Lecture 23: Curvilinear Coordinates (RHB 8.10)

It is often convenient to work with variables other than the Cartesian coordinates \( x, y, z \). For example in Lecture 15 we met spherical polar and cylindrical polar coordinates. These are two important examples of what are called curvilinear coordinates. In this lecture we set up a formalism to deal with these rather general coordinate systems.

Actually we have effectively covered much of the material in lectures 18 and 19 where we studied surface and volume integrals. There we considered parameterisation of surfaces and volumes. Here we do the same thing but think about how this realises non-Cartesian coordinate systems.

Suppose we change from the Cartesian coordinates \((x_1, x_2, x_3)\) to the curvilinear coordinates, which we denote \( u_i \), each of which are functions of the \( x_i \):

\[
\begin{align*}
  u_1 &= u_1(x_1, x_2, x_3) \\
  u_2 &= u_2(x_1, x_2, x_3) \\
  u_3 &= u_3(x_1, x_2, x_3)
\end{align*}
\]

The \( u_i \) should be single-valued, except possibly at certain points, so the reverse transformation,

\[
x_i = x_i(u_1, u_2, u_3)
\]

can be made. A point may be referred to by its Cartesian coordinates \( x_i \), or by its curvilinear coordinates \( u_i \). For example, in 2-D, we might have:

Now consider coordinate surfaces defined by keeping one coordinate constant.

- The Cartesian coordinate surfaces ‘\( x_i = \text{constant} \)’ are planes, with constant unit normal vectors \( \mathbf{e}_i \) (or \( \mathbf{e}_1, \mathbf{e}_2 \), and \( \mathbf{e}_3 \)), intersecting at right angles.
- The surfaces ‘\( u_i = \text{constant} \)’ do not, in general, have constant unit normal vectors, nor in general do they intersect at right angles.
Example: Spherical polar co-ordinates

\[ r = \sqrt{x^2 + y^2 + z^2} \quad ; \quad \theta = \cos^{-1} \left( \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right) \quad ; \quad \phi = \tan^{-1} \left( \frac{y}{x} \right). \]

The surfaces of constant \( r, \theta, \) and \( \phi \) are:

- \( r = \) constant \( \implies \) spheres centred at the origin \( \text{unit normal } \vec{e}_r \)
- \( \theta = \) constant \( \implies \) cones of semi-angle \( \theta \) and axis along the \( z \)-axis \( \text{unit normal } \vec{e}_\theta \)
- \( \phi = \) constant \( \implies \) planes passing through the \( z \)-axis \( \text{unit normal } \vec{e}_\phi \)

These surfaces are not all planes, but they do intersect at right angles.

If the coordinate surfaces intersect at right angles (i.e. the unit normals intersect at right angles), as in the example of spherical polars, the curvilinear coordinates are said to be orthogonal.

23. 1. Orthogonal Curvilinear Coordinates

Unit Vectors and Scale Factors

Suppose the point \( P \) has position \( r = r(u_1, u_2, u_3) \). If we change \( u_1 \) by a small amount, \( du_1 \), then \( r \) moves to position \( (r + dr) \), where

\[ dr = \frac{\partial r}{\partial u_1} du_1 \equiv h_1 \vec{e}_1 du_1 \]

where we have defined the unit vector \( \vec{e}_1 \) and the scale factor \( h_1 \) by

\[ h_1 = \left| \frac{\partial r}{\partial u_1} \right| \quad \text{and} \quad \vec{e}_1 = \frac{1}{h_1} \frac{\partial r}{\partial u_1}. \]

- The vector \( \vec{e}_1 \) is a unit vector in the direction of increasing \( u_1 \).
- The scale factor \( h_1 \) gives the magnitude of \( dr \) when we make the change \( u_1 \to u_1 + du_1 \). Thus for an infinitesimal change of \( u_1 \)

\[ |dr| = h_1 du_1 \]
Similarly, we can define $h_i$ and $e_i$ for $i = 2$ and $3$.

- The unit vectors $e_i$ are not constant vectors. In general they are non-Cartesian basis vectors, they depend on the position vector $r$, i.e. their directions change as the $u_i$ are varied.
- If $e_i \cdot e_j = \delta_{i j}$, then the $u_i$ are orthogonal curvilinear coordinates.

For Cartesian coordinates, the scale factors are unity and the unit vectors $e_i$ reduce to the Cartesian basis vectors we have used throughout the course:

$$
\frac{\partial r}{\partial x} = e_1, \quad \frac{\partial r}{\partial y} = e_2, \quad \frac{\partial r}{\partial z} = e_3
$$

Example: spherical polars: $u_1 = r$, $u_2 = \theta$ and $u_3 = \phi$ in that order:

$$
r = r \sin \theta \cos \phi \hat{i} + r \sin \theta \sin \phi \hat{j} + r \cos \theta \hat{k},
$$

where to avoid confusion in this section we use $i, j, k$ for the Cartesian basis vectors. (By this stage there should be no confusion with the suffices $i, j, k$). Therefore

$$
\frac{\partial r}{\partial \rho} = \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k} \quad \Rightarrow \quad h_\rho = 1 \\
\frac{\partial r}{\partial \theta} = r \cos \theta \cos \phi \hat{i} + r \cos \theta \sin \phi \hat{j} - r \sin \theta \hat{k} \quad \Rightarrow \quad h_\theta = r \\
\frac{\partial r}{\partial \phi} = -r \sin \theta \sin \phi \hat{i} + r \sin \theta \cos \phi \hat{j} \quad \Rightarrow \quad h_\phi = r \sin \theta
$$

Thus

$$
e_\rho = \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k} = \frac{r}{r} \\
e_\theta = \cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k} \\
e_\phi = -\sin \phi \hat{i} + \cos \phi \hat{j}
$$

These unit vectors are normal to the level surfaces described above (spheres, cones and planes) and are clearly orthogonal: $e_\rho \cdot e_\theta = e_\rho \cdot e_\phi = e_\theta \cdot e_\phi = 0$

and form a RH orthonormal basis: $e_\rho \times e_\theta = e_\phi, \; e_\theta \times e_\phi = e_\rho, \; e_\phi \times e_\rho = e_\theta$.

Example: Cylindrical polars: $u_1 = \rho$, $u_2 = \phi$ and $u_3 = z$ in that order:

$$
r = \rho \cos \phi \hat{i} + \rho \sin \phi \hat{j} + z \hat{k}.
$$

Thus

$$
\frac{\partial r}{\partial \rho} = \cos \phi \hat{i} + \sin \phi \hat{j} \quad ; \quad \frac{\partial r}{\partial \phi} = -\rho \sin \phi \hat{i} + \rho \cos \phi \hat{j} \quad ; \quad \frac{\partial r}{\partial z} = \hat{k}
$$

and

$$
h_\rho = 1 \quad ; \quad h_\phi = \rho \quad ; \quad h_z = 1
$$

Therefore

$$
e_\rho = \cos \phi \hat{i} + \sin \phi \hat{j} \quad ; \quad e_\phi = -\sin \phi \hat{i} + \cos \phi \hat{j} \quad ; \quad e_z = \hat{k}
$$
These unit vectors are normal to the level surfaces described by cylinders about the $z$-axis ($\rho = \text{constant}$), planes through the $z$-axis ($\phi = \text{constant}$), planes perpendicular to the $z$ axis ($z = \text{constant}$) and are clearly orthogonal.

**Remark:** An example of a curvilinear coordinate system which is **not** orthogonal is provided by the system of elliptical cylindrical coordinates (see tutorial 9.4).

$$\mathbf{r} = a \rho \cos \theta \hat{i} + b \rho \sin \theta \hat{j} + z \hat{k} \quad (a \neq b)$$

In the following we shall only consider orthogonal systems

**Arc Length**

The arc length $ds$ is the length of the infinitesimal vector $d\mathbf{r}$: $$(ds)^2 = d\mathbf{r} \cdot d\mathbf{r}.$$ 

In Cartesian coordinates $$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2.$$ 

In curvilinear coordinates, if we change all three coordinates $u_i$ by infinitesimal amounts $du_i$, then we have

$$d\mathbf{r} = \frac{\partial r}{\partial u_1} du_1 + \frac{\partial r}{\partial u_2} du_2 + \frac{\partial r}{\partial u_3} du_3$$

$$= h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3$$

For the case of orthogonal curvilinears, because the basis vectors are orthonormal we have $$(ds)^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$$

For spherical polars, we showed that $h_r = 1$, $h_\theta = r$, and $h_\phi = r \sin \theta$

therefore $$(ds)^2 = (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2 \theta (d\phi)^2$$
Lecture 23 continued:

23. 2. Elements of Area and Volume

Basically we just repeat using scale factors what we did in lectures 18 and 19.

**Vector Area**

If \( u_1 \rightarrow u_1 + du_1 \), then \( r \rightarrow r + dr_1 \), where \( dr_1 = h_1 e_1 du_1 \), and if \( u_2 \rightarrow u_2 + du_2 \), then \( r \rightarrow r + dr_2 \), where \( dr_2 = h_2 e_2 du_2 \). (Curvature greatly exaggerated in figure!)

On the surface of constant \( u_1 \) the **vector area** bounded by \( dr_2 \) and \( dr_3 \) is given by

\[
dS_1 = (dr_2) \times (dr_3) = (h_2 du_2 e_2) \times (h_3 du_3 e_3) = h_2 h_3 du_2 du_3 e_1,
\]

since \( e_2 \times e_3 = e_1 \) for orthogonal systems.

Thus \( dS_1 \) is a vector pointing in the direction of the **normal** to the surfaces ‘\( u_1 =\)constant’, its magnitude being the area of the small parallelogram with edges \( dr_2 \) and \( dr_3 \). Similarly, one can define \( dS_2 \) and \( dS_3 \).

For the case of **spherical polars**, if we vary \( \theta \) and \( \phi \), keeping \( r \) fixed, then

\[
dS_r = (h_\theta d\theta e_\theta) \times (h_\phi d\phi e_\phi) = h_\theta h_\phi d\theta d\phi e_r = r^2 \sin \theta d\theta d\phi e_r.
\]

Similarly for \( dS_\theta \) and \( dS_\phi \).

**Volume**

The volume contained in the parallelepiped with edges \( dr_1, dr_2 \) and \( dr_3 \), is

\[
dV = dr_1 \cdot dr_2 \times dr_3
\]

\[
= (h_1 du_1 e_1) \cdot (h_2 du_2 e_2) \times (h_3 du_3 e_3)
\]

\[
= h_1 h_2 h_3 du_1 du_2 du_3
\]

because \( e_1 \cdot e_2 \times e_3 = 1 \).

For **spherical polars**, we have

\[
dV = h_r h_\theta h_\phi dr d\theta d\phi = r^2 \sin \theta dr d\theta d\phi
\]
Components of a Vector Field in Curvilinear Coordinates

A vector field \( A(r) \) can be expressed in terms of **curvilinear components** \( A_i \), defined as:

\[
A(r) = \sum_i A_i(u_1, u_2, u_3) \varepsilon_i
\]

where \( \varepsilon_i \) is the \( i \)th basis vector for the curvilinear coordinate system.

For orthogonal curvilinear coordinates, the component \( A_i \) is obtained by taking the scalar product of \( A \) with the \( i \)th (curvilinear) basis vector \( \varepsilon_i \):

\[
A_i = \varepsilon_i \cdot A(r)
\]

**Example** If \( A = \hat{i} \) in Cartesian coordinates, then in spherical polars, \( A_r = A \cdot \varepsilon_r = \sin \theta \cos \phi \), etc.

\[
A(r, \theta, \phi) = \sin \theta \cos \phi \varepsilon_r + \cos \theta \cos \phi \varepsilon_\theta - \sin \phi \varepsilon_\phi.
\]

**Example** If \( A = r \) then in cylindrical polars \( A_\rho = A \cdot \varepsilon_\rho = \rho \cos^2 \phi + \rho \sin^2 \phi = \rho \), etc, and

\[
A(\rho, \phi, z) = r = \rho \varepsilon_\rho + z \varepsilon_z
\]

23. 3. **Grad, Div, Curl, and the Laplacian in Orthogonal Curvilinears**

We defined the vector operators grad, div, curl firstly in Cartesian coordinates, then most generally through integral definitions without regard to a coordinate system. Here we complete the picture by providing the definitions in any orthogonal curvilinear coordinate system.

**Gradient**

In section (2) we defined the gradient in terms of the change in a scalar field \( f \) when we let \( r \to r + dr \):

\[
\delta f = \nabla f(r) \cdot dr
\]

Now consider what happens when we write \( f \) in terms of orthogonal curvilinear coordinates \( f = f(u_1, u_2, u_3) \). As before, we denote the curvilinear basis vectors by \( \varepsilon_1, \varepsilon_2, \) and \( \varepsilon_3 \).

Let \( u_1 \to u_1 + du_1, \ u_2 \to u_2 + du_2, \) and \( u_3 \to u_3 + du_3 \).

By Taylor’s theorem, we have

\[
\delta f = \frac{\partial f}{\partial u_1} du_1 + \frac{\partial f}{\partial u_2} du_2 + \frac{\partial f}{\partial u_3} du_3
\]

We have already shown that

\[
\overrightarrow{dr} = h_1 \, du_1 \, \varepsilon_1 + h_2 \, du_2 \, \varepsilon_2 + h_3 \, du_3 \, \varepsilon_3.
\]
Using the orthogonality of the basis vectors, \( \varepsilon_i \cdot \varepsilon_j = \delta_{ij} \), we can write

\[
\delta f = \left( \frac{\partial f}{\partial u_1} \varepsilon_1 + \frac{\partial f}{\partial u_2} \varepsilon_2 + \frac{\partial f}{\partial u_3} \varepsilon_3 \right) \cdot (\varepsilon_1 du_1 + \varepsilon_2 du_2 + \varepsilon_3 du_3)
\]

\[
= \left( \frac{1}{h_1} \frac{\partial f}{\partial u_1} \varepsilon_1 + \frac{1}{h_2} \frac{\partial f}{\partial u_2} \varepsilon_2 + \frac{1}{h_3} \frac{\partial f}{\partial u_3} \varepsilon_3 \right) \cdot \left( h_1 \varepsilon_1 du_1 + h_2 \varepsilon_2 du_2 + h_3 \varepsilon_3 du_3 \right)
\]

\[
= \left( \frac{1}{h_1} \frac{\partial f}{\partial u_1} \varepsilon_1 + \frac{1}{h_2} \frac{\partial f}{\partial u_2} \varepsilon_2 + \frac{1}{h_3} \frac{\partial f}{\partial u_3} \varepsilon_3 \right) \cdot dr
\]

Comparing this result with equation (1) above, we obtain the following expression for \( \nabla f \) in orthogonal curvilinears:

\[
\nabla f = \frac{1}{h_1} \frac{\partial f}{\partial u_1} \varepsilon_1 + \frac{1}{h_2} \frac{\partial f}{\partial u_2} \varepsilon_2 + \frac{1}{h_3} \frac{\partial f}{\partial u_3} \varepsilon_3
\]

\[
= \sum_{i=1}^{3} \frac{1}{h_i} \frac{\partial f}{\partial u_i} \varepsilon_i
\]

For **spherical polars**, we obtain

\[
\nabla f(r, \theta, \phi) = \varepsilon_r \frac{\partial f}{\partial r} + \varepsilon_\theta \frac{1}{r} \frac{\partial f}{\partial \theta} + \varepsilon_\phi \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi}.
\]

For **cylindrical polars**, we obtain

\[
\nabla f(\rho, \phi, z) = \varepsilon_\rho \frac{\partial f}{\partial \rho} + \varepsilon_\phi \frac{1}{\rho} \frac{\partial f}{\partial \phi} + \varepsilon_z \frac{\partial f}{\partial z}.
\]

**Divergence**

In orthogonal curvilinear coordinates

\[
\text{div} \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} (A_1 h_2 h_3) + \frac{\partial}{\partial u_2} (A_2 h_3 h_1) + \frac{\partial}{\partial u_3} (A_3 h_1 h_2) \right\}
\]

This expression can be obtained by using the integral definition of \( \text{div} \mathbf{A} \), or alternatively using vector operator identities (see BK 4.13, RHB 8.10).

For Cartesian coordinates, we have \( h_i = 1 \), and we regain the usual expression for \( \nabla \cdot \mathbf{A} \) in Cartesians.

For **spherical polars** we have

\[
\text{div} \mathbf{A}(r, \theta, \phi) = \frac{1}{r^2 \sin \theta} \left\{ \frac{\partial}{\partial r} (r^2 \sin \theta A_r) + \frac{\partial}{\partial \theta} (r \sin \theta A_\theta) + \frac{\partial}{\partial \phi} (r A_\phi) \right\}
\]

\[
= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \left\{ \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{\partial}{\partial \phi} (A_\phi) \right\}.
\]
where \( A_r, A_\theta, \) and \( A_\phi \) are the components of the vector field \( \mathbf{A} \) in the basis \((\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)\).

For **cylindrical polars** we have

\[
\text{div}\mathbf{A}(\rho, \phi, z) = \frac{1}{\rho} \left\{ \frac{\partial}{\partial \rho} \left( \rho A_\rho \right) + \frac{\partial}{\partial \phi} A_\phi + \frac{\partial}{\partial z} \left( \rho A_z \right) \right\}
\]

\[
= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho A_\rho \right) + \frac{1}{\rho} \frac{\partial}{\partial \phi} A_\phi + \frac{\partial}{\partial z} A_z .
\]

where \( A_\rho, A_\phi, \) and \( A_z \) are the components of the vector field \( \mathbf{A} \) in the basis \((\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_z)\).

**Curl**

In orthogonal curvilinear co-ordinates, curl is most conveniently written as

\[
\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left| \begin{array}{ccc} h_1 \mathbf{e}_1 & h_2 \mathbf{e}_2 & h_3 \mathbf{e}_3 \\
\frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\
h_1 A_1 & h_2 A_2 & h_3 A_3 \\
\end{array} \right|
\]

*e.g.* the first component is given by

\[
\mathbf{e}_1 \cdot \nabla \times \mathbf{A} = \frac{1}{h_2 h_3} \left\{ \frac{\partial}{\partial u_2} (A_3 h_3) - \frac{\partial}{\partial u_3} (A_2 h_2) \right\}
\]

and the components of \( \nabla \times \mathbf{A} \) in the \( \mathbf{e}_2 \) and \( \mathbf{e}_3 \) directions may be obtained by cyclic permutations of the suffices.

The above formula can be demonstrated by using the line integral definition of curl, as used to prove Stokes’ theorem, (see tutorial) or by vector operator identities (BK. 4.13 or RHB 8.10).

For **spherical polars** we have

\[
\nabla \times \mathbf{A} = \frac{1}{r^2 \sin \theta} \left| \begin{array}{ccc} \mathbf{e}_r & r \mathbf{e}_\theta & r \sin \theta \mathbf{e}_\phi \\
\frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\
A_r & r A_\theta & r \sin \theta A_\phi \\
\end{array} \right|
\]

**Laplacian of a Scalar Field**

The Laplacian operator acting on a *scalar* field is defined by \( \nabla^2 f = \nabla \cdot (\nabla f) \), giving:

\[
\nabla^2 f = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} \left( h_2 h_3 \frac{\partial f}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left( h_3 h_1 \frac{\partial f}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left( h_1 h_2 \frac{\partial f}{\partial u_3} \right) \right\}
\]

In **spherical polars**, we have

\[
\nabla^2 f(r, \theta, \phi) = \frac{1}{r^2 \sin \theta} \left\{ \frac{\partial}{\partial r} \left( r^2 \sin \theta \frac{\partial f}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{\partial}{\partial \phi} \left( \frac{1}{\sin \theta} \frac{\partial f}{\partial \phi} \right) \right\}
\]

\[
= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \left\{ \sin \theta \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{\partial^2 f}{\partial \phi^2} \right\}.
\]