PP4: Example Sheet 5

QED, Dirac Equation

1. Show that the γ matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

   where 1 is a 2×2 unit matrix, 0 is a 2×2 null matrix, and σ^i are the Pauli matrices, fulfills the relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

2. Requiring the normalisation condition $u^\dagger u = 2 \mid E \mid$ show that the normalisation constant of the Dirac spinors for free particles (Equation 164) is $N = \sqrt{\mid E \mid + m}$.

3. Show that for the solutions of the Dirac equation for a free particle, $u^{(1)}$ and $u^{(2)}$, the lower components, u_B, are smaller than the upper ones, u_A, in the non-relativistic limit by a factor $\beta = v/c$. For simplicity, assume that the particle is travelling in the $+z$ direction, and has four-momentum $p^\mu = (E, 0, 0, p)$.

   Verify that for a particle moving along the z-axis the spinor $u^{(1)}$ satisfies

$$\vec{\Sigma} \cdot \hat{\vec{p}}\, u^{(1)} = u^{(1)}$$

   The definition of $\vec{\Sigma} \cdot \hat{\vec{p}}$ is given in the lecture notes.

4. Draw the lowest order Feynman diagrams for electron-muon scattering $e^-\mu^- \to e^-\mu^-$ and for electron-positron annhilation into a muon pair $e^+e^- \to \mu^+\mu^-$. Write down their respective Matrix element $\mathcal{M}$.

   In the relativistic limit where the masses can be neglected, the spin-averaged matrix element for unpolarised electron-muon scattering $e^-\mu^- \to e^-\mu^-$ is given by $\langle \mid \mathcal{M} \mid^2 \rangle = 2e^4 \left(\frac{s^2+u^2}{t^2}\right)$. Use the substitution rule for crossed reactions to show that the spin-averaged matrix element for $e^+e^- \to \mu^+\mu^-$ is

$$\langle \mid \mathcal{M} \mid^2 \rangle = 2e^4 \left(\frac{t^2+u^2}{s^2}\right) .$$

5. In the centre-of-mass (CoM) frame, the differential cross section for $e^+e^- \to \mu^+\mu^-$ is $\left.\frac{d\sigma}{d\Omega}\right|_{\mathrm{CoM}} = \frac{1}{64\pi^2 s}\frac{p_f}{p_i}\langle \mid \mathcal{M} \mid^2 \rangle$ where p_i and p_f are the magnitudes of the $e^\pm$ and $\mu^\pm$ momenta, respectively. For relativistic energies where the mass of the particles can be neglected, show that the differential cross section for $e^+e^- \to \mu^+\mu^-$ is

$$\left.\frac{d\sigma}{d\Omega}\right|_{\mathrm{CoM}} = \frac{\alpha^2}{4s}\left(1 + \cos^2\theta\right)$$

   where θ is the scattering angle between the incoming e^- and the outgoing μ^-.