9 Quantum Electrodynamics (QED)

In this chapter we will study **quantum electrodynamics (QED)** which is the theory describing the electromagnetic interaction of quarks and leptons. QED is the original theory of particle dynamics and also the most successful one. It allows us to derive the *matrix elements*, and cross section or decay rate for any electromagnetic reaction such as $e^+e^- \to \mu^+\mu^-(q\bar{q})$. These processes are relativistic and *antiparticles* occur. We will use perturbation theory for which QED is very well suited. This approach takes one-particle wavefunctions for free fermions and treats the scattering of particles as a perturbation. Antiparticles will be introduced as negative energy solutions. We will show how matrix elements can be calculated by applying a set of rules on all Feynman diagrams which produce a particular reaction.

9.1 Feynman Diagrams - Matrix Elements

The **dynamics** of a physical process is described by the **matrix element** or the invariant amplitude $\mathcal{M}$ of the interaction. We already discussed in Section 4.3 how, using Fermi's Golden rule, the matrix element is related to measurables quantities such as cross sections and decay rates (Equations 47, 50, 51). Now we want to calculate $\mathcal{M}$. We start out by drawing all **Feynman diagrams** which are possible for a particular physical process. The matrix element $\mathcal{M}$ is equal to the sum of the amplitudes of all diagrams. These amplitudes $\mathcal{M}$ can be calculated using relativistic quantum field theory. In this course, we show how to obtain the same results by using a set of rules which need to be applied to the relevant Feynman diagrams.

To illustrate this method let us have a look at the scattering of two spinless particles $a + a \to b + b$, shown in Figure 34. These are fictitious particles since the only spinless particles we know, e.g. the pions are actually composite $q\bar{q}$ states and the mediator bosons, e.g. the photons, have spin one. This exercise with toy particles will allow us to understand the principles of the method

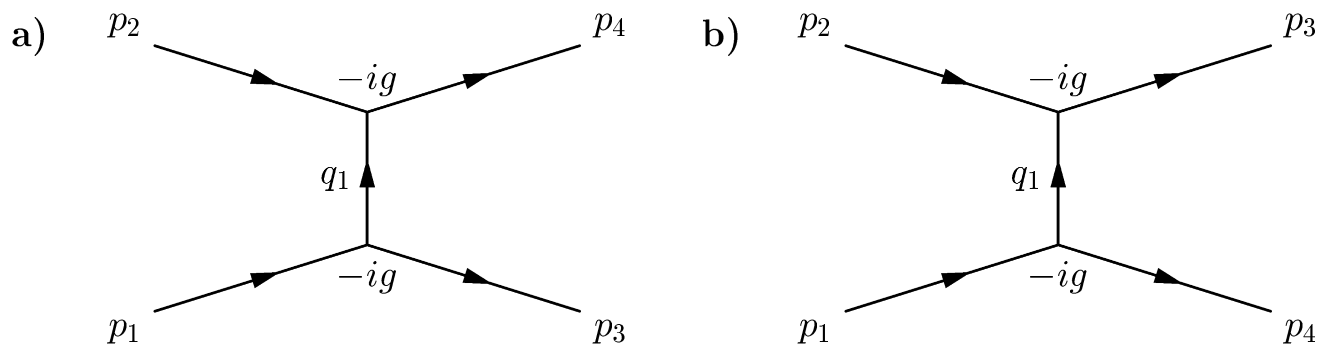

Figure 34: The two Feynman diagrams for scattering of two spinless particles.

without having to worry about calculational details.

A physical process is represented by the corresponding Feynman diagrams. The **Feynman rules** state that the matrix element $-i\mathcal{M}$ for a process involving only spinless particles can be written as the product of the following factors:

- *Notation*: Label all *external* lines with their four-momentum p_1, p_2, ... , label all *internal lines* with their four-momentum q_1, q_2, ... An arrow is put on each line with the following conventions: For particles the arrow points forward in time and for antiparticles it points backward, respectively. For internal lines the arrow is arbitrarily assigned.

- *External lines* come with a factor of 1 each.

- *Propagator*: For each internal line write a factor $\frac{i}{q_j^2 - m_j^2}$. Note that $q_j^2 \neq m_j^2$, since the internal particles are virtual and can have a mass different from their rest mass. We say they are not on their mass shell.

- *Coupling*: For each vertex there is a factor $-ig$ where g is the coupling constant.

In the example given above, there are two diagrams which result in the same final state and thus are indistinguishable from each other. In Figure 34a) p_1 turns into p_3 whereas in Figure 34b) p_1 turns into p_4. The amplitude for the two diagrams need to be added. We thus obtain for the matrix element of $a + a \to b + b$

$$\begin{aligned} -i\mathcal{M} &= (-i\,g)^2 \frac{i}{(p_4 - p_2)^2 - m^2} + (-i\,g)^2 \frac{i}{(p_3 - p_2)^2 - m^2} \\ \mathcal{M} &= \frac{g^2}{(p_4 - p_2)^2 - m^2} + \frac{g^2}{(p_3 - p_2)^2 - m^2} \end{aligned} \tag{148}$$

It is evident that the Matrix element is a Lorentz invariant, as it indeed must be.

We have demonstrated how, using Feynman rules, the matrix element can be calculated for a process involving spinless particles and exchange bosons. For particles with spin, the factors for external lines, propagators, and vertices obviously will be different and algebraic complications can arise. However, the basic rules stay the same and it is always possible to write down $-i\mathcal{M}$. In principle, there are an infinite number of additional diagrams with the same external lines. Fortunately each additional vertex introduces a factor of g^2. For small coupling constants, $g^2 \ll 1$, the perturbative sum of all diagrams leads to a finite result. As in the example above, we will often calculate the matrix element only for the simplest diagram of a given process and neglect these higher order corrections. We will now develop the Feynman rules for **quantum electrodynamics (QED)** which is the theory describing the electromagnetic interaction of quarks and leptons.

9.2 Klein-Gordon Equation

In non-relativistic quantum mechanics the **Schrödinger Equation** for a particle of mass can be obtained by substituting the classical energy-momentum relation

$$E = \frac{p^2}{2m} + V$$

with the differential operators

$$E \to i\hbar\frac{\delta}{\delta t} \qquad\qquad p \to -i\hbar\nabla$$

The resulting operator equation acts on the wavefunction $\psi(\vec{x}, t)$

$$i\frac{\delta\psi}{\delta t} = \left(-\frac{1}{2m}\nabla^2 + V\right)\psi \qquad \text{(Schrödinger Equation).} \tag{149}$$

where we set $\hbar = 1$. Free field solutions ($V = 0$) are given by $\psi(\vec{x}, t) \propto e^{-iEt}\psi(\vec{x})$. The probability density ρ of finding a particle in a given volume $\mathrm{d}x^3$ is related to the wavefunction by $\rho\mathrm{d}x^3 = \mid\psi\mid^2 \mathrm{d}x^3$. This density will be altered by the flux of particles, j, into or out of that volume. From the conservation of probability we find that ρ and $\vec{j}$ must satisfy the continuity equation

$$\frac{\delta\rho}{\delta t} + \nabla\cdot\vec{j} = 0\ . \tag{150}$$

By multiplying Equation 149 with $-i\psi^*$ and subtracting the conjugate Equation multiplied by $-i\psi$ we form $\delta\rho/\delta t$ on the left-hand side of the equation and can identify the flux density $\vec{j}$

$$\frac{\delta\rho}{\delta t} = \frac{\delta}{\delta t}\mid\psi\mid^2 = \frac{\delta\psi^*}{\delta t}\psi + \frac{\delta\psi}{\delta t}\psi^*$$

$$\vec{j} = -\frac{i}{2m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)\ . \tag{151}$$

The Schrödinger Equation is non-relativistic. A Lorentz invariant equation can be derived by substituting the relativistic energy-momentum relation

$$p_\mu \cdot p^\mu = E^2 - \vec{p}^2 = m^2$$

with the same differential operators, now written in covariant form:

$$p_\mu \to i\hbar\delta_\mu \tag{152}$$

$$\text{where} \qquad \delta_\mu = \frac{\delta}{\delta x^\mu} = \left(\frac{\delta}{\delta t}, \nabla\right) \qquad\qquad \delta^\mu = \left(\frac{\delta}{\delta t}, -\nabla\right)$$

We obtain the **Klein-Gordon Equation**[27], written as

$$\left(-\frac{\delta^2}{\delta t^2} + \nabla^2\right)\psi = m^2\psi \qquad \text{(Klein-Gordon Equation)} \tag{153}$$

$$\text{or} \qquad \left(\Box^2 + m^2\right)\psi = 0$$

[27]Schrödinger apparently discovered this equation even before the one which bears his name. He rejected it because it gave an incorrect expression for the fine structure of hydrogen.

where $\square^2 = \delta_\mu \delta^\mu$ is the D'Alembertian operator. Solutions are of the form

$$\psi(x^\mu) = N \exp\left(-ip_\nu x^\nu\right) = N \exp\left(-i(Et - \vec{p} \cdot \vec{x})\right)$$

However there are problems with the interpretation of Equation 153. Firstly, there are two energy solutions $E = \pm(p^2 + m^2)^{\frac{1}{2}}$. In addition to the positive solutions, $E > 0$, we have also *negative energy* solutions, $E < 0$. Secondly, these negative energy solutions are associated to *negative probability densities* ρ

$$\rho = i\left(\psi^* \frac{\delta\psi}{\delta t} - \psi \frac{\delta\psi^*}{\delta t}\right) = 2E \mid N \mid^2 \tag{154}$$

These problems led Dirac in 1928 to search for an alternative relativistic wave equation. Later the Klein-Gordon equation was restored in relativistic quantum field theory where it successfully describes spin 0 particles and antiparticles.

9.3 Dirac Equation

Dirac noticed that the Schrödinger Equation was first order in time derivatives. Relativity requires that time and space should be treated equivalently. Dirac then tried to find a solution in which space derivatives were also first order. We will not follow the historical derivation given in most textbooks which involves the matrices α and β, but start with writing down the **Dirac Equation** in its modern covariant form.

$$\left(i\gamma^0 \frac{\delta}{\delta t} + i\vec{\gamma} \cdot \vec{\nabla} - m\right)\psi = 0 \qquad \text{(Dirac Equation)} \tag{155}$$

$$\text{or} \qquad (i\gamma^\mu \delta_\mu - m)\,\psi = 0$$

where we have introduced the coefficients γ^0 and $\vec{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$. Note that $\gamma^\mu \delta_\mu = \gamma^0 \delta_0 + \vec{\gamma} \cdot \vec{\nabla}$. What are the γ^μ? Clearly, the Dirac Equation must satisfy energy-momentum conservation, i.e. the Klein-Gordon Equation. This can be achieved by multiplying Equation 155 by its conjugate equation

$$\begin{aligned} 0 &= \left(-i\gamma^0 \frac{\delta}{\delta t} - i\vec{\gamma} \cdot \vec{\nabla} - m\right)\left(i\gamma^0 \frac{\delta}{\delta t} + i\vec{\gamma} \cdot \vec{\nabla} - m\right)\psi \\ &= \left\{\left[(\gamma^0)^2 \frac{\delta^2}{\delta t^2} + \vec{\gamma}^2 \vec{\nabla}^2 + m^2\right] + \left(\left[\gamma^0 \vec{\gamma} + \vec{\gamma}\gamma^0\right] \frac{\delta}{\delta t} \vec{\nabla} + \left\{\left[\gamma^i \gamma^j + \gamma^j \gamma^i\right] \frac{\delta^2}{\delta x^i \delta x^j}\right\}_{i \neq j}\right)\right\}\psi \end{aligned}$$

and demanding consistency with Equation 153. From this it is clear that the γ^μ cannot be ordinary numbers. We then try matrices, which not necessarily need to commute, i.e. in general $a \cdot b \neq b \cdot a$. The smallest dimension of matrices which can satisfy the above relations is 4×4. We deduce the following conditions for γ^μ, where $i = 1, 2, 3$:

$$(\gamma^0)^2 = 1 \qquad (\gamma^i)^2 = -1 \qquad \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 0 \qquad \mu \neq \nu\ .$$

All this can be expressed as an *anticommutation* relation, defined by $\{a, b\} = a \cdot b + b \cdot a$:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \ . \tag{156}$$

There are a number of different representations for the 4-dimensional γ matrices. We use the following (Dirac-Pauli) representation:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{157}$$

where 1 denotes a 2×2 unit matrix, 0 denotes a 2×2 null matrix, and σ^i are the Pauli matrices, see Equation 67. Other useful properties of the γ matrices are:

$$\gamma^{0\dagger} = \gamma^0 \qquad \gamma^{i\dagger} = -\gamma^i \qquad \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$$

where $i = 1, 2, 3$ and $\gamma^{\nu\dagger}$ is the transpose conjugate or hermitian conjugate matrix.

9.4 Spinors

In effect the Dirac equation comprises of 4 coupled differential equations. Thus the wavefunction ψ on which the Dirac equation operates must have four components. However, this object is not a four-vector, and we call ψ a Dirac **spinor**:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{158}$$

First we want to construct a probability density and a current out of the Dirac spinors. By taking the hermitian conjugate of the Dirac equation, an multiplying it from the right by γ^0 we obtain the *adjoint* Dirac equation

$$i\delta_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} = 0 \tag{159}$$

where the adjoint spinor is defined as $\bar{\psi} \equiv \psi^\dagger\gamma^0$. The dagger signifies the transpose conjugate or hermitian conjugate row spinor $\psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$. By multiplying Equation 155 from the left by $\bar{\psi}$ and Equation 159 form the right by ψ, and adding, we can construct a current which satisfies the continuity Equation 150:

$$\delta_\mu(\bar{\psi}\gamma^\mu\psi) = \bar{\psi}\gamma^\mu\delta_\mu\psi + (\delta_\mu\bar{\psi})\gamma^\mu\psi = 0 \qquad \Longrightarrow \qquad j^\mu = \bar{\psi}\gamma^\mu\psi$$

The associated probability density ρ is now positive definite.

$$\rho = j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi = \sum_{i=1}^{4} \mid \psi_i \mid^2 > 0$$

It can be proved that the current $j^\mu = \bar{\psi}\gamma^\mu\psi$ has the properties of a four-vector, i.e. it transforms like a four-vector under Lorentz or symmetry (P, C) transformations. Moreover, we insert the charge $-e$ into j^μ and identify it as the charge current density of the electron:

$$j^\mu = -e\bar{\psi}\gamma^\mu\psi \, . \tag{160}$$

Now we try to find solutions of the Dirac equation. For a **free particle** we make the ansatz:

$$\psi(\vec{x},t) = N\ u(p)\exp\left(-ip_\nu x^\nu\right) \qquad u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \tag{161}$$

where $u(p)$ is a 4-component **spinor** independent of x. Substituting into the Dirac equation we obtain

$$\begin{aligned} (\gamma^\mu p_\mu - m)u(p) &= 0 \\ \text{or} \qquad (\not{p} \ - m)u(p) &= 0 \end{aligned} \tag{162}$$

where we have introduced the shorthand notation "a-slash": $\not{a} \ \equiv \gamma^\mu a_\mu = \gamma^0 a^0 - \vec{\gamma}\cdot\vec{a}$. The above equation is known as the *momentum space Dirac equation.*

Applying the Dirac Equation 155 for a particle at rest, $\vec{p} = \vec{0}$, we find the following equations - in 4×4 matrix notation - for the energy eigenvalues and eigenvectors:

$$\left(i\gamma^0\frac{\delta}{\delta t} - m\right)\psi = \left(i\gamma^0(-iE) - m\right)\psi = 0$$
$$Eu = \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix} u \, .$$

The four equations are now decoupled and we can immediately write down the eigenvalues, $E = m,\ m,\ -m,\ -m$ and eigenspinors:

$$u^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad u^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad u^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad u^{(4)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{163}$$

There are two solutions with $E = m$ and two with $E = -m$. Thus negative energy solutions still occur. The corresponding wavefunctions are:

$$\psi^{(1)} = e^{-imt}u^{(1)} \qquad \psi^{(2)} = e^{-imt}u^{(2)} \qquad \psi^{(3)} = e^{+imt}u^{(3)} \qquad \psi^{(4)} = e^{+imt}u^{(4)} \, .$$

9.5 Antiparticles

To explain the negative energy solutions, Dirac used the exclusion principle and postulated that the negative energy solutions are all occupied as sketched in Figure 35 and that the vacuum is a sea of $E < 0$ electrons. One can create a *hole* in this sea by exciting an electron from a negative to a positive energy state. The absence of an electron of charge $-e$ and energy $-E$ is then interpreted as the presence of an **antiparticle** (positron) of charge $+e$ and energy $+E$. Hence the excitation corresponds to the pair-production process $\gamma \to e^+e^-$. Only 3 years later (1931) the positron was discovered by Anderson, see Figure 2 thus beautifully fulfilling Dirac's prediction.

Today we use a prescription which was proposed by Stückelberg (1941) and Feynman (1948) to treat the negative energy states. The idea, illustrated in Figure 36, is that a negative energy solution describes a particle which propagates backward in time or equivalently, a positive energy **antiparticle** propagating forward in space-time. This elegant identification can be made because

$$\exp\left(-i\left[(-E)(-t) - (-\vec{p})\cdot(-\vec{x})\right]\right) = \exp\left(-i(Et - \vec{p}\cdot\vec{x})\right)$$

In summary, the Dirac equation with $\vec{p} = 0$ has four independent solutions. These describe, respectively, an electron with spin up, an electron with spin down, a positron with spin up and a positron with spin down.

Applying the momentum space Dirac equation for a free particle (Equations 161 and 162) and dividing u into two two-component spinors we obtain the following matrix equation

$$(\gamma^\mu p_\mu - m)\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} E - m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E - m \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$

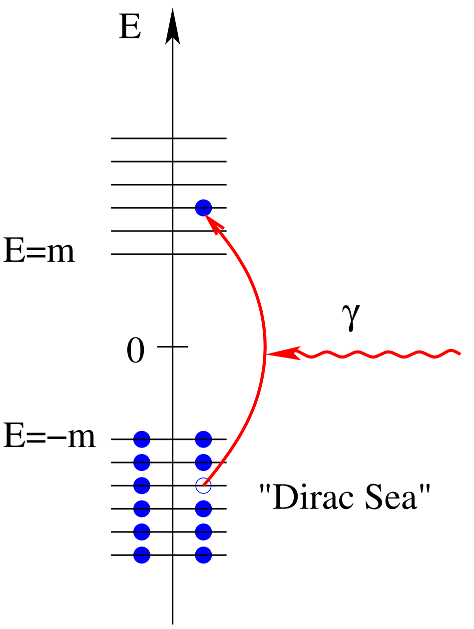

Figure 35: Electron energy states and Dirac sea.

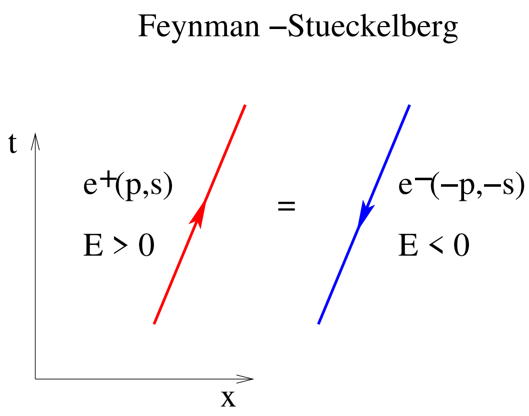

Figure 36: Stückelberg-Feynman interpretation of negative energies.

where u_A and u_B denote the two upper and two lower components of u respectively. We notice that the equations for u_A and u_B are coupled:

$$u_A = \frac{\vec{\sigma}\cdot\vec{p}}{E-m} u_B \qquad\qquad u_B = \frac{\vec{\sigma}\cdot\vec{p}}{E+m} u_A \; .$$

Two positive and two negative energy solutions are obtained by setting $u_A^{(1)} = \begin{pmatrix}1\\0\end{pmatrix}$, and $u_A^{(2)} = \begin{pmatrix}0\\1\end{pmatrix}$, and $u_B^{(1)} = \begin{pmatrix}1\\0\end{pmatrix}$, and $u_B^{(2)} = \begin{pmatrix}0\\1\end{pmatrix}$, respectively:

$$u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \quad u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} \quad v^{(2)} = N \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix} \quad v^{(1)} = N \begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \tag{164}$$

The first two solutions $u^{(1,2)}(p)\exp(-ip\cdot x)$ have energy $E = \sqrt{m^2+p^2}$ and describe a free electron of energy E and momentum $\vec{p}$. To describe the two solutions with negative energy, $E = -\sqrt{m^2+p^2}$, we have introduced the v spinors

$$v^{(1,2)}(p)\exp(ip\cdot x) = u^{(4,3)}(-p)\exp(-i(-p)\cdot x) \; .$$

The positron spinors $v^{(1)}$ and $v^{(2)}$ have now positive energies $E = \sqrt{m^2+p^2}$, corresponding to the physical energy and momentum of the positron, respectively. The normalisation constant is $N = \sqrt{\mid E \mid + m}$. It follows that the momentum space Dirac Equation for the v's have the sign of the four-momentum p_μ reversed:

$$(\not{p} + m)v(p) = 0 \; . \tag{165}$$

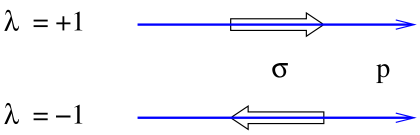

Figure 37: Helicity eigenvalues λ of the operator $\vec{\sigma}\cdot\hat{\vec{p}}$.

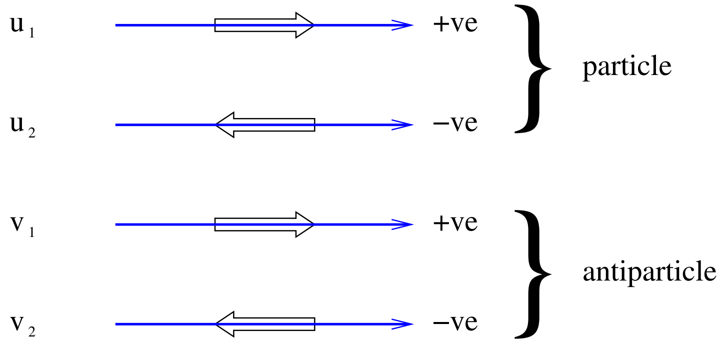

Figure 38: Helicity eigenstates for a particle or antiparticle travelling along the $+z$ axis.

For an electron of given momentum p, what is the correct interpretation of these spinor ($u^{(1,2)}$ and $v^{(1,2)}$) solutions? To understand this we introduce the 4×4 *spin matrices* for Dirac particles

$$S = \frac{\hbar}{2}\Sigma \qquad \text{where} \qquad \Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

In general, $u^{(1)}$ is not an eigenstate of Σ_z. However the spin projection in the direction of motion

$$\vec{\Sigma} \cdot \hat{\vec{p}} = \begin{pmatrix} \vec{\sigma} \cdot \hat{\vec{p}} & 0 \\ 0 & \vec{\sigma} \cdot \hat{\vec{p}} \end{pmatrix} \qquad \text{where} \qquad \hat{\vec{p}} = \frac{\vec{p}}{|\,\vec{p}\,|}\ . \tag{166}$$

commutes with the Hamiltonian. The spin component in the direction of motion, $\frac{1}{2}\vec{\sigma}\cdot\hat{\vec{p}}$, is therefore a good quantum number and an observable quantity. We call $\vec{\sigma} \cdot \hat{\vec{p}}$ the **helicity** of a state. The possible eigenvalues are $\lambda = \pm 1$ as is shown in Figure 37. For a particle or antiparticle travelling along the $+z$ axis the $u^{(1,2)}$ and $v^{(1,2)}$ are just the helicity eigenstates, see Figure 38.

We also define *chiral* spinor operators which project out the helicity ± 1 components or *handedness* of a spinor. We call the negative (positive) helicity projection the left- (right-) handed part of a particle.

$$\begin{aligned} u_L &\equiv \tfrac{1}{2}(1-\gamma^5)u^{(1,2)} & \qquad v_R &\equiv \tfrac{1}{2}(1-\gamma^5)v^{(1,2)} \\ u_R &\equiv \tfrac{1}{2}(1+\gamma^5)u^{(1,2)} & \qquad v_L &\equiv \tfrac{1}{2}(1+\gamma^5)v^{(1,2)} \end{aligned} \tag{167}$$

Here we have made use of the matrix γ^5 which is defined as

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\ . \tag{168}$$

The matrix γ^5 satisfies the following useful relations:

$$(\gamma^5)^2 = 1 \qquad \{\gamma^5, \gamma^\mu\} = 0 \qquad \gamma^{5\dagger} = \gamma^5$$

For massive particles the handedness of a particle is not a good quantum number, but this quantity will become very useful when discussing weak interactions.

9.6 Photons

We now describe the interaction of a Dirac particle with an electromagnetic field. An external electromagnetic potential $A^\mu = (\phi, \vec{A})$ can be coupled to an electron with charge $-e$ by the *gauge invariant* substitution

$$p^\mu \to p^\mu - \frac{-e}{c}A^\mu = p^\mu + \frac{e}{c}A^\mu \tag{169}$$

We insert the above into the Dirac Equation 155. In the non-relativistic case, we can factorise the spinor wavefunction - we keep $\hbar$ and c in this section - as follows:

$$\psi = \exp\left(\frac{-imc^2}{\hbar}t\right)\begin{pmatrix}\varphi\\ \chi\end{pmatrix}$$

where ϕ and χ have two components each. We obtain

$$i\hbar\frac{\delta}{\delta t}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} = c\vec{\sigma}\cdot(\vec{p}+\frac{e}{c}\vec{A})\begin{pmatrix}\chi\\ \varphi\end{pmatrix} - e\phi\begin{pmatrix}\varphi\\ \chi\end{pmatrix} - 2mc^2\begin{pmatrix}0\\ \chi\end{pmatrix} \tag{170}$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is a vector containing the Pauli matrices. For slowly varying φ and χ and small potential ϕ when compared to mc^2 the two components can be separated. We find

$$\chi = \frac{\vec{\sigma}\cdot(\vec{p}+\frac{e}{c}\vec{A})}{2mc}\varphi\ .$$

In the non-relativistic case, the χ component is small compared to φ by a factor of about $v/c \ll 1$. By inserting χ into the first line of Equation 170 we obtain the *Pauli Equation* for an electron

$$i\hbar\frac{\delta\varphi}{\delta t} = \left[\frac{(\vec{p}+\frac{e}{c}\vec{A})^2}{2m} + \frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B} - e\phi\right]\varphi \tag{171}$$

where the magnetic field is $\vec{B} = \vec{\nabla}\times\vec{A}$. The 2 components of φ are sufficient to describe the two spin states of an electron. Comparing Equation 171 with the intrinsic magnetic moment of an electron (Equation 141) $\vec{\mu}_e = -\frac{e\hbar}{2mc}\vec{\sigma} = -g\frac{e}{2mc}\vec{s}$ (remember $\vec{s} = \frac{1}{2}\hbar\vec{\sigma}$) the Dirac equation predicts that the *gyromagnetic* ratio is $g = 2$ which is in agreement with the experimental result of $g = 2.00232$. For the small discrepancy from 2, see Section 9.11.

The quantum mechanical wavefunction A^μ for a free photon can be written as

$$A^\mu = \varepsilon^\mu(q)\exp(-iq\cdot x) \tag{172}$$

where q^μ is the photon four-momentum and ε^μ is the polarisation four-vector of the photon. The Lorentz condition $\delta_\mu A^\mu = 0$ and the zero mass of the photon $m_\gamma^2 = q^2 = 0$ result in the condition $\vec{\varepsilon}\cdot\vec{q} = 0$. There are two independent polarisation vectors and they are both transverse to the photon direction, as is expected from classical electrodynamics.

9.7 Feynman Rules for QED

We are now in a position to write down the Feynman rules for QED which comprises of spin $\frac{1}{2}$ fermions and spin 1 photons. The fundamental $e^- e^- \gamma$ vertex is shown in Figure 39. All electromagnetic phenomena are reducible to this diagram. The different contributions to the matrix element are obtained by adapting the Feynman rules for spinless particles (see Section 9.1) to rules for Dirac spinors. While this is not a derivation, the different terms can be arrived at

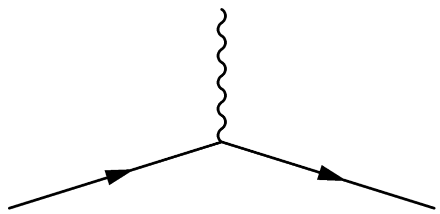

Figure 39: Feynman diagram for the $e^- \to e^- \gamma$ vertex.

by comparing the matrix elements for spinless particles and the solutions for the Klein-Gordon equation with the Dirac equation and its solutions. In QED, the Feynman rules state that the matrix element $-i\mathcal{M}$ of an electromagnetic process is the product of the following factors:

- Notation: as for spinless particles, but add spin states if known.
- External lines for fermions are spinors, photons have polarisation:
  - u for incoming particle
  - $\bar{u}$ for outgoing particle
  - $\bar{v}$ for incoming antiparticle
  - v for outgoing antiparticle
  - ϵ^μ for incoming photon
  - $\epsilon^{\mu *}$ for outgoing photon
- Each vertex has a factor of $ie\gamma^\mu$ where $\pm e$ is the charge of the fermion.
- Propagators:
  - $\frac{-ig_{\mu\nu}}{q^2}$ for photons
  - $\frac{i(\not{q} + m)}{q^2 - m^2}$ for spin $\frac{1}{2}$ particles with mass m
  - $\frac{-i(g_{\mu\nu} - q_\mu q_\nu / M^2)}{q^2 - M^2}$ for massive spin 1 bosons with mass M
- Antisymmetrisation: Include a minus sign between diagrams that differ only in the interchange of two identical fermions.

These rules ensure that energy and momentum is conserved at each vertex and that the matrix element is a Lorentz invariant quantity.

9.8 Møller Scattering

As an example let us write down the matrix element $-i\mathcal{M}$ for the scattering of two electrons $e^-e^- \to e^-e^-$. This process has two Feynman diagrams, shown in Figure 40. Collecting all

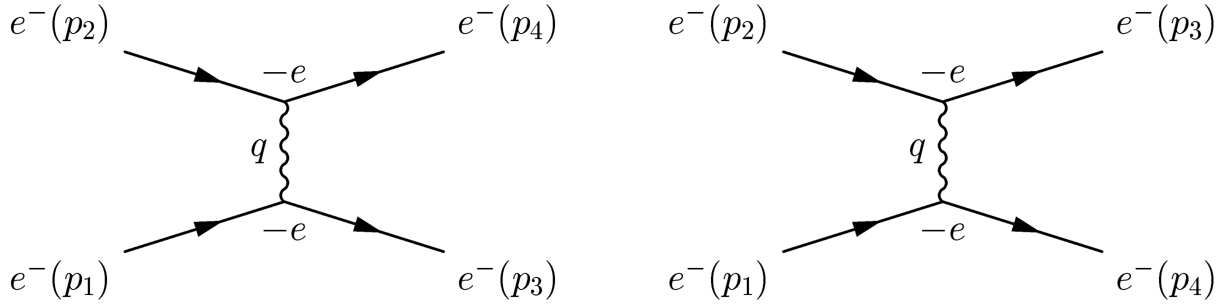

Figure 40: Moeller scattering $e^-e^- \to e^-e^-$.

the terms (ingoing and outgoing fermions, vertices, propagators) we can write down the matrix element

$$\begin{aligned} -i\mathcal{M} &= \frac{(ie\bar{u}_3\gamma^\mu u_1)\,(-ig_{\mu\nu})\,(ie\bar{u}_4\gamma^\nu u_2)}{(p_1-p_3)^2} - \frac{(ie\bar{u}_4\gamma^\mu u_1)\,(-ig_{\mu\nu})\,(ie\bar{u}_3\gamma^\nu u_2)}{(p_1-p_4)^2} \\ \mathcal{M} &= -e^2\frac{(\bar{u}_3\gamma^\mu u_1)\,(\bar{u}_4\gamma_\mu u_2)}{(p_1-p_3)^2} + e^2\frac{(\bar{u}_4\gamma^\mu u_1)\,(\bar{u}_3\gamma_\mu u_2)}{(p_1-p_4)^2} \end{aligned} \tag{173}$$

As promised, $\mathcal{M}$ is a Lorentz quantity. Note that due to Fermi statistics the second diagram gives a minus sign with respect to the first one. Once the spin states are specified, $\mathcal{M}$ is a number which can be calculated.

In an experiment we often do not record information about the spin of the particles. To calculate this *unpolarised* cross section we therefore allow for scattering of all possible spin configurations. To get the correct result we then average over the spins of the incoming particles and sum over all spins of the particles in the final state.

$$|\mathcal{M}|^2 \to \langle|\mathcal{M}|^2\rangle = \frac{1}{(2s_1+1)(2s_2+1)}\sum_{s_f}|\mathcal{M}|^2 \tag{174}$$

In the non-relativistic limit ($\vec{p}=\vec{0}$) the spinors are simple. Only upper components are non-zero and contributions are only possible if the spin directions are not changed.

$$\begin{aligned} \bar{u}^{(s)}\gamma^\mu u^{(s)} &= 2m \qquad \text{if} \qquad \mu = 0 \\ \bar{u}^{(s)}\gamma^\mu u^{(s')} &= 0 \qquad \text{if} \qquad s \neq s' \end{aligned}$$

Thus, only 6 out of the 16 possible spin configurations contribute to the result, namely

$$\begin{aligned} \mathcal{M}(\uparrow\uparrow\longrightarrow\uparrow\uparrow) = \mathcal{M}(\downarrow\downarrow\longrightarrow\downarrow\downarrow) &= -4e^2m^2\left(\frac{1}{t}-\frac{1}{u}\right) \\ \mathcal{M}(\uparrow\downarrow\longrightarrow\uparrow\downarrow) = \mathcal{M}(\downarrow\uparrow\longrightarrow\downarrow\uparrow) &= -4e^2m^2\,\frac{1}{t} \\ \mathcal{M}(\uparrow\downarrow\longrightarrow\downarrow\uparrow) = \mathcal{M}(\downarrow\uparrow\longrightarrow\uparrow\downarrow) &= 4e^2m^2\frac{1}{u} \end{aligned}$$

where the Mandelstam variables u and t, defined in Equation 24, are the photon propagators. We perform the spin summation (Equation 174) and obtain

$$\langle | \mathcal{M} |^2 \rangle = 8e^4 m^4 \left[\left(\frac{1}{t} - \frac{1}{u} \right)^2 + \frac{1}{t^2} + \frac{1}{u^2} \right]$$

In the centre-of-mass (CoM) frame we can express u and t as function of the initial and final state momentum $p = | \vec{p_i} | = | \vec{p_f} |$ and the scattering angle θ, where E is the electron energy:

$$t = (p_1 - p_3)^2 = 2(E^2 - p^2) - 2(E^2 - p^2 \cos\theta) = -2p^2(1 - \cos\theta) = -4p^2 \sin^2 \frac{\theta}{2}$$
$$u = (p_1 - p_4)^2 = 2(E^2 - p^2) - 2(E^2 - p^2 \cos(\pi - \theta)) = -2p^2(1 + \cos\theta) = -4p^2 \cos^2 \frac{\theta}{2} \; .$$

Using the spin-summed matrix element, $\langle | \mathcal{M} |^2 \rangle$, we can now determine the differential cross section. The scattering cross section in the CoM frame is given by Equation 54:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{p_f}{p_i} \langle | \mathcal{M} |^2 \rangle \; .$$

Using the QED coupling constant is $\alpha = e^2/4\pi \approx 1/137$. We obtain the differential cross section for Møller scattering $e^- e^- \to e^- e^-$ in the non-relativistic limit, where $s \approx 4m^2$:

$$\frac{d\sigma}{d\Omega} \left(e^- e^- \to e^- e^- \right) \Big|_{\mathrm{CoM}} = \frac{m^2 \alpha^2}{16 p^4} \left(\frac{1}{\sin^4 \frac{\theta}{2}} + \frac{1}{\cos^4 \frac{\theta}{2}} - \frac{1}{\sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2}} \right) \; . \qquad (175)$$

9.9 Electron-Muon Scattering

We now consider the process electron muon scattering $e^- \mu^- \to e^- \mu^-$. Here we only have one

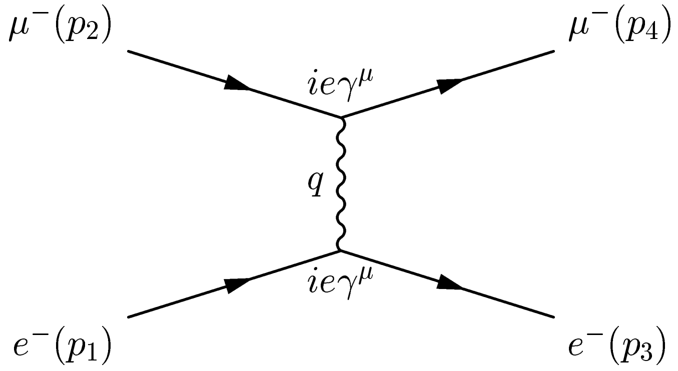

Figure 41: $e^- \mu^- \to e^- \mu^-$

diagram and the matrix element can be written as

$$\mathcal{M} = \frac{-e^2}{(p_1 - p_3)^2} \left(\bar{u}_3 \gamma^\mu u_1 \right) \left(\bar{u}_4 \gamma_\mu u_2 \right)$$

We can easily write down $|\mathcal{M}|^2$

$$|\mathcal{M}|^2 = \frac{e^4}{(p_1 - p_3)^4} \left(\bar{u}_3 \gamma^\mu u_1\right) \left(\bar{u}_4 \gamma_\mu u_2\right) \left(\bar{u}_3 \gamma^\nu u_1\right)^* \left(\bar{u}_4 \gamma_\nu u_2\right)^*$$

The full relativistic calculation goes beyond the scope of this course. Here I simply show the general idea and give results without deriving them. The quantities to handle in $|\mathcal{M}|^2$ are of the type

$$\begin{aligned} G &= \left(\bar{u}_a \Gamma_1 u_b\right) \left(\bar{u}_a \Gamma_1 u_b\right)^* \\ &= \left(\bar{u}_a \Gamma_1 u_b\right) \left(\bar{u}_b \bar{\Gamma}_1 u_a\right) \end{aligned}$$

where $\bar{\Gamma}_i = \gamma^0 \Gamma_i^\dagger \gamma^0$. Invoking the completeness relations

$$\begin{aligned} \sum_{s=1,2} u^{(s)} \bar{u}^{(s)} &= (\not p + m) \\ \sum_{s=1,2} v^{(s)} \bar{v}^{(s)} &= (\not p - m) \end{aligned}$$

for the sum over all spins, the spinors vanish and it all reduces to matrix multiplications and taking the trace ($TrA = \sum_i A_{ii}$)

$$\sum \left(\bar{u}_a \Gamma_1 u_b\right) \left(\bar{u}_b \bar{\Gamma}_2 u_a\right)^* = Tr\left(\Gamma_1 (\not p_b + m_b) \bar{\Gamma}_2 (\not p_a + m_a)\right)$$

We then use trace theorems[28] to carry out the matrix multiplications and obtain $\langle | \mathcal{M} |^2 \rangle$ for $e^- \mu^- \to e^- \mu^-$

$$\langle | \mathcal{M} |^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_3 \cdot p_2) - m_\mu^2 (p_1 \cdot p_3) - m_e^2 (p_2 \cdot p_4) + 2 m_e^2 m_\mu^2\right]$$

Using this result we can now calculate cross sections. For example, let us take a "heavy" muon with mass M at rest, e.g. a proton[29], where the recoil of M can be neglected. In this case

$$p_1 = (E, \vec{p}_1) \qquad p_2 = (M, 0) \qquad p_3 = (E, \vec{p}_3) \qquad p_4 = (M, 0)$$

where E is the energy of the electron. Using that the momenta of the incoming and scattered electron have equal magnitude $p = |\vec{p_1}| = |\vec{p_3}|$ and the scattering angle θ, we can calculate the kinematics of the reaction.

$$\begin{aligned} (p_1 - p_3)^2 &= -2p^2(1 - \cos\theta) = -4p^2 \sin^2\frac{\theta}{2} \\ (p_1 \cdot p_3) &= m^2 + 2p^2 \sin^2\frac{\theta}{2} \\ (p_1 \cdot p_2) &= (p_3 \cdot p_4) = ME \\ (p_2 \cdot p_4) &= M^2 \end{aligned}$$

[28] Trace theorems are listed in many books, e.g. *Introduction to Elementary Particle Physics* by D. Griffiths.

[29] This assumes that the proton is a point-like spin $\frac{1}{2}$ particle.

The scattering cross section in the laboratory frame is given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 M^2}\langle \mid \mathcal{M} \mid^2 \rangle$$

and we obtain for the differential cross section (Equation 54)

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4p^4 \sin^4\frac{\theta}{2}}[F(q^2)]^2 \left(m^2 + p^2\cos^2\frac{\theta}{2}\right) \tag{176}$$

This is the *Mott* formula which describes the differential cross section for electron-proton scattering. We have introduced the form factor $F(q^2)$ which treats the effect of finite size of the proton. For point-like particles $F(q^2) = 1$. In the non-relativistic limit the Mott formula reduces to the *Rutherford* cross section

$$\frac{d\sigma}{d\Omega} = \frac{m^2\alpha^2}{4p^4 \sin^4\frac{\theta}{2}} \tag{177}$$

In the extreme relativistic limit, mass terms can be neglected and the matrix element squared is

$$\langle \mid \mathcal{M} \mid^2 \rangle = 2e^4 \frac{s^2 + u^2}{t^2} \tag{178}$$

9.10 $e^+e^- \to \mu^+\mu^-$

The process $e^+e^- \to \mu^+\mu^-$ proceeds through the annihilation of the electron and positron into a virtual photon. Using the Feynman rules we obtain the following matrix element

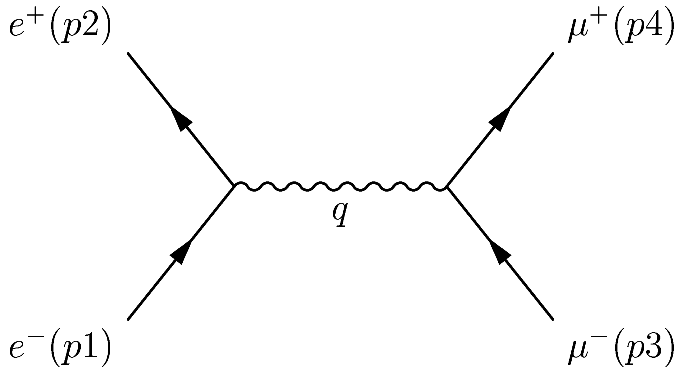

Figure 42: $e^+e^- \to \mu^+\mu^-$

$$\mathcal{M} = \frac{-e^2}{(p_1+p_2)^2}\left(\bar{v}_2\gamma^\mu u_1\right)\left(\bar{u}_3\gamma_\mu v_4\right)$$

For extreme relativistic energies the electron and muon mass can be neglected. There are 16 possible helicity configurations but not all of these contribute to the unpolarised cross section.

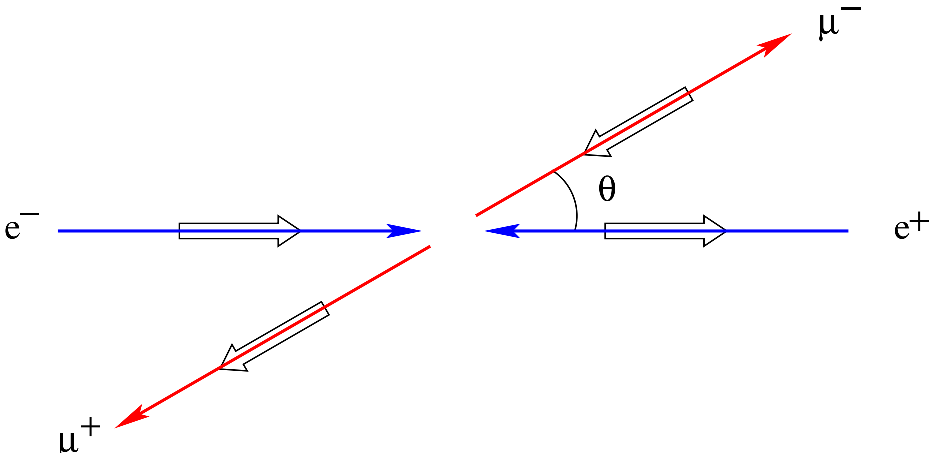

Figure 43: Allowed helicity configuration for $e^+e^- \to \mu^+\mu^-$.

For example if the e^- is right-handed, $u^{(1)}(p_1)$, the e^+ must be left-handed, $\bar{v}^{(1)}(p_2)$ to have $\mathcal{M}$ non-zero. Similarly, if the μ^- is left-handed, the μ^+ must be right-handed, as shown in Figure 43. In total 4 helicity configurations are allowed. Since at high energy the helicity also likes to be preserved at each vertex $\mathcal{M}$ depends on the scattering angle

Helicity				
e^+	e^-	μ^+	μ^-	$\langle \mid \mathcal{M} \mid^2 \rangle$
1	-1	1	-1	$e^4(1+\cos\theta)^2$
1	-1	-1	1	$e^4(1-\cos\theta)^2$
-1	1	1	-1	$e^4(1-\cos\theta)^2$
-1	1	-1	1	$e^4(1+\cos\theta)^2$

Hence, the total unpolarised matrix element squared and the differential cross section are

$$\langle \mid \mathcal{M} \mid^2 \rangle = e^4\left(1+\cos^2\theta\right)$$
$$\left.\frac{d\sigma}{d\Omega}\right|_{\mathrm{CoM}} = \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right)$$

The total cross section for $e^+e^- \to \mu^+\mu^-$ is

$$\sigma = \int \left.\frac{d\sigma}{d\Omega}\right|_{\mathrm{CoM}} d\Omega = \frac{4\pi\alpha^2}{3s} \tag{179}$$

9.11 Higher Order Corrections

This simple result is modified in QED by higher order corrections. There are four next to leading order diagrams.

- The vertex correction:
- The vacuum polarisation or "bubble" diagram:

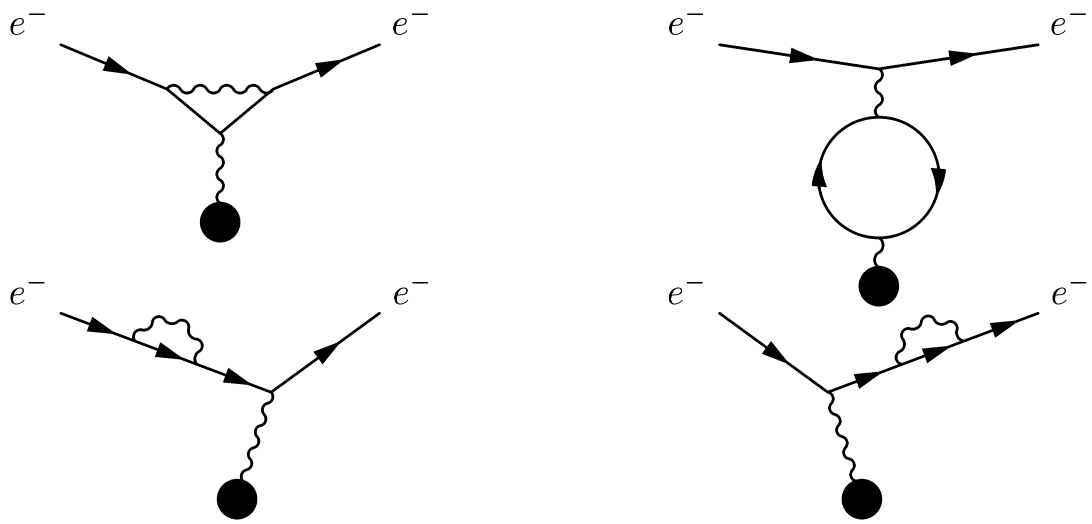

Figure 44: Higher order diagrams

- Two diagrams in which the fermion lines are "dressed":

The higher order diagrams are all smaller than the basic diagram by a factor of α^2. It is possible to continue the calculation to even higher order by adding more photon and fermion lines. The gyromagnetic ratio, g-2, for an electron is measured very precisely:

$$\left.\frac{g-2}{2}\right|_e = 0.0011596521869(41) \qquad \left.\frac{g-2}{2}\right|_\mu = 0.0011659160(6) \tag{180}$$

compared to the theoretical prediction from Quantum Electrodynamics:

$$\left.\frac{g-2}{2}\right|_e = 0.00115965213(3) \qquad \left.\frac{g-2}{2}\right|_\mu = 0.0011659203(20) \tag{181}$$

This is one of the most precise measurements and one of the most accurate tests of a theory in the whole of physics.