Lecture 15: Vector Operator Identities (*RHB 8.8*)

There are a large number of identities for div, grad, and curl. It's not necessary to know *all* of these, but you are advised to be able to produce from memory expressions for $\underline{\nabla} r$, $\underline{\nabla} \cdot \underline{r}$, $\underline{\nabla} \times \underline{r}$, $\underline{\nabla}\phi(r)$, $\underline{\nabla}(\underline{a} \cdot \underline{r})$, $\underline{\nabla} \times (\underline{a} \times \underline{r})$, $\underline{\nabla}(fg)$, and 1 2 3 4 below. You should be *familiar* with the rest and to be able to *derive* and *use* them when necessary!

Most importantly you should be at ease with div, grad and curl. This only comes through practice and deriving the various identities gives you just that. In these derivations the advantages of suffix notation, the summation convention and ϵ_{ijk} will become apparent.

In what follows, $\phi(\underline{r})$ is a scalar field; $\underline{A}(\underline{r})$ and $\underline{B}(\underline{r})$ are vector fields.

15. 1. Distributive Laws

1. $\underline{\nabla} \cdot (\underline{A} + \underline{B}) \;=\; \underline{\nabla} \cdot \underline{A} + \underline{\nabla} \cdot \underline{B}$
2. $\underline{\nabla} \times (\underline{A} + \underline{B}) \;=\; \underline{\nabla} \times \underline{A} + \underline{\nabla} \times \underline{B}$

The proofs of these are straightforward using suffix or 'x y z' notation and follow from the fact that div and curl are linear operations.

15. 2. Product Laws

The results of taking the div or curl of **products** of vector and scalar fields are predictable but need a little care:-

3. $\underline{\nabla} \cdot (\phi\,\underline{A}) \;=\; \phi\,\underline{\nabla} \cdot \underline{A} + \underline{A} \cdot \underline{\nabla}\phi$
4. $\underline{\nabla} \times (\phi\,\underline{A}) \;=\; \phi\,(\underline{\nabla} \times \underline{A}) + (\underline{\nabla}\phi) \times \underline{A} \;=\; \phi\,(\underline{\nabla} \times \underline{A}) - \underline{A} \times \underline{\nabla}\phi$

Proof of (4): first using ϵ_{ijk}

$$
\begin{aligned}
\underline{\nabla} \times (\phi\,\underline{A}) &= \underline{e}_i\,\epsilon_{ijk}\,\frac{\partial}{\partial x_j}(\phi\,A_k) \\
&= \underline{e}_i\,\epsilon_{ijk}\left(\phi\left(\frac{\partial}{\partial x_j}A_k\right) + A_k\left(\frac{\partial}{\partial x_j}\phi\right)\right) \\
&= \phi\,(\underline{\nabla} \times \underline{A}) + (\underline{\nabla}\phi) \times \underline{A}
\end{aligned}
$$

or avoiding ϵ_{ijk} and using 'x y z' notation: $\underline{\nabla} \times (\phi\,\underline{A}) = \begin{vmatrix} \underline{e}_x & \underline{e}_y & \underline{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \phi A_x & \phi A_y & \phi A_z \end{vmatrix}$.

The x component is given by

$$
\begin{aligned}
\frac{\partial(\phi A_z)}{\partial y} - \frac{\partial(\phi A_y)}{\partial z} &= \phi\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \left(\frac{\partial\phi}{\partial y}\right)A_z - \left(\frac{\partial\phi}{\partial z}\right)A_y \\
&= \phi(\underline{\nabla} \times \underline{A})_x + \left[(\underline{\nabla}\phi) \times \underline{A}\right]_x
\end{aligned}
$$

A similar proof holds for the y and z components.

Although we have used Cartesian coordinates in our proofs, the identities hold in all coordinate systems.

15. 3. Products of Two Vector Fields

Things start getting complicated!

5. $\underline{\nabla}\,(\underline{A}\cdot\underline{B}) \;=\; (\underline{A}\cdot\underline{\nabla})\,\underline{B} \,+\, (\underline{B}\cdot\underline{\nabla})\underline{A} \,+\, \underline{A}\times(\underline{\nabla}\times\underline{B}) \,+\, \underline{B}\times(\underline{\nabla}\times\underline{A})$
6. $\underline{\nabla}\cdot(\underline{A}\times\underline{B}) \;=\; \underline{B}\cdot(\underline{\nabla}\times\underline{A}) \,-\, \underline{A}\cdot(\underline{\nabla}\times\underline{B})$
7. $\underline{\nabla}\times(\underline{A}\times\underline{B}) \;=\; \underline{A}\,(\underline{\nabla}\cdot\underline{B}) \,-\, \underline{B}\,(\underline{\nabla}\cdot\underline{A}) \,+\, (\underline{B}\cdot\underline{\nabla})\,\underline{A} \,-\, (\underline{A}\cdot\underline{\nabla})\,\underline{B}$

Proof of (6):

$$\begin{aligned}
\underline{\nabla}\cdot(\underline{A}\times\underline{B}) &= \frac{\partial}{\partial x_i}\,\epsilon_{ijk}\,A_j\,B_k \\
&= \epsilon_{ijk}\left(\frac{\partial A_j}{\partial x_i}\right)B_k \,+\epsilon_{ijk}\,A_j\left(\frac{\partial B_k}{\partial x_i}\right) \\
&= B_k\epsilon_{kij}\,\frac{\partial A_j}{\partial x_i} \,-A_j\epsilon_{jik}\,\frac{\partial B_k}{\partial x_i}
\end{aligned}$$

The proofs of (5) and (7) involve the product of two epsilon ijks. For example, this is why there are four terms on the rhs of (7).

All other results involving one $\underline{\nabla}$ can be derived from the above identities.

Example: If $\underline{a}$ is a *constant* vector, and $\underline{r}$ is the position vector, show that

$$\underline{\nabla}\,(\underline{a}\cdot\underline{r}) = (\underline{a}\cdot\underline{\nabla})\,\underline{r} = \underline{a}$$

In lecture 13 we showed that $\underline{\nabla}\,(\underline{a}\cdot\underline{r}) = \underline{a}$ for constant $\underline{a}$. Hence, we need only evaluate

$$(\underline{a}\cdot\underline{\nabla})\,\underline{r} \;=\; a_i\,\frac{\partial}{\partial x_i}\,\underline{e}_j\,x_j \;=\; a_i\,\underline{e}_j\,\delta_{ij} \;=\; a_i\,\underline{e}_i \;=\; \underline{a} \tag{1}$$

and the identity holds.

Example: Show that $\underline{\nabla}\cdot(\underline{\omega}\times\underline{r}) = 0$ where $\underline{\omega}$ is a *constant* vector.
Using 6. $\underline{\nabla}\cdot(\underline{\omega}\times\underline{r}) = \underline{\omega}\cdot(\underline{\nabla}\times\underline{r}) \,-\, \underline{r}\cdot(\underline{\nabla}\times\underline{\omega}) = 0 - 0$

Example: Show that $\underline{\nabla}\cdot(r^{-3}\underline{r}) = 0$, (where $r = |\underline{r}|$ as usual).
Using identity (3), we have

$$\underline{\nabla}\cdot(r^{-3}\underline{r}) \;=\; r^{-3}\,(\underline{\nabla}\cdot\underline{r}) \,+\, \underline{r}\cdot\underline{\nabla}(r^{-3})$$

We have previously shown that $\underline{\nabla} \cdot \underline{r} = 3$ and that $\underline{\nabla}(r^n) = n\, r^{n-2}\, \underline{r}$. Hence

$$\begin{aligned}
\underline{\nabla} \cdot (r^{-3}\underline{r}) &= r^{-3}\,(\underline{\nabla} \cdot \underline{r}) + \underline{r} \cdot \underline{\nabla}(r^{-3}) \\
&= \frac{3}{r^3} + \underline{r} \cdot \left(\frac{-3}{r^5}\, \underline{r} \right) \\
&= \frac{3}{r^3} + \frac{-3}{r^5}\, r^2 \;=\; 0 \qquad \text{(except at } r = 0\text{)}
\end{aligned}$$

15. 4. Identities involving 2 $\underline{\nabla}$'s

8. $\underline{\nabla} \times (\underline{\nabla}\phi) = 0$ — curl grad ϕ is always zero.
9. $\underline{\nabla} \cdot (\underline{\nabla} \times \underline{A}) = 0$ — div curl $\underline{A}$ is always zero.
10. $\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{A}) - \nabla^2 \underline{A}$

Proofs are easily obtained in Cartesian coordinates using suffix notation:-

Proof of (8)

$$\begin{aligned}
\underline{\nabla} \times (\underline{\nabla}\,\phi) &= \underline{e}_i\, \epsilon_{ijk}\, \frac{\partial}{\partial x_j}\, (\underline{\nabla}\,\phi)_k \;=\; \underline{e}_i\, \epsilon_{ijk}\, \frac{\partial}{\partial x_j}\, \frac{\partial}{\partial x_k}\, \phi \\
&= \underline{e}_i\, \epsilon_{ijk}\, \frac{\partial}{\partial x_k}\, \frac{\partial}{\partial x_j}\, \phi \qquad \left(\text{since } \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x} \text{ etc} \right) \\
&= \underline{e}_i\, \epsilon_{ikj}\, \frac{\partial}{\partial x_j}\, \frac{\partial}{\partial x_k}\, \phi \qquad \text{(interchanging labels } j \text{ and } k\text{)} \\
&= -\underline{e}_i\, \epsilon_{ijk}\, \frac{\partial}{\partial x_j}\, \frac{\partial}{\partial x_k}\, \phi \qquad (ikj \to ijk \text{ gives minus sign}) \\
&= -\underline{\nabla} \times (\underline{\nabla}\,\phi) \;=\; 0
\end{aligned}$$

since any vector equal to minus itself is must be zero. Proof of (9) is similar. It is important to understand how these two identities stem from the anti-symmetry of ϵ_{ijk} hence the anti-symmetry of the curl curl operation.

(10) can be proven using the identity for the product of two ϵ_{ijk}. Although the proof is tedious it is far simpler than trying to use 'xyz' (try both and see!)

(10) is an important result and is used frequently in electromagnetism, fluid mechanics, and other 'field theories'.

Finally, when a scalar field ϕ depends only on the magnitude of the position vector $r = |\underline{r}|$, we have

$$\nabla^2\, \phi(r) = \phi''(r) + \frac{2\phi'(r)}{r}$$

where the prime denotes differentiation with respect to r. Proof of this relation is left to the tutorial.

Before commencing with integral vector calculus we review here polar co-ordinate systems.

Polar Co-ordinate Systems Here dV indicates a volume element and dA an area element. Note that different conventions, *e.g.* for the angles ϕ and θ, are sometimes used, in particular in the Mathematics 'Several Variable Calculus' Module.

Plane polar co-ordinates

Cylindrical polar co-ordinates

Spherical polar co-ordinates