Lecture 23: Curvilinear Coordinates (RHB 8.10)

It is often convenient to work with variables other than the Cartesian coordinates x_i ($= x$, y, z). For example in Lecture 15 we met spherical polar and cylindrical polar coordinates. These are two important examples of what are called **curvilinear coordinates**. In this lecture we set up a formalism to deal with these rather general coordinate systems.

Actually we have effectively covered much of the materical in lectures 18 and 19 where we studied surface and volume integrals. There we considered parameterisation of surfaces and volumes. Here we do the same thing but think about how this realises non-Cartesian Coordinate systems.

Suppose we change from the Cartesian coordinates (x_1, x_2, x_3) to the curvilinear coordinates, which we denote u_i, each of which are functions of the x_i:-

$$\begin{aligned} u_1 &= u_1(x_1,\, x_2,\, x_3) \\ u_2 &= u_2(x_1,\, x_2,\, x_3) \\ u_3 &= u_3(x_1,\, x_2,\, x_3) \end{aligned}$$

The u_i should be single-valued, except possibly at certain points, so the reverse transformation,

$$x_i = x_i(u_1,\, u_2,\, u_3)$$

can be made. A point may be referred to by its Cartesian coordinates x_i, or by its curvilinear coordinates u_i. For example, in 2-D, we might have:-

Now consider **coordinate surfaces** defined by keeping one coordinate constant.

- The Cartesian coordinate surfaces 'x_i = constant' are **planes**, with **constant** unit normal vectors $\underline{e}_i$ (or $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$), intersecting at right angles.
- The surfaces 'u_i = constant' do **not**, in general, have constant unit normal vectors, nor in general do they intersect at right angles.

Example: Spherical polar co-ordinates

$$r \;=\; \sqrt{x^2+y^2+z^2} \quad ; \quad \theta \;=\; \cos^{-1}\left\{\frac{z}{\sqrt{x^2+y^2+z^2}}\right\} \quad ; \quad \phi \;=\; \tan^{-1}\left(\frac{y}{x}\right) \; .$$

The surfaces of constant r, θ, and ϕ are:-

r = constant $\Longrightarrow$ spheres centred at the origin   unit normal $\underline{e}_r$

θ = constant $\Longrightarrow$ cones of semi-angle θ and axis along the z-axis   unit normal $\underline{e}_\theta$

ϕ = constant $\Longrightarrow$ planes passing through the z-axis   unit normal $\underline{e}_\phi$

These surfaces are **not** all planes, but they **do** intersect at right angles.

If the coordinate surfaces intersect at right angles (*i.e. the unit normals intersect at right angles*), as in the example of spherical polars, the curvilinear coordinates are said to be **orthogonal**.

23. 1. Orthogonal Curvilinear Coordinates

Unit Vectors and Scale Factors

Suppose the point P has position $\underline{r} = \underline{r}(u_1, u_2, u_3)$. If we change u_1 by a small amount, du_1, then $\underline{r}$ moves to position $(\underline{r} + \underline{dr}\,)$, where

$$\underline{dr} \;=\; \frac{\partial \underline{r}}{\partial u_1}\, du_1 \;\equiv\; h_1\, \underline{e}_1\, du_1$$

where we have defined the **unit vector** $\underline{e}_1$ and the **scale factor** h_1 by

$$\boxed{\; h_1 \;=\; \left|\frac{\partial \underline{r}}{\partial u_1}\right| \quad \text{and} \quad \underline{e}_1 \;=\; \frac{1}{h_1}\,\frac{\partial \underline{r}}{\partial u_1}\; . \;}$$

- The vector $\underline{e}_1$ is a unit vector in the **direction of increasing** u_1.
- The scale factor h_1 gives the **magnitude** of $\underline{dr}$ when we make the change $u_1 \to u_1 + du_1$. Thus for an infinitessimal change of u_1

$$|\underline{dr}| = h_1\, du_1$$

Similarly, we can define h_i and $\underline{e}_i$ for $i = 2$ and 3.

- The unit vectors $\underline{e}_i$ are **not constant vectors**. In general they are **non-Cartesian** basis vectors, they depend on the position vector $\underline{r}$, *i.e.* their directions change as the u_i are varied.
- If $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$, then the u_i are **orthogonal curvilinear coordinates**.

For Cartesian coordinates, the scale factors are unity and the unit vectors $\underline{e}_i$ reduce to the Cartesian basis vectors we have used throughout the course:

$$\underline{r} \;=\; x\,\underline{e}_1 \,+\, y\,\underline{e}_2 \,+\, z\,\underline{e}_3 \qquad \text{so that} \qquad h_1\,\underline{e}_1 \;=\; \frac{\partial \underline{r}}{\partial x} \;=\; \underline{e}_1, \quad \text{etc.}$$

Example: spherical polars: $u_1 = r$, $u_2 = \theta$ and $u_3 = \phi$ in that order:

$$\underline{r} \;=\; r\,\sin\theta\,\cos\phi\,\underline{i} \,+\, r\,\sin\theta\,\sin\phi\,\underline{j} + r\,\cos\theta\,\underline{k},$$

where to avoid confusion in this section we use $\underline{i}, \underline{j}, \underline{k}$ for the *Cartesian* basis vectors. (By this stage there should be no confusion with the suffices i, j, k). Therefore

$$\begin{aligned}
\frac{\partial \underline{r}}{\partial r} &= \sin\theta\,\cos\phi\,\underline{i} + \sin\theta\,\sin\phi\,\underline{j} + \cos\theta\,\underline{k} &&\Longrightarrow\; h_r \;=\; 1\\
\frac{\partial \underline{r}}{\partial \theta} &= r\,\cos\theta\,\cos\phi\,\underline{i} + r\,\cos\theta\,\sin\phi\,\underline{j} - r\,\sin\theta\,\underline{k} &&\Longrightarrow\; h_\theta \;=\; r\\
\frac{\partial \underline{r}}{\partial \phi} &= -r\,\sin\theta\,\sin\phi\,\underline{i} + r\,\sin\theta\,\cos\phi\,\underline{j} &&\Longrightarrow\; h_\phi \;=\; r\,\sin\theta
\end{aligned}$$

Thus

$$\begin{aligned}
\underline{e}_r &= \sin\theta\cos\phi\,\underline{i} + \sin\theta\sin\phi\,\underline{j} + \cos\theta\,\underline{k} \;=\; \underline{r}\,/\,r\\
\underline{e}_\theta &= \cos\theta\cos\phi\,\underline{i} + \cos\theta\sin\phi\,\underline{j} - \sin\theta\,\underline{k}\\
\underline{e}_\phi &= -\sin\phi\,\underline{i} + \cos\phi\,\underline{j}
\end{aligned}$$

These unit vectors are normal to the level surfaces described above (spheres, cones and planes) and are clearly orthogonal: $\underline{e}_r \cdot \underline{e}_\theta \;=\; \underline{e}_r \cdot \underline{e}_\phi \;=\; \underline{e}_\theta \cdot \underline{e}_\phi \;=\; 0$
and form a RH orthonormal basis: $\underline{e}_r \times \underline{e}_\theta \;=\; \underline{e}_\phi,\;\; \underline{e}_\theta \times \underline{e}_\phi \;=\; \underline{e}_r,\;\; \underline{e}_\phi \times \underline{e}_r \;=\; \underline{e}_\theta$.

Example: Cylindrical polars: $u_1 = \rho$, $u_2 = \phi$ and $u_3 = z$ in that order:

$$\underline{r} \;=\; \rho\,\cos\phi\,\underline{i} \,+\, \rho\,\sin\phi\,\underline{j} + z\,\underline{k}\,.$$

Thus
$$\frac{\partial \underline{r}}{\partial \rho} \;=\; \cos\phi\,\underline{i} + \sin\phi\,\underline{j} \quad ; \quad \frac{\partial \underline{r}}{\partial \phi} \;=\; -\rho\,\sin\phi\,\underline{i} + \rho\,\cos\phi\,\underline{j} \quad ; \quad \frac{\partial \underline{r}}{\partial z} \;=\; \underline{k}$$

and
$$h_\rho \;=\; 1 \quad ; \quad h_\phi \;=\; \rho \quad ; \quad h_z \;=\; 1$$

Therefore
$$\underline{e}_\rho = \cos\phi\,\underline{i} + \sin\phi\,\underline{j} \quad ; \quad \underline{e}_\phi = -\sin\phi\,\underline{i} + \cos\phi\,\underline{j} \quad ; \quad \underline{e}_z = \underline{k}$$

These unit vectors are normal to the level surfaces described by cylinders about the z-axis ($\rho =$ constant), planes through the z-axis ($\phi =$ constant), planes perpendicular to the z axis ($z =$ constant) and are clearly orthogonal.

Remark: An example of a curvilinear coordinate system which is **not** orthogonal is provided by the system of elliptical cylindrical coordinates (see tutuorial 9.4).

$$\underline{r} \;=\; a\,\rho\,\cos\theta\;\underline{i} \,+\, b\,\rho\,\sin\theta\;\underline{j} \,+\, z\;\underline{k} \qquad (a \;\neq\; b)$$

In the following we shall only consider orthogonal systems

Arc Length

The arc length ds is the length of the infinitesimal vector $\underline{dr}$:-

$$(ds)^2 = \underline{dr} \cdot \underline{dr}\ .$$

In Cartesian coordinates

$$(ds)^2 \;=\; (dx)^2 \,+\, (dy)^2 \,+\, (dz)^2\,.$$

In curvilinear coordinates, if we change all three coordinates u_i by infinitesimal amounts du_i, then we have

$$\begin{aligned} \underline{dr} \;&=\; \frac{\partial \underline{r}}{\partial u_1}\,du_1 \,+\, \frac{\partial \underline{r}}{\partial u_2}\,du_2 \,+\, \frac{\partial \underline{r}}{\partial u_3}\,du_3 \\ &=\; h_1\,du_1\,\underline{e}_1 \,+\, h_2\,du_2\,\underline{e}_2 \,+\, h_3\,du_3\,\underline{e}_3 \end{aligned}$$

For the case of orthogonal curvilinears, because the basis vectors are orthonormal we have

$$(ds)^2 \;=\; h_1^2\,du_1^2 \,+\, h_2^2\,du_2^2 \,+\, h_3^2\,du_3^2$$

For spherical polars, we showed that

$$h_r \;=\; 1, \qquad h_\theta \;=\; r, \qquad \text{and} \qquad h_\phi \;=\; r\,\sin\theta$$

therefore

$$(ds)^2 \;=\; (dr)^2 + r^2\;(d\theta)^2 + r^2\;\sin^2\theta\;(d\phi)^2$$

Lecture 23 continued:

23. 2. Elements of Area and Volume

Basically we just repeat using scale factors what we did in lectures 18 and 19.

Vector Area

If $u_1 \to u_1 + du_1$, then $\underline{r} \to \underline{r} + \underline{dr}_1$, where $\underline{dr}_1 = h_1\, \underline{e}_1\, du_1$, and if $u_2 \to u_2 + du_2$, then $\underline{r} \to \underline{r} + \underline{dr}_2$, where $\underline{dr}_2 = h_2\, \underline{e}_2\, du_2$. (Curvature greatly exaggerated in figure!)

On the surface of constant u_1 the **vector area** bounded by $\underline{dr}_2$ and $\underline{dr}_3$ is given by

$$\underline{dS}_1 \;=\; (\underline{dr}_2) \times (\underline{dr}_3) \;=\; (h_2\, du_2\, \underline{e}_2) \times (h_3\, du_3\, \underline{e}_3) \;=\; h_2\, h_3\, du_2\, du_3\, \underline{e}_1 \;,$$

since $\underline{e}_2 \times \underline{e}_3 = \underline{e}_1$ for orthogonal systems.

Thus $\underline{dS}_1$ is a vector pointing in the direction of the **normal** to the surfaces 'u_1 =constant', its magnitude being the area of the small parallelogram with edges $\underline{dr}_2$ and $\underline{dr}_3$. Similarly, one can define $\underline{dS}_2$ and $\underline{dS}_3$.

For the case of **spherical polars**, if we vary θ and ϕ, keeping r fixed, then

$$\underline{dS}_r \;=\; \left(h_\theta\, d\theta\, \underline{e}_\theta\right) \times \left(h_\phi\, d\phi\, \underline{e}_\phi\right) \;=\; h_\theta\, h_\phi\, d\theta\, d\phi\, \underline{e}_r \;=\; r^2\, \sin\theta\, d\theta\, d\phi\, \underline{e}_r \;.$$

Similarly for $\underline{dS}_\theta$ and $\underline{dS}_\phi$.

Volume

The volume contained in the parallelepiped with edges $\underline{dr}_1$, $\underline{dr}_2$ and $\underline{dr}_3$, is

$$\begin{aligned} dV &= \underline{dr}_1 \cdot \underline{dr}_2 \times \underline{dr}_3 \\ &= (h_1\, du_1\, \underline{e}_1) \cdot (h_2\, du_2\, \underline{e}_2) \times (h_3\, du_3\, \underline{e}_3) \\ &= h_1\, h_2\, h_3\, du_1\, du_2\, du_3 \end{aligned}$$

because $\underline{e}_1 \cdot \underline{e}_2 \times \underline{e}_3 = 1$.

For **spherical polars**, we have

$$dV \;=\; h_r\, h_\theta\, h_\phi\, dr\, d\theta\, d\phi \;=\; r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Components of a Vector Field in Curvilinear Coordinates
A vector field $\underline{A}(\underline{r})$ can be expressed in terms of **curvilinear components** A_i, defined as:–

$$\underline{A}(\underline{r}) \;=\; \sum_i A_i(u_1, u_2, u_3)\, \underline{e}_i$$

where $\underline{e}_i$ is the ith basis vector for the curvilinear coordinate system.

For **orthogonal** curvilinear coordinates, the component A_i is obtained by taking the scalar product of $\underline{A}$ with the ith (curvilinear) basis vector $\underline{e}_i$

$$\boxed{A_i = \underline{e}_i \cdot \underline{A}(\underline{r})}$$

NB A_i must be expressed in terms of u_i (not x, y, z) when working in the u_i basis.

Example If $\underline{A} = \underline{i}$ in Cartesian coordinates, then in spherical polars,$A_r = \underline{A}\cdot\underline{e}_r = \sin\theta\cos\phi$, etc.

$$\underline{A}(r,\theta,\phi) \;=\; \sin\theta\cos\phi\,\underline{e}_r \;+\; \cos\theta\cos\phi\,\underline{e}_\theta \;-\; \sin\phi\,\underline{e}_\phi\ .$$

Example If $\underline{A} = \underline{r}$ then in cylindrical polars $A_\rho = \underline{A}\cdot\underline{e}_\rho = \rho\cos^2\phi + \rho\sin^2\phi = \rho$ etc, and

$$\underline{A}(\rho,\phi,z) \;=\; \underline{r} \;=\; \rho\,\underline{e}_\rho \;+\; z\,\underline{e}_z$$

23. 3. Grad, Div, Curl, and the Laplacian in Orthogonal Curvilinears

We defined the vector operators grad, div, curl firstly in Cartesian coordinates, then most generally through integral definitions without regard to a coordinate system. Here we complete the picture by providing the definitions in any orthogonal curvilinear coordinate system.

Gradient
In section (2) we defined the gradient in terms of the change in a scalar field f when we let $\underline{r} \to \underline{r} + \underline{dr}$

$$\delta f \;=\; \underline{\nabla}\, f(\underline{r}) \cdot \underline{dr} \qquad (1)$$

Now consider what happens when we write f in terms of orthogonal curvilinear coordinates $f = f(u_1,\, u_2,\, u_3)$. As before, we denote the curvilinear basis vectors by $\underline{e}_1$, $\underline{e}_2$, and $\underline{e}_3$.

Let $u_1 \to u_1 + du_1,\ \ u_2 \to u_2 + du_2,\ $ and $u_3 \to u_3 + du_3$.
By Taylor's theorem, we have

$$\delta f \;=\; \frac{\partial f}{\partial u_1}\, du_1 \;+\; \frac{\partial f}{\partial u_2}\, du_2 \;+\; \frac{\partial f}{\partial u_3}\, du_3$$

We have already shown that

$$\underline{dr} \;=\; h_1\, du_1\, \underline{e}_1 \;+\; h_2\, du_2\, \underline{e}_2 \;+\; h_3\, du_3\, \underline{e}_3\ .$$

Using the orthogonality of the basis vectors, $\underline{e}_i \cdot \underline{e}_j \; = \; \delta_{ij}$, we can write

$$\begin{aligned}
\delta f \;&=\; \left(\frac{\partial f}{\partial u_1}\,\underline{e}_1 \,+\, \frac{\partial f}{\partial u_2}\,\underline{e}_2 \,+\, \frac{\partial f}{\partial u_3}\,\underline{e}_3\right)\cdot\left(\underline{e}_1\,du_1 \,+\, \underline{e}_2\,du_2 \,+\, \underline{e}_3 du_3\right)\\
&=\; \left(\frac{1}{h_1}\frac{\partial f}{\partial u_1}\,\underline{e}_1 \,+\, \frac{1}{h_2}\frac{\partial f}{\partial u_2}\,\underline{e}_2 \,+\, \frac{1}{h_3}\frac{\partial f}{\partial u_3}\,\underline{e}_3\right)\cdot\left(h_1\,\underline{e}_1\,du_1 \,+\, h_2\,\underline{e}_2\,du_2 \,+\, h_3\,\underline{e}_3\,du_3\right)\\
&=\; \left(\frac{1}{h_1}\frac{\partial f}{\partial u_1}\,\underline{e}_1 \,+\, \frac{1}{h_2}\frac{\partial f}{\partial u_2}\,\underline{e}_2 \,+\, \frac{1}{h_3}\frac{\partial f}{\partial u_3}\,\underline{e}_3\right)\cdot\underline{dr}
\end{aligned}$$

Comparing this result with equation (1) above, we obtain the following expression for $\underline{\nabla}\, f$ in orthogonal curvilinears

$$\boxed{\begin{aligned}
\underline{\nabla}\, f \;&=\; \frac{1}{h_1}\frac{\partial f}{\partial u_1}\,\underline{e}_1 \,+\, \frac{1}{h_2}\frac{\partial f}{\partial u_2}\,\underline{e}_2 \,+\, \frac{1}{h_3}\frac{\partial f}{\partial u_3}\,\underline{e}_3\\
&=\; \sum_{i=1}^{3}\frac{1}{h_i}\frac{\partial f}{\partial u_i}\,\underline{e}_i
\end{aligned}}$$

For **spherical polars**, we obtain

$$\underline{\nabla}\, f(r,\theta,\phi) \;=\; \underline{e}_r\,\frac{\partial f}{\partial r} \,+\, \underline{e}_\theta\,\frac{1}{r}\frac{\partial f}{\partial \theta} \,+\, \underline{e}_\phi\,\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\;.$$

For **cylindrical polars**, we obtain

$$\underline{\nabla}\, f(\rho,\phi,z) \;=\; \underline{e}_\rho\,\frac{\partial f}{\partial \rho} \,+\, \underline{e}_\phi\,\frac{1}{\rho}\frac{\partial f}{\partial \phi} \,+\, \underline{e}_z\,\frac{\partial f}{\partial z}\;.$$

Divergence
In orthogonal curvilinear coordinates

$$\boxed{\mathrm{div}\underline{A} \;=\; \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}(A_1h_2h_3) \,+\, \frac{\partial}{\partial u_2}(A_2h_3h_1) \,+\, \frac{\partial}{\partial u_3}(A_3h_1h_2)\right\}}$$

This expression can be obtained by using the integral definition of div$\underline{A}$, or alternatively using vector operator identities (see BK 4.13, RHB 8.10).

For Cartesian coordinates, we have $h_i = 1$, and we regain the usual expression for $\underline{\nabla}\cdot\underline{A}$ in Cartesians.

For **spherical polars** we have

$$\begin{aligned}
\mathrm{div}\underline{A}(r,\theta,\phi) \;&=\; \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}\left(r^2\sin\theta A_r\right) \,+\, \frac{\partial}{\partial\theta}\left(r\sin\theta A_\theta\right) \,+\, \frac{\partial}{\partial\phi}\left(rA_\phi\right)\right\}\\
&=\; \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right) \,+\, \frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) \,+\, \frac{\partial}{\partial\phi}\left(A_\phi\right)\right\}\;.
\end{aligned}$$

where A_r, A_θ, and A_ϕ are the components of the vector field $\underline{A}$ in the basis $(\underline{e}_r,\ \underline{e}_\theta,\ \underline{e}_\phi)$.

For **cylindrical polars** we have

$$\begin{aligned}\mathrm{div}\underline{A}(\rho,\phi,z) &= \frac{1}{\rho}\left\{\frac{\partial}{\partial\rho}\Big(\rho A_\rho\Big) + \frac{\partial}{\partial\phi}A_\phi + \frac{\partial}{\partial z}\left(\rho A_z\right)\right\} \\ &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\Big(\rho A_\rho\Big) + \frac{1}{\rho}\frac{\partial}{\partial\phi}A_\phi + \frac{\partial}{\partial z}A_z\ .\end{aligned}$$

where A_ρ, A_ϕ, and A_z are the components of the vector field $\underline{A}$ in the basis $(\underline{e}_\rho,\ \underline{e}_\phi,\ \underline{e}_z)$.

Curl

In orthogonal curvilinear co-ordinates, curl is most conveniently written as

$$\boxed{\underline{\nabla}\times\underline{A} = \frac{1}{h_1\,h_2\,h_3}\begin{vmatrix} h_1\,\underline{e}_1 & h_2\,\underline{e}_2 & h_3\,\underline{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1\,A_1 & h_2\,A_2 & h_3\,A_3 \end{vmatrix}}$$

e.g. the first component is given by

$$\underline{e}_1\cdot\underline{\nabla}\times\underline{A} = \frac{1}{h_2\,h_3}\left\{\frac{\partial}{\partial u_2}(A_3h_3) - \frac{\partial}{\partial u_3}(A_2h_2)\right\}$$

and the components of $\underline{\nabla}\times\underline{A}$ in the $\underline{e}_2$ and $\underline{e}_3$ directions may be obtained by cyclic permutations of the suffices.

The above formula can be demonstrated by using the line integral definition of curl, as used to proof Stokes' theorem, (see tutorial) or by vector operator identities (BK. 4.13 or RHB 8.10).

For **spherical polars** we have

$$\underline{\nabla}\times\underline{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \underline{e}_r & r\,\underline{e}_\theta & r\,\sin\theta\,\underline{e}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta\,A_\phi \end{vmatrix}$$

Laplacian of a Scalar Field

The Laplacian operator acting on a *scalar* field is defined by $\nabla^2 f = \underline{\nabla}\cdot(\underline{\nabla}\,f)$, giving:–

$$\boxed{\nabla^2 f = \frac{1}{h_1\,h_2\,h_3}\left\{\frac{\partial}{\partial u_1}\left(\frac{h_2\,h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3\,h_1}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1\,h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right\}}$$

In **spherical polars**, we have

$$\begin{aligned}\nabla^2 f(r,\theta,\phi) &= \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial\phi}\right)\right\} \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\left\{\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{\partial^2 f}{\partial\phi^2}\right\}\ .\end{aligned}$$