Lecture 22: Stokes' Theorem and Applications *(RHB 9.9, Dawber chapter 6)*

22. 1. Stokes' Theorem

If S is an **open** surface, bounded by a simple **closed** curve C, and $\underline{A}$ is a vector field defined on S, then

$$\boxed{\oint_C \underline{A}\cdot\underline{dr} \;=\; \int_S (\underline{\nabla}\times\underline{A})\cdot\underline{dS}}$$

where C is traversed in a right-hand sense about $\underline{dS}$. (As usual $\underline{dS} = \underline{n}dS$ and $\underline{n}$ is the unit normal to S).

Proof *(D 6.1; RHB 9.9)*:
Divide the surface area S into N adjacent small surfaces as indicated in the diagram. Let $\underline{\Delta S^i} = \Delta S^i\,\underline{n}^i$ be the vector element of area at $\underline{r}^i$. Using the integral definition of curl,

$$\underline{n}\cdot(\text{curl }\underline{A}) \;=\; \underline{n}\cdot(\underline{\nabla}\times\underline{A}) \;=\; \lim_{\Delta S\to 0}\frac{1}{\Delta S}\oint_C \underline{A}\cdot\underline{dr}$$

we multiply by ΔS^i and sum over all i to get

$$\sum_{i=1}^{N}\Big(\underline{\nabla}\times\underline{A}(\underline{r}^i)\Big)\cdot\underline{n}^i\,\Delta S^i \;=\; \sum_{i=1}^{N}\oint_{C^i}\underline{A}\cdot\underline{dr} + \sum_{i=1}^{N}\epsilon^i\,\Delta S^i$$

where C^i is the curve enclosing the area ΔS^i, and the quantity $\epsilon^i \to 0$ as $\Delta S^i \to 0$.

Since each small closed curve C^i is traversed in the same sense, then, from the diagram, all contributions to $\sum_{i=1}^{N}\oint_{C^i}\underline{A}\cdot\underline{dr}$ **cancel**, except on those curves where part of C^i lies on the curve C. For example, the line integrals along the common sections of the two small closed curves C^1 and C^2 **cancel exactly**. Therefore

$$\sum_{i=1}^{N}\oint_{C^i}\underline{A}\cdot\underline{dr} \;=\; \oint_C \underline{A}\cdot\underline{dr}$$

Hence

$$\oint_C \underline{A}\cdot\underline{dr} \;=\; \int_S(\underline{\nabla}\times\underline{A})\cdot\underline{dS} \;=\; \int_S \underline{n}\cdot(\underline{\nabla}\times\underline{A})\;dS$$

Mathematical Note: For those worried about how to analyse 'the error term', note that for finite N, we can put an upper bound

$$\sum_{i=1}^{N} \epsilon^i \, \Delta S^i \;\le\; S \max_i \left\{\epsilon^i\right\}$$

This tends to zero in the limit $N \to \infty$, because $\epsilon^i \to 0$ and S is finite.

22. 2. Physical Applications of Stokes' Theorem

In lecture 17 it was stated that if a vector field is irrotational (curl vanishes) then a line integral is independent of path. We can now prove this statement using Stokes' theorem.

Proof

Let $\underline{\nabla} \times \underline{A}(\underline{r}) \;=\; 0$ in R, and consider the **difference** of two line integrals from the point $\underline{r}_0$ to the point $\underline{r}$ along the two curves C_1 and C_2 as shown:

$$\int_{C_1} \underline{A}(\underline{r}') \cdot \underline{dr}' \;-\; \int_{C_2} \underline{A}(\underline{r}') \cdot \underline{dr}'$$

We use $\underline{r}'$ as integration variable to distinguish it from the **limits** of integration $\underline{r}_0$ and $\underline{r}$.

We can rewrite this as the integral around the **closed** curve $C = C_1 - C_2$:

$$\begin{aligned}
\int_{C_1} \underline{A}(\underline{r}') \cdot \underline{dr}' - \int_{C_2} \underline{A}(\underline{r}') \cdot \underline{dr}' \;&=\; \oint_C \underline{A}(\underline{r}') \cdot \underline{dr}' \\
&=\; \int_S \underline{\nabla} \times \underline{A} \cdot \underline{dS} \;=\; 0
\end{aligned}$$

In the above, we have used Stokes' theorem to write the *line* integral of $\underline{A}$ around the closed curve $C = C_1 - C_2$, as the *surface* integral of $\underline{\nabla} \times \underline{A}$ over an open surface S bounded by C. This integral is zero because $\underline{\nabla} \times \underline{A} = 0$ everywhere in R. Hence

$$\underline{\nabla} \times \underline{A}(\underline{r}) \;=\; 0 \quad \Rightarrow \quad \oint_C \underline{A}(\underline{r}') \cdot \underline{dr}' \;=\; 0$$

for *any* closed curve C in R as claimed.

Clearly, the converse is also true i.e. if the line integral between two points is path independent then the line integral around any closed curve (connecting the two points) is zero. Therefore

$$0 = \oint_C \underline{A}(\underline{r}') \cdot \underline{dr}' = \int_S \underline{\nabla} \times \underline{A} \cdot dS$$

where we have used Stokes' theorem and since this holds for any S the field must be irrotational.

Ampère's Law

In Physics 2 you will have met the integral form of Ampère's law, which describes the magnetic field $\underline{B}$ produced by a steady current $\underline{J}$:

$$\oint_C \underline{B} \cdot \underline{dr} = \mu_0 \int_S \underline{J} \cdot \underline{dS}$$

where the closed curve C bounds the surface S *i.e.* the rhs is the current flux across S. We can rewrite the lhs using Stokes' theorem to obtain

$$\int_S (\underline{\nabla} \times \underline{B}) \cdot \underline{dS} = \mu_0 \int_S \underline{J} \cdot \underline{dS}\,.$$

Since this holds for any surface S we must have

$$\underline{\nabla} \times \underline{B} - \mu_0\, \underline{J} = 0$$

which is the differential form of Ampère's law and is one of Maxwell's equations (see next year).

Planar Areas

Consider a planar surface in the $\underline{e}_1$–$\underline{e}_2$ plane and the vector field

$$\underline{A} = \frac{1}{2}\left[-y\underline{e}_1 + x\underline{e}_2\right]\;.$$

We find $\underline{\nabla} \times \underline{A} = \underline{e}_3$. Since a vector element of area normal to a planar surface in the $\underline{e}_1$–$\underline{e}_2$ plane is $\underline{dS} = dS\,\underline{e}_3$ we can obtain the area in the following way

$$\int_S \underline{\nabla} \times \underline{A} \cdot \underline{dS} = \int_S \underline{e}_3 \cdot \underline{dS} = \int_S dS = S$$

Now we can use Stokes' theorem to find

$$\begin{aligned} S \;&=\; \oint_C \underline{A} \cdot \underline{dr} = \frac{1}{2} \oint_C (-y\underline{e}_1 + x\underline{e}_2) \cdot (\underline{e}_1 dx + \underline{e}_2 dy) \\ &=\; \frac{1}{2} \oint_C (x\, dy - y\, dx) \end{aligned}$$

where C is the closed curve bounding the surface.

e.g. To find the area inside the curve

$$x^{2/3} + y^{2/3} = 1$$

use the substitution $x = \cos^3\phi$, $y = \sin^3\phi$, $0 \le \phi \le 2\pi$ then

$$\frac{dx}{d\phi} = -3\cos^2\phi\,\sin\phi \quad ; \quad \frac{dy}{d\phi} = 3\sin^2\phi\,\cos\phi$$

and we obtain

$$\begin{aligned} S &= \frac{1}{2}\oint_C \left(x\,\frac{dy}{d\phi} - y\,\frac{dx}{d\phi} \right) d\phi \\ &= \frac{1}{2}\int_0^{2\pi} \left(3\cos^4\phi\sin^2\phi + 3\sin^4\phi\cos^2\phi\right) d\phi \\ &= \frac{3}{2}\int_0^{2\pi} \sin^2\phi\cos^2\phi\; d\phi = \frac{3}{8}\int_0^{2\pi} \sin^2 2\phi\, d\phi = \frac{3\pi}{8} \end{aligned}$$

22. 3. Example on joint use of Divergence and Stokes' Theorems

Example: show that $\underline{\nabla} \cdot \underline{\nabla} \times \underline{A} \equiv 0$ independent of co-ordinate system:

Let S be a closed surface, enclosing a volume V. Applying the divergence theorem to $\underline{\nabla} \times \underline{A}$, we obtain

$$\int_V \underline{\nabla} \cdot (\underline{\nabla} \times \underline{A})\; dV \;=\; \int_S (\underline{\nabla} \times \underline{A}) \cdot \underline{dS}$$

Now divide S into two surfaces S_1 and S_2 with a **common** boundary C as shown below

Now use Stokes' theorem to write

$$\int_S (\underline{\nabla} \times \underline{A}) \cdot \underline{dS} \;=\; \int_{S_1} (\underline{\nabla} \times \underline{A}) \cdot \underline{dS} + \int_{S_2} (\underline{\nabla} \times \underline{A}) \cdot \underline{dS} \;=\; \oint_C \underline{A} \cdot \underline{dr} - \oint_C \underline{A} \cdot \underline{dr} \;=\; 0$$

where the second line integral appears with a minus sign because it is traversed in the **opposite** direction. (Recall that Stokes' theorem applies to curves traversed in the right hand sense with respect to the outward normal of the surface.)

Since this result holds for arbitrary volumes, we must have

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{A} \equiv 0$$