and/or curl vanish.

First consider the radial field $\underline{A} = \underline{r}$; $\underline{\nabla} \cdot \underline{A} = 3$; $\underline{\nabla} \times \underline{A} = 0$. We sketch the vector field $\underline{A}(\underline{r})$ by drawing at selected points vectors of the appropriate direction and magnitude. These give the tangents of 'flow lines'. Roughly speaking, in this example the divergence is positive

because bigger arrows come out of a point than go in. So the field 'diverges'. (Once the concept of flux of a vector field is understood this will make more sense.)

Now consider the field $\underline{v} = \underline{\omega} \times \underline{r}$ where $\underline{\omega}$ is a constant vector. One can think of $\underline{v}$ as the velocity of a point in a rigid rotating body. We sketch a cross-section of the field $\underline{v}$ with $\underline{\omega}$ chosen to point out of the page. We can calculate $\underline{\nabla} \times \underline{v}$ as follows:

$$\begin{aligned} \underline{\nabla} \times \left(\underline{\omega} \times \underline{r}\right) &= \underline{e}_i \, \epsilon_{ijk} \frac{\partial}{\partial x_j} \left(\underline{\omega} \times \underline{r}\right)_k \quad = \underline{e}_i \, \epsilon_{ijk} \frac{\partial}{\partial x_j} \epsilon_{klm} \, \omega_l \, x_m \\ &= \underline{e}_i \left(\delta_{il} \, \delta_{jm} - \delta_{im} \, \delta_{jl}\right) \omega_l \, \delta_{jm} \qquad \left(\text{since } \frac{\partial \omega_l}{\partial x_j} = 0\right) \\ &= \underline{e}_i \left(\omega_i \, \delta_{jj} - \delta_{ij} \, \omega_j\right) = \underline{e}_i \, 2 \, \omega_i = 2\underline{\omega} \end{aligned}$$

Thus $\boxed{\underline{\nabla} \times (\underline{\omega} \times \underline{r}) = 2\underline{\omega}}$ which is yet another *very* useful & important result!

To understand intuitively the non-zero curl imagine a small ball centred on a flow line of the field. The centre of the ball will follow the flow line. However the effect of the neighbouring flow lines is to make the ball rotate. Therefore the field has non-zero 'curl' and the axis of rotation gives the direction of the curl. In the previous example ($\underline{A} = \underline{r}$) the ball would just move away from origin without rotating therefore the field $\underline{r}$ has zero curl.

Also, as we shall see next lecture, $\underline{\nabla} \cdot \underline{v}=0$ for this example. To understand this note from the sketch that the same size of arrow goes into a point as comes out.

Terminology:

1. If $\underline{\nabla} \cdot \underline{A}(\underline{r}) = 0 \; \forall \underline{r}$, or in some region R, $\underline{A}$ is said to be **solenoidal** there.
2. If $\underline{\nabla} \times \underline{A}(\underline{r}) = 0 \; \forall \underline{r}$, or in some region R, $\underline{A}$ is said to be **irrotational** there.

14. 4. The Laplacian Operator ∇^2

We may take the *divergence* of the *gradient* of a scalar field $\phi(\underline{r})$

$$\underline{\nabla} \cdot (\underline{\nabla}\, \phi(\underline{r})) \;=\; \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i}\, \phi(\underline{r}) \;\equiv\; \nabla^2 \phi(\underline{r})$$

∇^2 is the **Laplacian operator** "del-squared". In Cartesian coordinates

$$\nabla^2 \;=\; \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i}$$

More explicitly

$$\boxed{\nabla^2\, \phi(\underline{r}) \;=\; \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} \quad \text{or} \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2}}$$

It may be shown that the Laplacian of a scalar field $\nabla^2\, \phi$ is also a scalar field, *i.e.* the Laplacian is a **scalar operator**.

Example

$$\nabla^2\, r^2 = \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} x_j x_j = \frac{\partial}{\partial x_i}(2x_i) = 2\delta_{ii} = 6\ .$$

In Cartesian coordinates, the effect of the Laplacian on a vector field $\underline{A}$ is *defined* to be

$$\nabla^2 \underline{A}(\underline{r}) \;=\; \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i}\, \underline{A}(\underline{r}) \;=\; \frac{\partial^2}{\partial x_1^2}\, \underline{A}(\underline{r}) \;+\; \frac{\partial^2}{\partial x_2^2}\, \underline{A}(\underline{r}) \;+\; \frac{\partial^2}{\partial x_3^2}\, \underline{A}(\underline{r})$$

The Laplacian acts on a vector field to produce another vector field.