Lagrangian

- Lagrangian field theory: useful framework to discuss symmetries and conservation laws
- Lagrangian density $\mathcal{L}$ defined as

$$\mathcal{L} = T - V \qquad\qquad L = \int \mathcal{L} \mathrm{d}^3 x$$

T and V are kinetic and potential energy densities

- Euler-Lagrange Equation

$$\frac{\delta}{\delta x_\mu} \left(\frac{\delta \mathcal{L}}{\delta \left(\frac{\delta \phi}{\delta x_\mu} \right)} \right) - \frac{\delta \mathcal{L}}{\delta \phi} = 0$$

- Start with Lagrangian, eg.

$$\begin{aligned} \mathcal{L} &= \tfrac{1}{2}(\delta_\mu \phi)(\delta^\mu \phi) - \tfrac{1}{2} m^2 \phi^2 \\ &= \tfrac{1}{2} \left(\delta_0 \phi \delta_0 \phi - \delta_1 \phi \delta_1 \phi - \delta_2 \phi \delta_2 \phi - \delta_3 \phi \delta_3 \phi \right) - \tfrac{1}{2} m^2 \phi^2 \end{aligned}$$

This gives the Klein-Gordon equation

$$\delta_\mu \delta^\mu \phi + m^2 \phi = 0$$

- Dirac equation follows from

$$\begin{aligned} \mathcal{L} &= i \bar{\psi} \gamma^\mu \delta_\mu \psi - m \bar{\psi} \psi \\ 0 &= \left(i \gamma^\mu \delta_\mu - m \right) \psi \end{aligned}$$

Gauge Transformations

- Maxwell's equations are invariant under transformations

$$A_\mu \to A'_\mu = A_\mu + \delta_\mu \chi$$

for any field $\chi = \chi(x^\mu)$

- Global phase transformation for e^- spinor fields

$$\psi(x) \to U(\alpha)\psi(x) = e^{i\alpha}\psi(x)$$

α is independent of x, group U(1) symmetry

- Gauge principle
Demand $\mathcal{L}$ invariant under U(1) rotations
Sufficient to look at infinitesimal rotations

$$U(\alpha) = 1 + i\alpha$$

- Noether theorem $\delta\mathcal{L} = 0$
Global gauge invariance $\Longleftrightarrow$ Charge conservation

$$j^\mu = -e\bar{\psi}\gamma^\mu\psi$$
$$\delta_\mu j^\mu = 0$$

- What if $\alpha = \alpha(x)$?

Local Gauge Invariance

- Local phase transformation

$$\psi(x) \rightarrow U(\alpha(x))\psi(x) = e^{i\alpha(x)}\psi(x)$$
$$\bar{\psi}(x) \rightarrow \bar{U}(\alpha(x))\bar{\psi}(x) = e^{-i\alpha(x)}\bar{\psi}(x)$$

- $\mathcal{L}$ is not invariant, kinetic term is

$$\delta_\mu\psi \rightarrow e^{i\alpha(x)}\delta_\mu\psi(x) + ie^{i\alpha(x)}\psi\delta_\mu\alpha(x)$$

- Need to modify derivative
  Introduce covariant derivative

$$D_\mu\psi \rightarrow e^{i\alpha(x)}D_\mu\psi(x)$$
$$D_\mu \equiv \delta_\mu - ieA_\mu$$
$$A_\mu \rightarrow A_\mu + \frac{1}{e}\delta_\mu\alpha(x)$$

  Now $\mathcal{L}$ invariant

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\delta_\mu - m)\psi + e\bar{\psi}\gamma^\mu A_\mu\psi$$

- Price to pay $\Longrightarrow$ Trophy
  Gauge field A_μ, photons
  QED Lagrangian add kinetic energy

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\delta_\mu - m)\psi + e\bar{\psi}\gamma^\mu A_\mu\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$
$$F_{\mu\nu} = \delta_\mu A_\nu - \delta_\nu A_\mu$$

QCD Gauge Invariance

- Colour vectors $c(x) = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$, λ matrices $T_i = \frac{1}{2}\lambda_i$

- Demand local SU(3) phase invariance

$$c(x) \to U c(x) = e^{-i\vec{\theta}(x)\cdot\vec{T}} c(x)$$

  8 rotation angles $\vec{\theta}(x) = (\theta_1(x), \theta_2(x), ..., \theta_8(x))$

- Analogous to QED $\Longrightarrow$ Must introduce 8 gauge fields G_μ^a which correspond to 8 massless gluons $\Longrightarrow$ QCD Lagrangian

$$\mathcal{L} = \bar{q}(i\gamma^\mu \delta_\mu - m)q - g(\bar{q}\gamma^\mu T^a q)G_\mu^a - \frac{1}{4}G_{\mu\nu}^a G^{\mu\nu a}$$

- SU(3) is non abelian group

  Generators $\left[\lambda^a, \lambda^b\right] = 2if^{abc}\lambda^c$

  Covariant derivative introduces gluon self coupling
  3-gluon and 4-gluon vertices

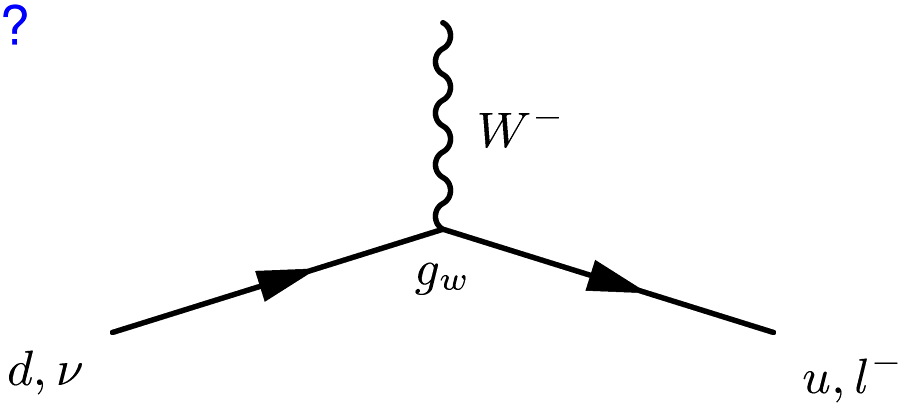
$$G_\mu^a \to G_\mu^a - \frac{1}{g}\delta_\mu \theta^a - f^{abc}\theta^b G_\mu^c$$

$$G_{\mu\nu}^a \to \delta_\mu G_\nu^a - \delta_\nu G_\mu^a - gf^{abc}G_\mu^b G_\nu^c$$

$$\mathcal{L} = \text{“}\bar{q}q\text{”} + \text{“}G^2\text{”} + g\text{“}\bar{q}qG\text{”} + g\text{“}G^3\text{”} + g^2\text{“}G^4\text{”}$$

Weak Isospin

- How are charged and neutral weak currents related?



- Charged $W^\pm$ current left-handed , e.g.

$$j^\mu \equiv (j^\mu)^+ = \bar{\nu}\gamma^\mu \tfrac{1}{2}(1-\gamma^5)e = \bar{\nu}_L\gamma^\mu e_L$$

$$j^{\mu\dagger} \equiv (j^\mu)^- = \bar{e}\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\nu = \bar{e}_L\gamma^\mu \nu_L$$

- Introduce **Weak Isospin**
  Left-handed doublets $(I_W = \frac{1}{2})$

$$\chi_L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L \quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L \qquad \begin{matrix} I_{W,3} = +\frac{1}{2} \\ I_{W,3} = -\frac{1}{2} \end{matrix}$$

  Right-handed singlets $(I_W = 0)$

$$\chi_R = \left(e^-\right)_R \quad \left(\mu^-\right)_R \quad \left(\tau^-\right)_R \qquad I_{W,3} = 0$$

- Charged currents re-written as

$$j^\mu = \bar{\chi}_L\gamma^\mu\sigma_+\chi_L \qquad j^{\mu\dagger} = \bar{\chi}_L\gamma^\mu\sigma_-\chi_L$$

$$\sigma_\pm = \tfrac{1}{2}(\sigma_1 \pm i\sigma_2) \qquad \text{Pauli matrices}$$

Weak Gauge Invariance

- Demand local SU(2) phase (gauge) invariance

$$\psi \to U\psi = e^{-i\vec{W}(x)\cdot\vec{\sigma}}$$

$\vec{W}(x) = (W_1(x), W_2(x), W_3(x))$

invariance under rotations in weak isospin space

- Must introduce 3 gauge fields which correspond to 3 massless gauge bosons W_i^μ, W_2^μ, W_3^μ
Triplet $I_W = 1$ combinations

$$W^{\mu +} = \frac{1}{\sqrt{2}}(W_1^\mu + iW_2^\mu)$$
$$W^{\mu 0} = W_3^\mu$$
$$W^{\mu -} = \frac{1}{\sqrt{2}}(W_1^\mu - iW_2^\mu)$$

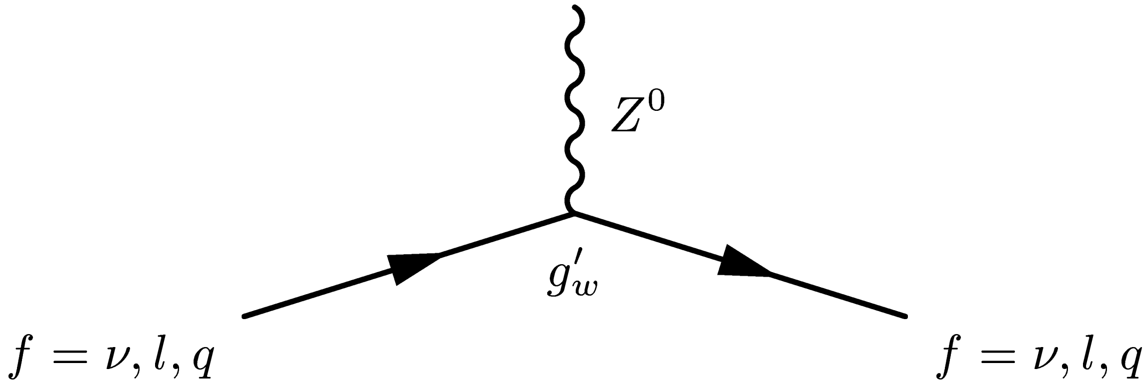
- Identify $W^{\mu\pm}$ with charged $W^\pm$
Problem mass $M_W \neq 0$
$\Longrightarrow$ Higgs mechanism

- Weak isospin conserved for charged current

	$\nu_e \to e^- W^+$	$e^- \to \nu_e W^-$
$I_{W,3}$:	$\frac{1}{2} = -\frac{1}{2} + 1$	$-\frac{1}{2} = +\frac{1}{2} - 1$

Weak Hypercharge

- SU(2) gauge invariance requires weak neutral currents mediated by neutral gauge boson W_3



- Neutral weak current

$$j_3^\mu = \bar{\chi}_L \gamma^\mu \tfrac{1}{2}\sigma_3\, \chi_L = \tfrac{1}{2}\bar{\nu}_L \gamma^\mu \nu_L - \tfrac{1}{2}\bar{e}_L \gamma^\mu e_L$$

couples only to left-handed parts, V-A structure

- Weak isospin conserved for neutral current

$$\begin{array}{lccc} & \nu_e \to \nu_e W_3 & e^- \to e^- W_3 & u \to u W_3 \\ I_{W,3}: & \frac{1}{2} = \frac{1}{2} + 0 & -\frac{1}{2} = -\frac{1}{2} + 0 & \frac{1}{2} = \frac{1}{2} + 0 \end{array}$$

- Include el. mag. current j_{em}^μ, charge Q

$$j_{em}^\mu = \bar{e}_L \gamma^\mu Q\, e_L + \bar{e}_R \gamma^\mu Q\, e_R$$

$\Longrightarrow$ Introduce **Weak Hypercharge** Y_W

$$Q = I_{W,3} + \tfrac{1}{2} Y_W$$

$$j_{em}^\mu = j_3^\mu + \tfrac{1}{2} j_Y^\mu$$

$$j_Y^\mu = 2(j_{em}^\mu - j_3^\mu) \quad = -2\bar{e}_R \gamma^\mu e_R - \bar{e}_L \gamma^\mu e_L - \bar{\nu}_L \gamma^\mu \nu_L$$

Electroweak Mixing

- Weak isospin and hypercharge quantum numbers

lepton	I_W	$I_{W,3}$	Q	Y_W	quark	I_W	$I_{W,3}$	Q	Y_W
ν_L	$\frac{1}{2}$	$\frac{1}{2}$	0	-1	u_L	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{3}$
e_L	$\frac{1}{2}$	$-\frac{1}{2}$	-1	-1	d_L	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{3}$
					u_R	0	0	$\frac{2}{3}$	$\frac{4}{3}$
e_R	0	0	-1	-2	d_R	0	0	$-\frac{1}{3}$	$-\frac{2}{3}$

- Unification of electromagnetic and weak forces (Glashow 1961)
  Local $U(1)_Y$ gauge invariance requires gauge field $B(x) \perp W_3^\mu$

$$-iej^\mu_{em}A_\mu \qquad \text{QED}$$

$$-ig_w J_i^\mu W_{i,\mu} - i\frac{g'_w}{2}J_Y^\mu B_\mu \qquad \text{Electroweak}$$

- Is the W_3 the Z^0 ? $\quad (m_W \neq m_Z <> 0)$
  Is it the γ?

- Z^0 and γ have same quantum numbers
  Z^0 and γ are linear combinations of W_3 and B

$$A^\mu = \cos\theta_W B^\mu + \sin\theta_W W_3^\mu$$
$$Z^\mu = -\sin\theta_W B^\mu + \cos\theta_W W_3^\mu$$

Weak mixing or Weinberg angle θ_W
from experiment $\sin^2\theta_W = 0.23124 \pm 0.00024$

Neutral Weak Currents

- Unified neutral current interaction

$$-ig_w J_3^\mu W_{3,\mu} - i\frac{g'_w}{2} J_Y^\mu B_\mu =$$
$$-i\left(g_w \sin\theta_W J_3^\mu + \frac{g'_w}{2}\cos\theta_W J_Y^\mu\right) A_\mu$$
$$-i\left(g_w \cos\theta_W J_3^\mu - \frac{g'_w}{2}\sin\theta_W J_Y^\mu\right) Z_\mu$$

- Weak and el. mag. coupling constants must be identical up to mixing angle

$$g_w \sin\theta_W = g'_w \cos\theta_W \sin\theta_W = e$$

- Neutral weak current interaction

$$= -i\frac{g_w}{\cos\theta_W} j_{NC}^\mu Z_\mu$$
$$j_{NC}^\mu = J_3^\mu - \sin^2\theta_W J_{em}^\mu$$

- Z^0 couples also to right-handed fermions
  Feynman rule for Z^0 vertex

$$-i\frac{g'_w}{2}\gamma^\mu(c_V - c_A\gamma^5)$$

- Vector and axial vector coupling constants

$$c_V = g_L + g_R \qquad g_L = I_{W,3} - Q\sin^2\theta_W$$
$$c_A = g_L - g_R \qquad g_R = Q\sin^2\theta_W$$

Standard Model

- Standard Model of electroweak interactions
  Lagrangian is invariant under symmetries

$$SU(2) \otimes U(1)$$

- Why are the W and Z boson massive?
- Higgs mechanism and spontaneous symmetry breaking (SSB) $\Longrightarrow$ Particles acquire mass
- Gauge boson masses

$$M_W = \sqrt{\frac{\sqrt{2}e^2}{8G_F \sin^2\theta_W}} = 80.4 \text{ GeV} \quad M_Z = \frac{M_W}{\cos\theta_W} \approx 91.2 \text{ GeV}$$

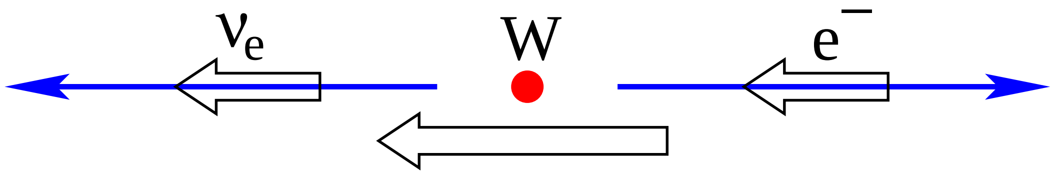
  Lepton and quark masses not predicted

- Only SSB gauge theories renormalisable
  (t'Hooft and Veltmann)
- 1974: Neutral currents discovered
  1983: $W^\pm$ and Z^0 boson observed
  LEP: Preciscion tests of electroweak model

$W^{\pm}$ Boson Decay

- Matrix element for $W^- \to e^- \bar{\nu}_e$

$$-i\mathcal{M} = -i\frac{g_w}{\sqrt{2}}\bar{e}\tfrac{1}{2}\gamma^{\mu}(1-\gamma^5)\nu = -i\frac{g_w}{\sqrt{2}}\bar{e}_L\gamma^{\mu}\nu_L$$



- V-A interaction, only one W helicity state contributes in limit $m_e = 0$
  For unpolarised W's

$$\mathcal{M} = g_w M_W$$

$$\langle|\,\mathcal{M}\,|^2\rangle = \tfrac{1}{3}g_w^2 M_W^2$$

$$\Gamma(W^- \to e^- \bar{\nu}_e) = \frac{g_w^2 M_W}{48\pi} = \frac{G_F M_W^3}{\sqrt{2}\,6\pi} = 230 \text{ GeV}$$

- Other possible W decay widths
  in units of $\Gamma(W^- \to e^- \bar{\nu}_e)$

$\Gamma(W^- \to e^- \bar{\nu}_e) = 1$	$\Gamma(W^- \to \bar{u}d) = 3\lvert V_{ud}\rvert^2$	$\Gamma(W^- \to \bar{c}d) = 3\lvert V_{cd}\rvert^2$
$\Gamma(W^- \to \mu^- \bar{\nu}_\mu) = 1$	$\Gamma(W^- \to \bar{u}s) = 3\lvert V_{us}\rvert^2$	$\Gamma(W^- \to \bar{c}s) = 3\lvert V_{cs}\rvert^2$
$\Gamma(W^- \to \tau^- \bar{\nu}_\tau) = 1$	$\Gamma(W^- \to \bar{u}b) = 3\lvert V_{ub}\rvert^2$	$\Gamma(W^- \to \bar{c}b) = 3\lvert V_{cb}\rvert^2$

- Total W Decay width

$$\Gamma(W^- \to \text{anything}) = 9\Gamma(W^- \to e^- \bar{\nu}_e) = \frac{9g_w^2 M_W}{48\pi} = 2.07 \text{ GeV}$$

Experiment: $\Gamma_W = 2.07 \pm 0.05$ GeV

Z^0 Boson Decay

- $Z^0 \rightarrow f\bar{f}$ couplings, not purely left-handed

$$-i\frac{g'_w}{2}\gamma^\mu(c_V - c_A\gamma^5) = -\,i\frac{g_w}{2\cos\theta_W}\gamma^\mu(c_V - c_A\gamma^5)$$

Fermion	c_V	c_A
ν_e, ν_μ, ν_τ	$\frac{1}{2}$	$\frac{1}{2}$
e^-, μ^-, τ^-	$-\frac{1}{2} + 2\sin^2\theta_W \approx -0.04$	$-\frac{1}{2}$
u, c, t	$\frac{1}{2} - \frac{4}{3}\sin^2\theta_W \approx +0.019$	$\frac{1}{2}$
d, s, b	$-\frac{1}{2} + \frac{2}{3}\sin^2\theta_W \approx -0.35$	$-\frac{1}{2}$

- Decay rates for unpolarised $Z^0 \rightarrow f\bar{f}$

$$\Gamma(Z^0 \rightarrow f\bar{f}) = \frac{g'^2_w M_Z}{48\pi}(c_V^2 + c_A^2)$$

Branching ratios $Z^0 \rightarrow f\bar{f}$ determined by $c_V^2 + c_A^2$

- Invisible $Z^0 \rightarrow \nu\bar{\nu}$ width

$$\Gamma(Z^0 \rightarrow \nu\bar{\nu}) = \frac{g'^2_w M_Z}{96\pi} = \frac{g_w^2 m_Z}{\cos^2\theta_W 96\pi}$$

$$= \frac{G_F M_Z M_W^2}{\sqrt{2}\,12\pi\cos^2\theta_W} = \frac{G_F M_Z^3}{\sqrt{2}\,12\pi} = 0.159\ \mathrm{GeV}$$

- Total Z^0 Decay width, experiment:

$\Gamma_W = 2.4952 \pm 0.0026\ \mathrm{GeV}$