8 Quark Model Hadrons

In this chapter we discuss the hypothesis that hadrons are built from fundamental fermions called *quarks*. In total six flavours of quarks were introduced in the 1960s and 1970s. We will discuss in detail their properties, and how they can be described by symmetry groups which then lead to invariance principles such as isospin. We will also describe the interactions between quarks and introduce the concept of *colour*.

8.1 u, d and s Quarks

In 1961 M. Gell-Mann introduced the *Eightfold Way* in which baryons and mesons were arranged in geometrical patterns according to their charge and strangeness[22]. The Eightfold Way provided an organisational structure to classify the hadrons. It already made use of the symmetry group $SU(3)$. Since it gives the same results as the quark model we will not discuss it any further. An understanding of why the Eightfold Way worked so well came already in 1964, when M. Gell-Mann and G. Zweig independently proposed the hypothesis that all hadrons are composed of more elementary constituents, which Gell-Mann called **quarks**. The three quarks consist of a strangeness $S = 0$ and isospin $I = \frac{1}{2}$ doublet, labelled u and d, standing for isospin "**up**" and "**down**", respectively. The third quark is a $S = -1$ isosinglet, labelled s for **strange**. The assignments u, d, and s are called the **flavour** of the quark. Baryons are bound states of three quarks, qqq, and mesons are composed of a quark and an antiquark, $q\bar{q}$, respectively. At the time, three flavours of quarks plus the corresponding antiquarks $\bar{u}$, $\bar{d}$, and $\bar{s}$ were needed to explain the spectrum of observed hadrons.

Before we discuss the quark model with three flavours, let us have a closer look at the u and d quarks. The lightest baryons, p, n, and mesons, π, are composed of u and d quarks only. The u and d quarks are grouped and into a isospin $I = \frac{1}{2}$ multiplet, and the antiquarks $\bar{u}$ and $\bar{d}$ into another $I = \frac{1}{2}$ doublet.

$$\begin{aligned} u &= \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle & \bar{d} &= -\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle \\ d &= \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle & \bar{u} &= \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle \end{aligned} \tag{114}$$

Note the sign convention for the $\bar{d}$ quark. These quark and antiquark doublets transform in exactly the same way as a spin $s = \frac{1}{2}$ doublet and we call this a $SU(2)$ **flavour symmetry**.

We can combine these doublets just like any other two isospin multiplets using the rules of angular momentum addition. If we combine a (u, d) quark doublet with a $(-\bar{d}, \bar{u})$ antiquark doublet we obtain a $I = 1$ triplet and $I = 0$ singlet of $q\bar{q}$ mesons:

[22] Independently, Ne'eman proposed a similar scheme.

$$\begin{aligned} \pi^+ &: & |1,1\rangle &= -u\bar{d} & & \\ \pi^0 &: & |1,0\rangle &= \sqrt{\tfrac{1}{2}}(u\bar{u} - d\bar{d}) & &\text{triplet} \\ \pi^- &: & |1,-1\rangle &= d\bar{u} & & \\ X^0 &: & |0,0\rangle &= \sqrt{\tfrac{1}{2}}(u\bar{u} + d\bar{d}) & &\text{singlet} \end{aligned} \tag{115}$$

The triplet is then identified with the three pions. In this way, we can construct isospin multiplets for non-strange mesons and baryons, starting from the constituent u and d quarks. In other words, we have a quark-model "explanation" for isospin. The $SU(2)$ isospin symmetry works very well. This is essentially due to the near-equal mass of the u and d quark constituents.

How can this very successful model based on two quarks be extended to incorporate strangeness? An additional s quark is needed, leading a larger symmetry group, $SU(3)$.

8.2 SU(3) Flavour Symmetry

The u and d quarks, which have similar mass, are grouped into a isospin multiplet, whereas the s quark and, as we will see later, the c, b, and t quark are isosinglets. Their masses are quite different from each other. However, the s quark mass is not too dissimilar from the effective mass of the u and d quark. In the quark model of 1964, Gell-Mann and Zweig introduced an approximate $SU(3)$ **flavour symmetry.** The three quarks u, d, and s form a triplet which is the fundamental representation of the group. The strong force is invariant under the rotations

$$U = e^{-i\vec{\theta}\cdot\vec{T}} \qquad \text{in} \qquad \begin{pmatrix} u \\ d \\ s \end{pmatrix} \qquad \text{space.}$$

The group $SU(3)$ is formed by a set of unitary 3×3 matrices with determinant detU = 1. These matrices have 9 complex elements, but 9 unitarity constraints and detU = 1 reduces the number of real free parameters to 8. Thus the group $SU(3)$ has 8 *generators* which may be taken as any set of 8 linearly independent hermitian traceless matrices with

$$[\lambda_i, \lambda_j] = 2i f_{ijk}\lambda_k \qquad\qquad T_i = \tfrac{1}{2}\lambda_i \tag{116}$$

where f_{ijk} are the $SU(3)$ structure constants which, for example, are given in Halzen and Martin, *Quarks and Leptons*, p.44. The generators matrices are usually labelled λ_i:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{117}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The matrices λ_1, λ_2, and λ_3 have the Pauli matrices as the top left 2×2 sub-matrices. As a consequence, the group $SU(2)$ is a subgroup of $SU(3)$, and two of the λ_i matrices are simultaneously diagonal. This number is called the *rank* of the group, so that $SU(3)$ has rank 2 and $SU(2)$ has rank 1. It follows that there are 2 operators with simultaneous eigenstates

$$I_3 = \tfrac{1}{2}\lambda_3 \qquad \text{and} \qquad \mathrm{Y} = \frac{1}{\sqrt{3}}\lambda_8 \tag{118}$$

and corresponding eigenvalues, additive quantum numbers, **Isospin** I_3 and **Hypercharge** Y. Operating on $u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ quarks we obtain the eigenvalues for I_3 and Y

$$\begin{array}{lll} I_3 u = +\frac{1}{2}u & I_3 d = -\frac{1}{2}d & I_3 s = 0 \\ Y u = +\frac{1}{3}u & Y d = +\frac{1}{3}d & Y s = -\frac{2}{3}s \end{array}$$

The choice of hypercharge rather than strangeness S has no physical significance. They are equivalent and related via

$$Y = S + B \tag{119}$$

where B is the baryon number. Note that all quarks are assigned $B = \frac{1}{3}$ which guarantees that $B = 1$ for baryons and $B = 0$ for mesons, respectively. We can rewrite the Gell-Mann Nishijima formula (Equation 112) in terms of hypercharge

$$Q = I_3 + \tfrac{1}{2}\,Y \tag{120}$$

to determine the charges q_u, q_d and q_s of the quarks. We obtain

$$q_u = +2/3 \qquad \text{and} \qquad \mathrm{q_d} = q_s = -1/3 \tag{121}$$

The quark charges are rational fractions of the electron charge[23]. Despite a huge experimental effort no fractional quark charge has ever been measured. Quarks are confined within hadrons.

Since there are two quantum numbers, isospin and hypercharge, graphical represenations of $SU(3)$ multiplets are 2-dimensional. In Figure 19, we plot the fundamental triplet, 3, and the fundamental anti-triplet, $\bar{3}$, as a function of Y and I_3 where 3 and $\bar{3}$ symbolically denote the

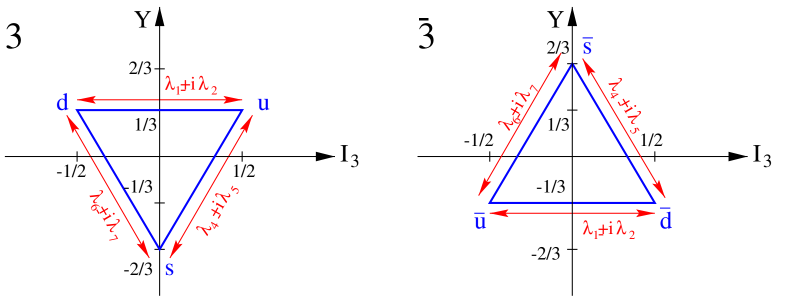


Figure 19: $SU(3)$ quark and antiquark multiplets.

dimension of these representations of $SU(3)$. Note that the quark and antiquark triplets are different, $3 \neq \bar{3}$. Also shown are the ladder operators which step from one quark to another.

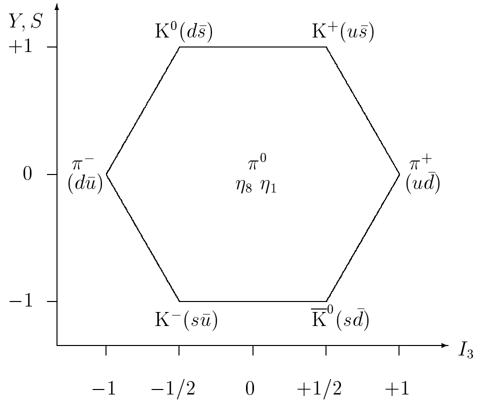
8.3 Quark-Antiquark Mesons

In the quark model, mesons are bound states of a quark and an antiquark, $q\bar{q}$. For the lightest mesons, the angular momentum between the $q\bar{q}$ pair is $L = 0$. If the spin vectors of the two quarks, $\vec{S}_1$ and $\vec{S}_2$, are antiparallel, the spin of the meson is $S = 0$. We obtain pseudoscalar mesons with $J^{PC} = 0^{-+}$. If the two quark spins are parallel, $S = 1$, we have vector mesons with $J^{PC} = 1^{--}$. For two quark flavours, $q = u$ or d, we have already produced the triplet and singlet combinations of $q\bar{q}$ pairs, see Equation 115. For three flavours of quarks, u, d, and s, there are nine possible $q\bar{q}$ combinations. The meson multiplet can be obtained geometrically by superimposing the antiquark triplet on top of every quark in the triplet (see Figure 19). The resulting multiplet structure is shown in Figure 20 for pseudoscalar mesons.

The quark model explains naturally that kaons are grouped in two isospin $I = \frac{1}{2}$ doublets, a particle doublet with $Y = S = 1$ and an antiparticle doublet with $Y = S = -1$.

$$K^+ = u\bar{s} \qquad K^0 = d\bar{s} \qquad \bar{K}^0 = -s\bar{d} \qquad K^- = s\bar{u}$$

[23]The fact that quark charges were fractional was a feature which made physicists, including Gell-Mann, extremely reluctant in accepting the physical consequences of the quark model. At the time, quarks were considered merely mathematical constructs.



Pseudoscalar mesons
$J^{PC} = 0^{-+}$

$\pi^0 = (d\bar{d} - u\bar{u})/\sqrt{2}$
$\eta_8 = (d\bar{d} + u\bar{u} - 2s\bar{s})/\sqrt{6}$
$\eta_1 = (d\bar{d} + u\bar{u} + s\bar{s})/\sqrt{3}$

Figure 20: The $q\bar{q}$ pseudoscalar meson nonet as a function of hypercharge versus third component of isospin I_3.

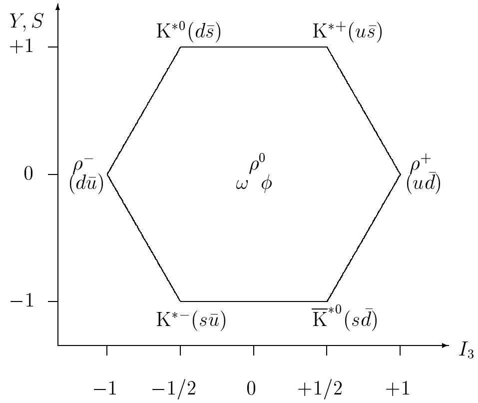
Three of the nine $q\bar{q}$ states are at the centre of the nonet and have $I_3 = Y = 0$. We have already identified one of these states as the neutral pion, π^0, see Equation 115. A second state η_8 can be found which is orthogonal to the π^0 and has unit norm. A generalisation of the $SU(2)$ singlet state (Equation 115) to include the s quark also yields a singlet state η_1 which is unaffected by $SU(3)$ transformations. The combination of u, d, s quarks with $\bar{u}$, $\bar{d}$, $\bar{s}$ quarks produces an *octet* and a *singlet* of states. In the language of group theory, see Equation 84, we have the decomposition

$$3 \otimes \bar{3} = 8 \oplus 1 \tag{122}$$

and the central states with $I_3 = Y = 0$ are summarised below.

$$\begin{array}{ll} \text{Octet} & \pi^0 = \frac{1}{\sqrt{2}}\left(u\bar{u} - d\bar{d}\right) \\ \text{Octet} & \eta_8 = \frac{1}{\sqrt{6}}\left(u\bar{u} + d\bar{d} - 2s\bar{s}\right) \\ \text{Singlet} & \eta_1 = \frac{1}{\sqrt{3}}\left(u\bar{u} + d\bar{d} + s\bar{s}\right) \end{array} \tag{123}$$

The vector meson nonet obeys the same symmetry properties ($SU(3)$) and decomposes accordingly, as is shown in Figure 21.



Vector mesons
$J^{PC} = 1^{--}$

$\rho^0 = (d\bar{d} - u\bar{u})/\sqrt{2}$
$\omega = (d\bar{d} + u\bar{u})/\sqrt{2}$
$\phi = -s\bar{s}$

Figure 21: The $q\bar{q}$ vector meson nonet as a function of hypercharge versus third component of isospin I_3.

8.4 Three Quark Baryons

When discussing baryons we will make use of the **Spin-Statistics theorem** for total wavefunctions. Under the interchange of two identical particles 1 and 2 the probability $|\psi|^2$ must stay the same, since the particles are indistinguishable. It follows that ψ is either symmetric or antisymmetric:

$$|\psi(1,2)|^2 = |\psi(2,1)|^2 \qquad \Longrightarrow \qquad \psi(1,2) \to \pm\psi(2,1)\ .$$

The spin-statistics theorem is a fundamental principle of quantum field theory which states that the total wavefunction is symmetric under the exchange of identical bosons and antisymmetric under the exchange of identical fermions, respectively.

$$\begin{array}{llll} \text{Bosons}: & S = 0, 1, 2, ... & \psi(1,2) = +\psi(2,1) & \text{symmetric} \\ \text{Fermions}: & S = \frac{1}{2}, \frac{3}{2}, ... & \psi(1,2) = -\psi(2,1) & \text{antisymmetric} \end{array} \tag{124}$$

It follows that the total wavefunction for baryons must be antisymmetric. A direct consequence of the spin-statistics theorem is the **Pauli exclusion principle**: Two or more identical fermions cannot occupy the same quantum state. This gives rise to the atomic electron structure. Note that many bosons can be in the same quantum state; an example of this is the laser.

There are 27 possible ways to combine the three light quarks. The quarks are triplets of $SU(3)$.

Combining three triplets (3) gives the following decomposition[24]:

$$3 \otimes 3 \otimes 3 = (6 \oplus \bar{3}) \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1 \tag{125}$$

where we again have used the notation introduced in Equation 84. We obtain a decuplet, two octets and a singlet. The properties of the wavefunctions for the different multiplets under the exchange of two quarks within a baryon are as follows

$$\begin{array}{llll} \text{Decuplet} & \psi_S & \text{symmetric} & \text{e.g. } uuu, (udu + duu + uud)/\sqrt{3} \\ \text{Octet} & \psi_{12} & \text{mixed} - \text{symmetric} & \text{e.g. } (ud - du)u/\sqrt{2} \\ \text{Octet} & \psi_{23} & \text{mixed} - \text{symmetric} & \text{e.g. } u(ud - du)/\sqrt{2} \\ \text{Singlet} & \psi_A & \text{antisymmetric} & (uds - usd + dsu - dus + sdu - sdu)/\sqrt{6} \end{array} \tag{126}$$

There is a third octet with wavefunction ψ_{13} which is antisymmetric under the exchange of quarks 1 and 3 but this is not independent of ψ_{12} and ψ_{23}.

We also need to combine the spins of the three quarks. Figure 14 illustrates how, using Clebsch-Gordan rules, to add the spins of three quarks. We obtain the following baryon spin compositions

$$2 \otimes 2 \otimes 2 = (3 \oplus 1) \otimes 2 = 4 \oplus 2 \oplus 2$$

We obtain a symmetric $s = \frac{3}{2}$ quadruplet and 2 mixed-symmetric $s = \frac{1}{2}$ doublets which are antisymmetric under the exchange of the spins of either quarks 1 and 2 or quarks 2 and 3, respectively.

8.5 SU(3) Colour Symmetry

When the quark model was introduced, some physicists objected to it for a good theoretical reason. Let us explain why. The total baryon wavefunction must be antisymmetric. It consists of different parts, a spatial component, plus spin and flavour contribution.

$$\psi(\text{total}) = \psi(\text{space})\psi(\text{spin})\psi(\text{flavour})$$

Now according to the quark model, the Δ^{++} resonance consists of three identical u quarks with total angular momentum $L = 0$ and a symmetric spin-flavour wavefunction

$$\left|\Delta^{++}, J_3 = \tfrac{3}{2}\right\rangle = |u\uparrow\rangle\,|u\uparrow\rangle\,|u\uparrow\rangle$$

Since $L = 0$, it follows that ψ_{space} is also symmetric. The Δ^{++} seems to have a symmetric total wavefunction which would violate the Pauli exclusion principle. The same argument can also be made for the ddd (Δ^-) and sss (Ω^-) particle states.

[24]Derivations of decompositions of $SU(3)$ multiplets are outside the scope of this course and won't be examined. The consequences, however, are important and must be understood.

A solution to this dilemma was proposed by Greenberg in 1964. He suggested that quarks come not only in different flavours u, d, s, but that each quark occurs in three different internal states which are called **colours**, say red r, green g and blue b. All hadrons must be colourless. Baryons consist of one quark of each of the three colours r, g, b. Thus, baryons have an equal mixture of r, g, and b colour which adds up to zero net colour. In mesons, the colour of the quark is cancelled by the anti-colour of the antiquark. Of course, colour is just a name and has no physical connection with the colours as we know them.

This hypothesis solved the Δ^{++} puzzle, but the introduction of yet another internal property for quarks must have sounded farfetched at the time. But nothing beats success like success and today we have ample experimental confirmation for the colour hypothesis. As an example, colour provides a natural explanation for the fact that no particle consisting of qq or $qqqq$ bound states has ever been observed and why no free quarks q have been detected.

The three colours are actually the three **colour charges** of the strong interaction force for the quarks in much the same way as their electric charges are specified by $\pm\frac{2}{3}e$ and $\pm\frac{1}{3}e$. The strong interactions are invariant under the interchange of colour assignments to quarks, thus we have a symmetry. The three colour charges (r, g, b) form a triplet under an exact $SU(3)$ symmetry. The colour wavefunction of hadrons are exactly analogous to the flavour wavefunctions when making the substitutions

$$u \to r \qquad\qquad d \to g \qquad\qquad s \to b$$

Since we do not observe the colour states of hadrons, only *colour singlet* states are observed in Nature. For mesons this state (see Equation 123) is a symmetric colour singlet

$$\frac{1}{\sqrt{3}}\,(r\bar{r} + g\bar{g} + g\bar{g}) \tag{127}$$

whereas for baryons the colour wavefunction chosen by Nature is an antisymmetric colour singlet (Equation 126)

$$(rgb - rbg + gbr - grb + brg - bgr)/\sqrt{6}\,. \tag{128}$$

By adding ψ(colour), the total wavefunction for $L = 0$ baryons is indeed antisymmetric if it is required that the product of ψ(spin) $\times$ ψ(flavour) is symmetric.

$$\psi(\text{total}) = \psi(\text{space})\psi(\text{spin})\psi(\text{flavour})\psi(\text{colour}) \tag{129}$$

$$\begin{array}{llll} \psi(\text{space}): & \text{symmetric} & \psi(\text{colour}): & \text{antisymmetric} \\ \psi(\text{spin}) \quad \times & \psi(\text{flavour}): & \text{symmetric} & \end{array}$$

In practice the colour wavefunctions are usually omitted. All that needs to be remembered is that the colour wavefunction is antisymmetric for baryons and symmetric for mesons, respectively.

8.6 Baryon Multiplets

The baryons in the decuplet have the symmetric spin-flavour wavefunction

$$\psi(\text{spin, flavour}) = \psi_S(\text{spin s} = \tfrac{3}{2})\psi_S(\text{flavour})$$

These have $L = 0$ and spin-parity $J^P = \frac{3}{2}^+$. In Figure 22 we plot quark contents, and the hypercharge Y of the decuplet baryons versus their isospin I_3. Note that the indicated quark contents are a shorthand notation for properly symmetrised expressions, e.g. duu stands for $(udu + duu + uud)/\sqrt{3}$, similarly dus stands for $(dus + uds + sud + sdu + dsu + usd)/\sqrt{6}$, and so on. The decuplet consists of a $S = 0$, isospin $I = \frac{3}{2}$ quadruplet, the already familiar $\Delta(1232)$; a

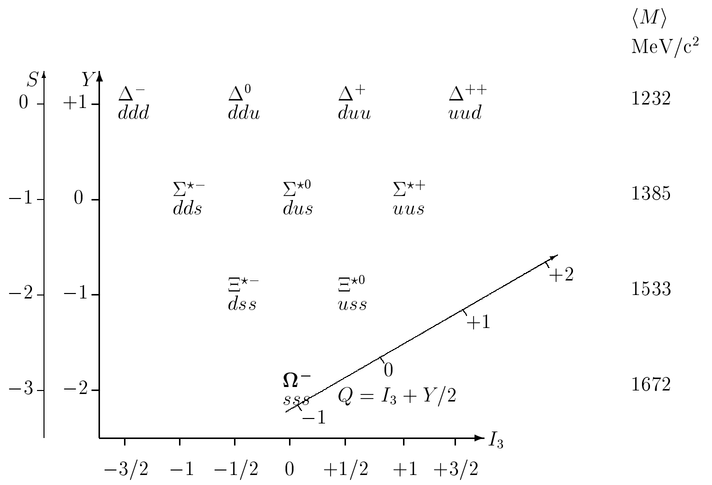


Figure 22: The baryon decuplet as a function of hypercharge Y versus the third component of isospin I_3. Also shown are the strangeness S and electric charge Q.

$S = -1$, $I = 1$ triplet, the Σ^*; a $S = -2$, $I = \frac{1}{2}$ doublet, the Ξ^*; and a $S = -3$, $I = 0$ singlet, the Ω^-. The members of a isospin multiplet can be created by applying the ladder operator $I_\pm$ which exchanges d and u quarks. Similarly successive rows are created by applying the ladder operator $\lambda_4 \pm i\lambda_5$ which replaces u with s quarks.

The members of each isospin multiplet have essentially the same mass within a few MeV, as is expected for electromagnetic mass splittings. The average mass within a multiplet, $\langle M \rangle$, is also given in Figure 22. The mass splittings between the different isospin multiplets which differ in strangeness (or hypercharge) obey a simple mass relation $\langle M \rangle = M_0 + YM_1$.

When the decuplet was first proposed, the state with three s quarks, the Ω^- had not yet been observed. However, the quark model predicted its mass and most likely decay channels. It was a first success for the $SU(3)$ flavour symmetry when the Ω^- particle was discovered in 1964, see Figure 23.

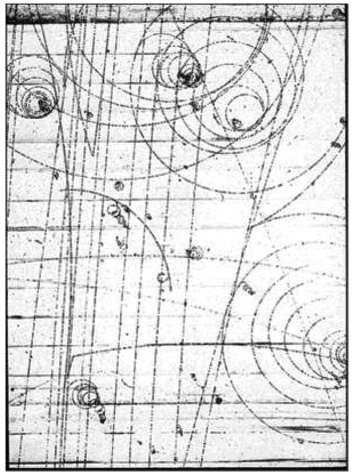


Figure 23: A bubble chamber picture of the first Ω^- event. The chain of reactions is: $\mathrm{K^-p} \longrightarrow \Omega^-\mathrm{K^+K^0}$ followed by $\Omega^- \longrightarrow \Xi^0\pi^-$, $\Xi^0 \longrightarrow \Lambda\pi^0$, $\Lambda \longrightarrow \mathrm{p}\pi^-$ and $\pi^0 \longrightarrow 2\gamma \longrightarrow 2(\mathrm{e^+e^-})$.

The construction of the baryon octet with spin-parity $J^P = \frac{1}{2}^+$ needs a little care since mixed symmetric flavour and spin wavefunctions must be combined. We obtain a properly normalised sum of the different octet wavefunction (Equation 126) each multiplied by the corresponding spin wavefunction:

$$\begin{aligned}\psi(\text{spin, flavour}) = \frac{\sqrt{2}}{3}\big[&\psi_{12}(\text{spin})\psi_{12}(\text{flavour}) \\ &+ \psi_{23}(\text{spin})\psi_{23}(\text{flavour}) \\ &+ \psi_{31}(\text{spin})\psi_{31}(\text{flavour})\big]\end{aligned} \tag{130}$$

In Figure 24, we plot quark contents, and the hypercharge Y of the octet baryons versus their isospin I_3. The indicated quark contents are again a shorthand notation for properly symmetrised expressions. The baryon octet[25] contains the lightest baryons which we have already encountered: a $S = 0$, $I = \frac{1}{2}$ nucleon (p, n) doublet; a $S = -1$, $I = 1$ triplet, the Σ; a $S = -1$, $I = 0$ singlet,

[25] It is the baryon and meson octets prompted Gell-Mann to coin the name “Eighfold Way”.

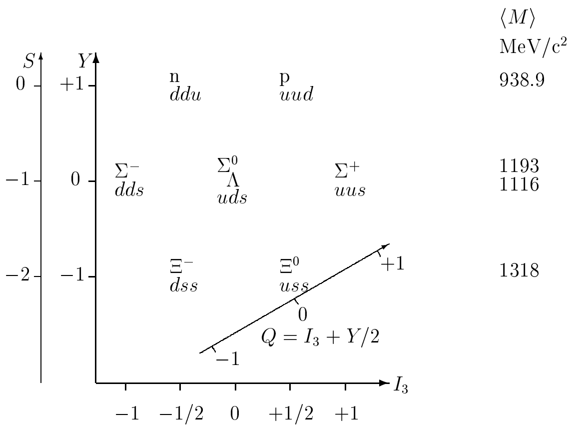


Figure 24: The baryon octet as a function of hypercharge Y versus the third component of isospin I_3. Also shown are the strangeness S and electric charge Q.

the Λ; and a $S = -2$, $I = \frac{1}{2}$ doublet, the cascade particle Ξ. Note that the antibaryons ($\bar{p}$, $\bar{n}$, $\bar{\Sigma}$, $\bar{\Lambda}$ and $\bar{\Xi}$) also form an SU(3) multiplet.

8.7 Hadron and Quark Masses

We now attempt to explain the observed hadron mass spectrum. There is evidence that the *bare* mass of the u and d quarks, m_u and m_d is only a few MeV, and that the mass of the s quark, m_s is about 150 MeV. However their *effective* or *constituent* mass within hadrons is much greater. The precise value varies. For example, it is slightly higher in baryons when compared with mesons. The effective mass of a quark within a hadron exceeds the bare mass by about 350 MeV.

If $SU(3)$ symmetry were exact, all the particles in a given multiplet, for example in the meson octet should have the same mass. This is clearly not the case. The mass of kaons is about three times the mass of pions. If it were not for the different masses, the strong interaction between quarks would be exactly the same for all flavours. But the s quark is much heavier than the u and d quark, $m_s > m_d \approx m_u$. This is why isospin is such a good symmetry, and why the $SU(3)$ flavour symmetry breaks down when exchanging an s quark with a u or d quark, respectively. For the reminder of this section we assume perfect isospin invariance and set $m_d = m_u = m_n$.

Before the introduction of quarks, Gell-Mann and Okubo introduced an empirical mass relation

which takes the $SU(3)$ symmetry breaking into account:

$$\langle M \rangle = M_0 + Y M_1 + M_2 \left[I(I+1) - Y^2/4 \right] \tag{131}$$

As expected this formula depends on the hypercharge Y and on the total isospin I. For the $J^P = \frac{3}{2}^+$ baryon decuplet we find the following relation between isospin and hypercharge $Y = B + S = 2(I-1)$. We can show that this reduces the Gell-Mann-Okubo formula to a linear mass dependence in Y:

$$\langle M \rangle = M_0 + Y M_1$$

which implies that the mass splitting between the different isomultiplets in the baryon decuplet is constant. We find good agreement with the experiment.

$$\begin{aligned} M_\Omega - M_{\Xi^*} &= 153 \text{ MeV} \\ M_{\Xi^*} - M_{\Sigma^*} &= 149 \text{ MeV} \\ M_{\Sigma^*} - M_\Delta &= 139 \text{ MeV} \\ \langle m_s - m_n \rangle &= 147 \text{ MeV} \end{aligned}$$

For the $\frac{1}{2}^+$ baryon octet, a linear mass relation does not work, e.g. it predicts $M_\Lambda = M_\Sigma$ which is clearly not the case. We then use the Gell-Mann-Okubo mass formula (Equation 131) to derive the following mass relation:

$$\begin{array}{ccc} 3M_\Lambda + M_\Sigma & = & 2M_N + 2M_\Xi \\ 4540 \text{ MeV} & & 4514 \text{ MeV} \end{array} \tag{132}$$

This agrees to within 0.6% with the experiment, which is quite impressive.

For the meson octet the same empirical relation as above is used, but it is applied to the square of the meson masses:

$$\begin{array}{ccccc} 3M_\eta^2 + M_\pi^2 & = & 2M_K^2 + 2M_{\bar{K}}^2 & = & 4M_K^2 \\ 0.918 \text{ GeV}^2 & & 0.983 \text{ GeV}^2 & & \end{array} \tag{133}$$

The comparison with experiment gives a 7% discrepancy.

Mass differences in hadron multiplets cannot be simply attributed to the the different quark constituents. The pseudoscalar π^+ and the vector ρ^+ meson have the same quark content ($u\bar{d}$) but differ in mass by more than a factor of five. Both the Λ and Σ^0 are a uds state but their masses differ by 77 MeV. This discrepancy must arise due to the quark-quark interactions. The full theory, quantum chromodynamics will be discussed later. Here we introduce the **chromo-magnetic hyperfine** interaction which is caused by the spin-spin coupling between two quarks. This produces a splitting of the hadron energy levels. The terminology and mathematics is the same as for the the hydrogen atom where each level is split into two very close hyperfine doublets.

In the ground state ($1S$), the transition between the two states gives rise to the famous 1420 MHz (21 cm) line in radio astronomy. For quark interaction the strong couplings are much larger than for the electron and the mass splittings turn out to be large.

The chromomagnetic spin-spin coupling between quarks can be turned into mass formulas for mesons and baryons

$$m(q\bar{q}) = m_1 + m_2 + a\,[\vec{\sigma_1}\cdot\vec{\sigma_2}/m_1 m_2]$$
$$m(q_1 q_2 q_3) = m_1 + m_2 + m_3 + b\left[\sum_{i>j}(\vec{\sigma_i}\cdot\vec{\sigma_j})/m_i m_j\right] \quad (134)$$

Using the relation $\vec{s}_i = \frac{\hbar}{2}\vec{\sigma}_i$, and recalling that the eigenvalue for the operator $\vec{S}^2$ is $S(S+1)$, one obtains

$$\begin{array}{llll} \vec{\sigma_1}\cdot\vec{\sigma_2} = 1 & S = 1 & \sum_{i>j}\vec{\sigma_i}\cdot\vec{\sigma_j} = 3 & S = \frac{3}{2} \\ \vec{\sigma_1}\cdot\vec{\sigma_2} = -3 & S = 0 & \sum_{i>j}\vec{\sigma_i}\cdot\vec{\sigma_j} = -3 & S = \frac{1}{2} \end{array} \quad (135)$$

for the different spin-spin dot products. In Table 12 we give a comparison of predicted and measured baryon masses where we have used the following values for the quark masses:

$$m_n = m_u = m_d = 363 \text{ MeV} \qquad m_s = 538 \text{ MeV} \qquad b = m_n^2\, 50 \text{ MeV}\ . \quad (136)$$

	$\Delta m/b$	Predicted	Measured mass [MeV]
$n,\ p$	$-3/m_n^2$	939	939 (input)
Λ	$-3/m_n^2$	1116	1116 (input)
Σ	$1/m_n^2 - 4/m_n m_s$	1179	1193
Ξ	$1/m_s^2 - 4/m_n m_s$	1327	1318
Δ	$3/m_n^2$	1239	1232
$\Sigma^\star$	$1/m_n^2 + 2/m_n m_s$	1381	1385
$\Xi^\star$	$1/m_s^2 + 2/m_n m_s$	1529	1533
Ω	$3/m_s^2$	1682	1672

Table 12: Predicted and measured baryon octet and decuplet masses in MeV, using the mass of the nucleon and of the Λ as inputs.

8.8 Meson Mixing

For pseudoscalar mesons, the octet states η_8 and the singlet state η_1 have identical quantum numbers. Hence the physical states η and η' are mixed. The mixing is usually written as

$$\begin{pmatrix}\eta\\ \eta'\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\eta_8\\ \eta_1\end{pmatrix}$$

Using the quadratic mass formula (Equation 133) we can calculate the mass M_8 of the η_8 to be

$$M_8^2 = \tfrac{1}{3}(4M_K^2 - M_\pi^2)\ . \quad (137)$$

In this case the energy operator for a particle at rest also uses the square of the mass and can be written as $\langle\psi|\,M^2\,|\psi\rangle$. We define the elements of the mass matrix, M_8^2, M_1^2, and M_{18}^2, as follows:

$$M_8^2 = \langle\eta_8|\,M^2\,|\eta_8\rangle$$
$$M_1^2 = \langle\eta_1|\,M^2\,|\eta_1\rangle$$
$$M_{18}^2 = \langle\eta_8|\,M^2\,|\eta_1\rangle$$

We can now rewrite the masses M_η^2 and $M_{\eta'}^2$ of the physical states η and η'. Using $M_{\eta,\eta'} = \langle\eta|\,M^2\,|\eta'\rangle = 0$ we can eliminate M_1^2 and M_{18}^2 and and obtain the following relation for the mixing angle θ:

$$\tan^2\theta = \left(\frac{M_\eta^2 - M_8^2}{M_8^2 - M_{\eta'}^2}\right) \qquad \Longrightarrow \qquad \theta = -10^\circ\ . \quad (138)$$

The mixing angle between the pseudoscalar mesons η and η' is relatively small.

Due to the same argument the physical vector mesons ϕ and ω are also mixtures of ϕ_8 and ϕ_1. Using the mass relation

$$M_8^2 = \tfrac{1}{3}(4M_{K^*}^2 - M_\rho^2)$$

the mixing angle is found to be close to the value of $\sin\theta = \frac{1}{\sqrt{3}}$. This is called *ideal mixing* and corresponds to

$$\phi = -s\bar{s} \quad (139)$$
$$\omega = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) \quad (140)$$

We find that the ϕ is indeed mostly a $s\bar{s}$ state and the ω a $u\bar{u}, d\bar{d}$ state, when comparing the following measured branching ratios for ϕ and ω mesons:

$$\begin{array}{lll} \phi \;\rightarrow & K^+K^- & 49\% \\ \;\;\rightarrow & K_L^0 K_S^0 & 34\% \\ \;\;\rightarrow & \pi^+\pi^-\pi^0 & 16\% \end{array} \qquad \begin{array}{lll} \omega \;\rightarrow & \pi^+\pi^-\pi^0 & 89\% \\ \;\;\rightarrow & \pi^0\gamma & 9\% \\ \;\;\rightarrow & \pi^+\pi^- & 2\% \end{array}$$

The branching ratio for $\phi \rightarrow K\overline{K}$ is surprisingly large given that this reaction has very little phase space, $M_\phi - 2M_K \approx 25$ MeV when compared to the decay into three pions, $M_\phi - 3m_\pi \approx 600$ MeV. However this can be understood when looking at the Feynman diagrams for the decays shown in Figure 25. In the suppressed decay $\phi \rightarrow \pi^+\pi^-\pi^0$ there are no quark lines that are common to both, the initial and final state whereas in the favoured decay $\phi \rightarrow K\overline{K}$, the quark lines are connected. Thus decays with flavour disconnected quark lines are suppressed. This suppression is known as **OZI rule**, courtesy Okubo-Zweig-Iizuka. In the context of QCD the suppression is due to the coupling constants associated with the exchange of three gluons which are necessary to mediate this decay.

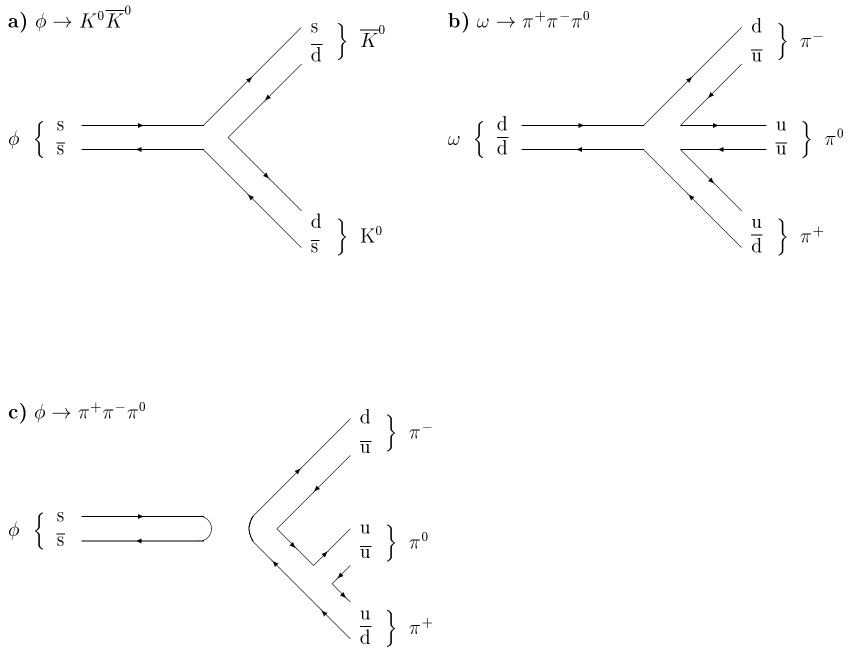


Figure 25: Quark line Feynman diagrams: In **a)** for the flavour connected decay $\phi \longrightarrow K\overline{K}$; in **c)** the flavour disconnected decay $\phi \longrightarrow 3\pi$; in **b)** the decay $\omega \longrightarrow 3\pi$ which is also flavour connected.

8.9 Magnetic Moments

A charged particle with mass m in an orbit with angular momentum L generates a magnetic dipole moment $\vec{\mu} = \frac{q}{2mc}\vec{L}$. In the presence of a magnetic field a magnetic dipole has an interaction energy $H_{\text{int}} = \vec{\mu} \cdot B$. Thus when atoms are placed in a magnetic field, the magnetic moment will cause the spectral lines to split. This is the *Zeeman* effect.

When in 1926 Uhlenbeck and Goudsmit introduced an *intrinsic angular momentum*, spin $s = \frac{1}{2}$, for the electron, it had a consequence, namely that the electron also has an **intrinsic magnetic moment** which will, due to the two possible spin states of an electron, cause a further split of the spectral lines. We write the magnetic dipole moment for the electron as

$$\vec{\mu_e} = -g\frac{e}{2mc}\vec{S} \qquad\qquad \mu_e = -\frac{g}{2}\frac{e\hbar}{2mc} \tag{141}$$

where μ_e is the component along the direction of the spin-up state, for which $s_z = \hbar/2$. Since

the charge of the electron negative, so is μ_e. The gyromagnetic ratio g was measured to have the value $g = 2$. In 1928, Dirac proposed an Equation which was able predict to predict the correct g value for relativistic spin $\frac{1}{2}$ particles.

However, measurements of the magnetic moments of the proton, μ_p, and the neutron, μ_n, gave the following values in units of *the nuclear magneton* μ_N:

$$\mu_p = +2.793\,\mu_N \neq \mu_N \qquad\qquad \mu_N = \frac{e\hbar}{2m_N c}$$

$$\mu_n = -1.913\,\mu_N \neq 0 \qquad ?$$

where $m_N \equiv m_p$ is the mass of the nucleon. These results are clearly incompatible with n or p being pointlike particles. It also indicates that there is a charge distribution inside a neutron. This implies that these n and p are not elementary particles.

The quark model allows us to calculate the magnetic moments of hadrons. If quarks are pointlike particles they should have a magnetic dipole moment $\vec{\mu}$ which for a spin $s = \frac{1}{2}$ particle of charge q and mass m can be written as

$$\vec{\mu} = \frac{q}{mc}\vec{s} \qquad\qquad \mu = \frac{q\hbar}{2mc} \tag{142}$$

For the quarks, we have

$$\mu_u = \frac{2}{3}\frac{e\hbar}{2m_u c} \qquad\qquad \mu_d = -\frac{1}{3}\frac{e\hbar}{2m_d c} \qquad\qquad \mu_s = -\frac{1}{3}\frac{e\hbar}{2m_s c}\ .$$

The magnetic moment for baryons μ_B is given by the vector sum of the individual quark magnetic moments:

$$\vec{\mu} = \vec{\mu_1} + \vec{\mu_2} + \vec{\mu_3}$$

$$\mu_B = \frac{2}{\hbar}\sum_{i=1}^{3}\langle\psi|\,\mu_i\vec{\sigma_i}\,|\psi\rangle$$

Using constituent quark masses $m_u = m_d = m_N/3$ and writing out the fully symmetrised spin wavefunction for a nucleon we obtain for μ_p and μ_n

$$\mu_p = \tfrac{1}{3}(4\mu_u - \mu_d) \qquad\qquad \mu_n = \tfrac{1}{3}(4\mu_d - \mu_u) \tag{143}$$

$$\Longrightarrow \qquad \frac{\mu_n}{\mu_p} = -2/3 \qquad\qquad \frac{\mu_n}{\mu_{p\,\text{exp}}} = -0.6848$$

which is in good agreement with the experimental measurements.

In this way all the magnetic moments of the octet baryons can be calculated. For example, the Λ hyperon is a uds bound state, where the u and d quark form a antisymmetric isosinglet $I = 0$. Thus they must also be in a antisymmetric spin state $S = 0$. Hence the u and d do not contribute to the magnetic moment and we have $\mu_\Lambda = \mu_s$. By fixing μ_p and μ_Λ to their measured values we can calculate the effective quark masses and obtain

$$m_n = m_u = m_d = 336\ \text{MeV} \qquad \text{and} \qquad \text{m}_\text{s} = 509\ \text{MeV}\ . \tag{144}$$

Using these quark masses we can now predict the magnetic moments for all octet baryons. In Table 13 we compare these predictions with the measurements and find agreement at the 10 - 20 % level.

	Magnetic moment	Prediction (μ_N)	Measured
p	$(4\mu_u - \mu_d)/3$	+2.79	+2.793
n	$(4\mu_d - \mu_u)/3$	−1.86	−1.913
Σ^+	$(4\mu_u - \mu_s)/3$	+2.69	+2.46±0.01
Σ^0	$(2\mu_u + 2\mu_d - \mu_d)/3$	+0.82	—
Σ^-	$(4\mu_d - \mu_s)/3$	−1.04	-1.12 ± 0.03
Ξ^0	$(4\mu_s - \mu_u)/3$	−1.44	-1.25 ± 0.01
Ξ^-	$(4\mu_s - \mu_d)/3$	−0.51	-0.651 ± 0.003
Λ	μ_s	−0.61	-0.613 ± 0.004
Ω	$3\mu_s$	−1.83	-2.02 ± 0.05

Table 13: Baryon magnetic moments in units of nuclear magnetons μ_N.

8.10 Charm Quark

In November 1974, two experimental groups simultaneously announced the discovery of a very narrow resonance with a mass of about 3100 MeV. In an experiment at Brookhaven, one group studied the production of e^+e^- pairs in the collision of a proton beam onto a Beryllium target. Another group carried out an experiment at SLAC where they recorded the production rate of hadrons in the annihilation of e^+e^- pairs:

BNL	$p + Be \to e^+e^-X$
SLAC	$e^+e^- \to$ hadrons
	$\to e^+e^-X, \mu^+\mu^-X$

The two results are shown in Figure 26. A clear new resonance had been established.

Soon after a second resonance ψ' was found at a mass of 3700 MeV. These new particles had extremely narrow widths when compared with light quark (u, d, s) resonances such as the ρ which has a width of 150 MeV. The true width of the J/ψ is actually much smaller than the experimental width, it can be determined form the Breit-Wigner formula and gives $\Gamma = 0.087$ MeV. This indicated that these particles cannot be built from the three known quarks. In fact, Glashow, Iliopoulos, and Maiani had postulated in 1970 that an additional quark flavour should exist. This quark would carry a new quantum number, C for **charm**, Like strangeness, the charm flavour would be conserved in strong and electromagnetic interactions. Thus it was postulated that the J/ψ and the ψ' particles consisted of a combination of $c\bar{c}$ quarks, called charmonium. Their masses

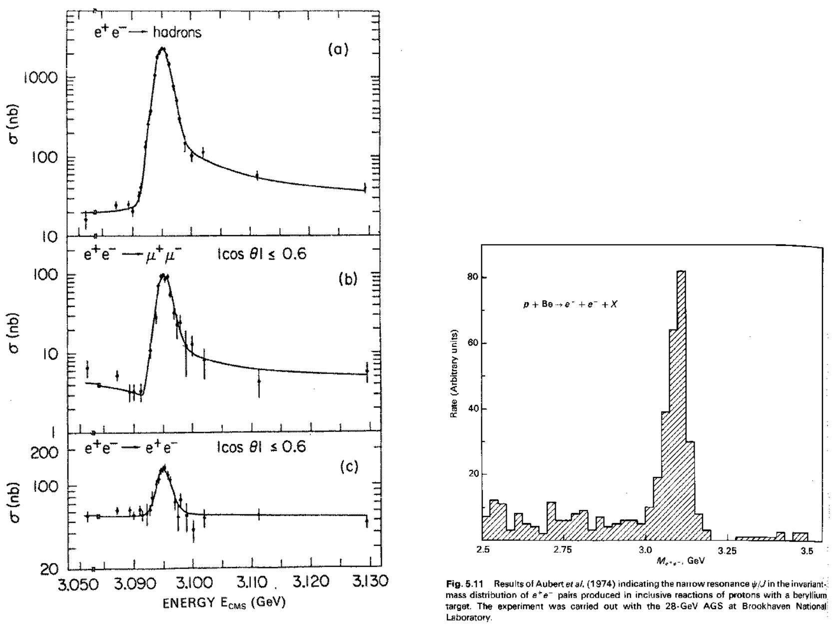


Figure 26: Discovery of the J/ψ meson at Brookhaven (right plot) and at SLAC (left plot).

indicate that the c quark would be much heavier than the known quarks u, d, s and have a mass $m_c \sim 1.4$ GeV.

The spin-parity assignment of the J/ψ and ψ' is $J^{PC} = 1^{--}$. Often their spectroscopic names $J/\psi(1S)$ and $\psi(2S)$ are used. In Figure 27 we show a Feynman diagram of the decay $J/\psi(1S)$ to light hadrons. The decay rates of these charmonia states are suppressed by the OZI rule which explains their narrow width. Subsequently, a third state $\psi(3S)$ was discovered at a mass of 3770

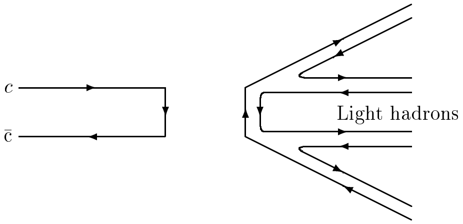


Figure 27: The decay of the 1^{--} J/ψ via three gluons to light hadrons.

MeV. This resonance has a much larger width of 23.6 MeV. It is the first state above the threshold to produce a flavour connected decay into a pair of D mesons containing the charm quark (open charm). The quark content of the best known charm mesons and baryons are:

$$\begin{array}{llll} J^P = 0^- & D^+ = c\bar{d} & D^0 = c\bar{u} & D_s^+ = c\bar{s} \\ J^P = 1^- & D^{*+} = c\bar{d} & D^{*0} = c\bar{u} & D_s^{*+} = c\bar{s} \\ J^P = \frac{1}{2}^+ & \Lambda_c^+ = udc & & \end{array} \quad (145)$$

These meson can be grouped in $SU(4)$ multiplets, see Figure 28, and have 3 simultaneous eigenstates (I_3, Y, C). But $SU(4)$ is obviously a heavy broken symmetry and we do not observe mixing between charm and light-quark states. Nevertheless, the discovery of the charm quark convinced the physicists that quarks really existed.

8.11 Bottom or Beauty Quark

The discovery of the narrow charmonium states was followed in 1977 by the observation of a similar narrow resonances at an experiment at Fermilab. Peaks were found in the invariant mass of $\mu^+\mu^-$ pairs in the region 9.0 - 10.5 GeV, see Figure 29:

$$\text{FNAL} \qquad\qquad p + Be, Cu, Pt \to \mu^+\mu^- X$$

This resonance was also confirmed in the production of hadrons in e^+e^- collisions. In Figure 30 we show the cross section for $e^+e^- \to$ hadrons as a function of the centre-of-mass energy as measured by the CUSB detector at the CESR storage ring. The four Upsilon states $\Upsilon(1S)$, $\Upsilon(2S)$, $\Upsilon(3S)$,

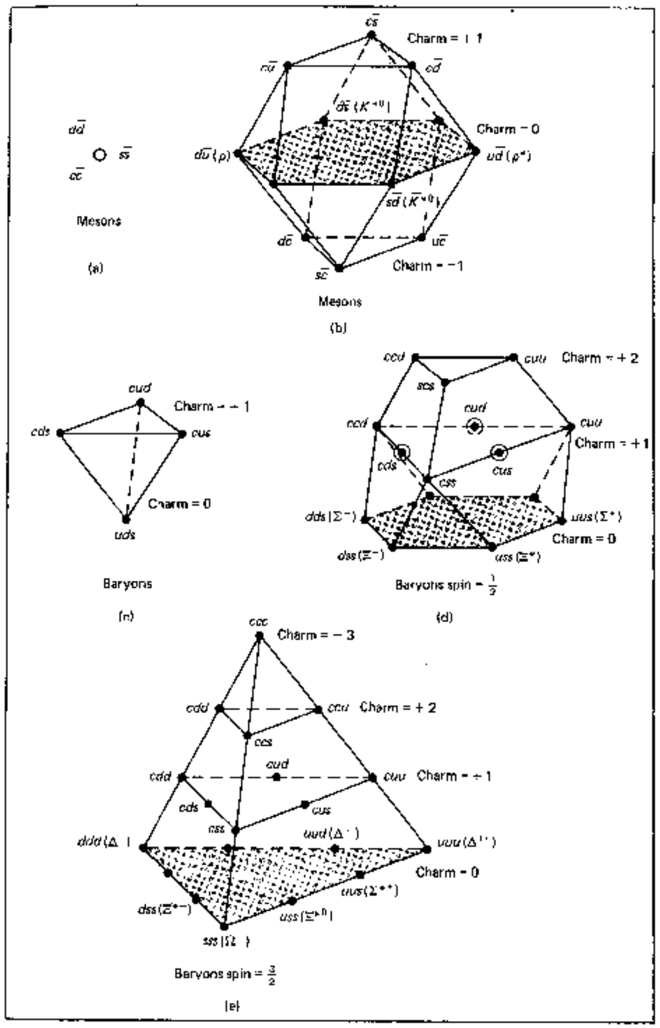


Figure 1.13 Supermultiplets constructed with four quarks. (From "Quarks with Color and Flavor," by S. Glashow. Copyright © Oct. 1975 by Scientific American, Inc. All rights reserved.)

Figure 28: $SU(4)$ Multiplets with four quarks.

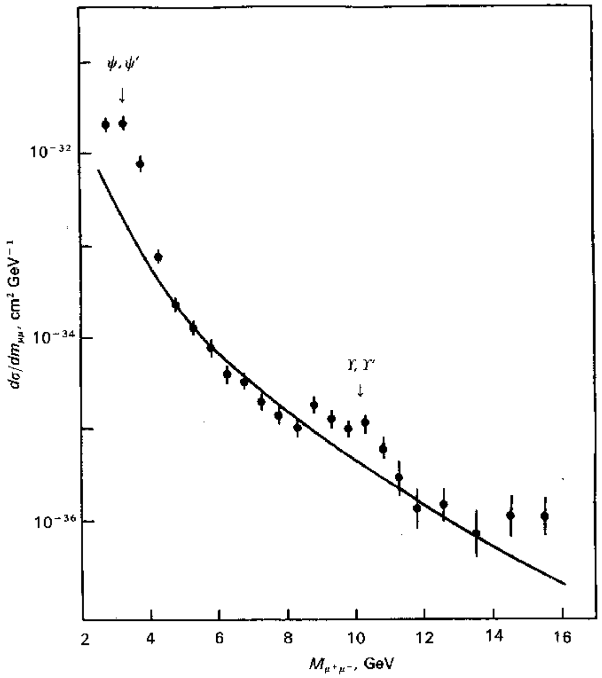


Figure 29: First evidence for the Υ and Υ' resonances.

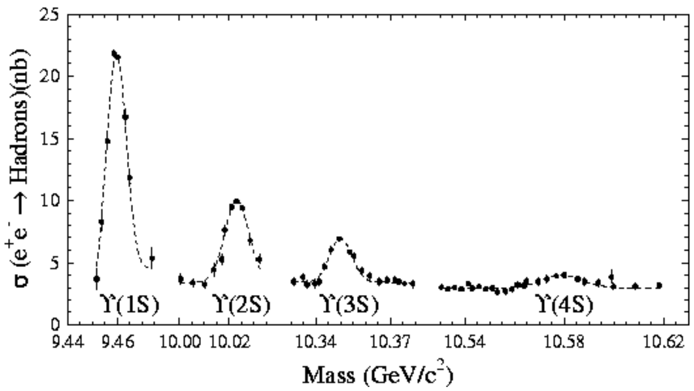


Figure 30: Cross section $\sigma(e^+e^- \to \text{hadrons})$ in the region of the Υ resonances.

and $\Upsilon(4S)$ are clearly visible. These Υ resonances were subsequently attributed to bound states of a pair of even heavier **bottom** quarks[26] which must have a mass $m_b \sim 4.5$ GeV. In Figure 31, we show the spectrum of $b\bar{b}$ states.

Similar considerations as for charmonium apply for the $b\bar{b}$ states. The $\Upsilon(1S)$, $\Upsilon(2S)$, and $\Upsilon(3S)$ resonances have only OZI suppressed strong decay modes and hence a very narrow width. The $\Upsilon(4S)$ resonance has a larger width of about 15 MeV (Figure 30) and its mass is just above the threshold for the production of B mesons. It has the following decay modes:

$$\Upsilon(4S) \to B^+B^- \text{ or } B^0\bar{B}^0$$

The quark content of the lightest beautiful B mesons and baryons are:

$$\begin{array}{llll} J^P = 0^- : & B^+ = u\bar{b} & B^0 = d\bar{b} & B_s^0 = s\bar{b} \\ J^P = 1^- : & B^{*+} = u\bar{b} & B^{*0} = d\bar{b} & B_s^{*0} = s\bar{b} \\ J^P = \frac{1}{2}^+ : & \Lambda_b^0 = udb & & \end{array} \tag{146}$$

8.12 Top Quark

The **top** quark was not discovered until 1995. It is unexpectedly massive with $m_t = 174\ GeV$. It was finally observed at the Fermilab $p\bar{p}$ collider operating at a centre-of-mass energy of 1800 GeV. Figure 32 shows a picture of the CDF detector, one of the two experiments that have made the first observations. Top quarks are produced in only about 1 in 10^{10} of all detected collisions. A

[26] It is regrettable the name *beauty* quark did not prevail.

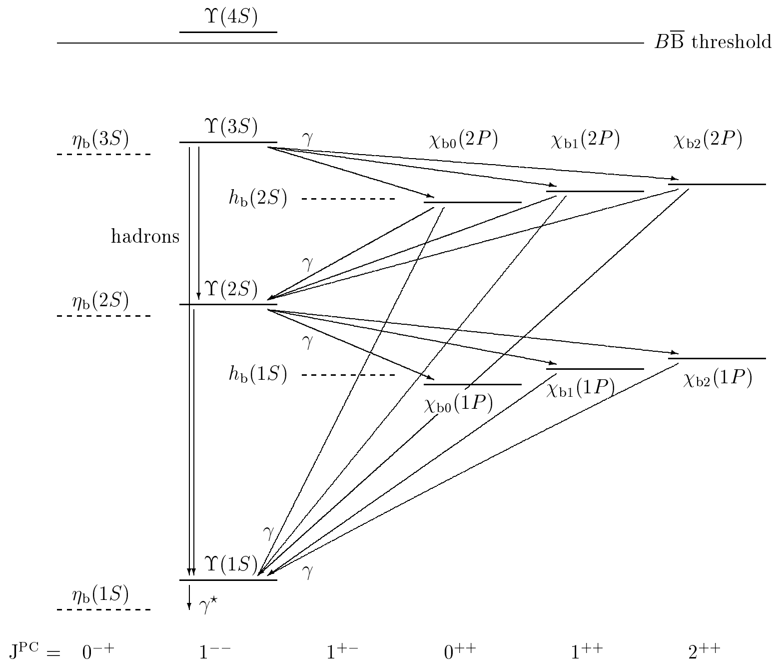


Figure 31: The spectrum of $b\bar{b}$ mesons and their dominant decay modes; shown as dashed lines are states yet to be established.

pairs of $t\bar{t}$ quarks is created, and is expected to decay weakly, predominantly in the decay mode

$$t\bar{t} \rightarrow W^+b + W^-\bar{b} \tag{147}$$

In Figure 33 we show an event picture of a "golden" top event where the two t quarks have produced the following decay chain:

$$t\bar{t} \rightarrow W^+b + W^-\bar{b} \qquad W^+ \rightarrow e^+\nu_e$$
$$b, \bar{b} \rightarrow b - \text{jets 1 \& 4} \qquad W^- \rightarrow q\bar{q} \text{ (jets 2 \& 3)}$$

The e^+ is clearly reconstructed, and the neutrino manifests itself as sizable missing (transverse) energy. In Jet 1 and Jet 4, secondary vertices with a decay length of a few millimetres are found by the CDF Silicon Vertex detector. This is expected for hadrons containing b quarks which have a long lifetime of ~ 1.5 ps.

Figure 32: Top quark discovery: CDF detector.

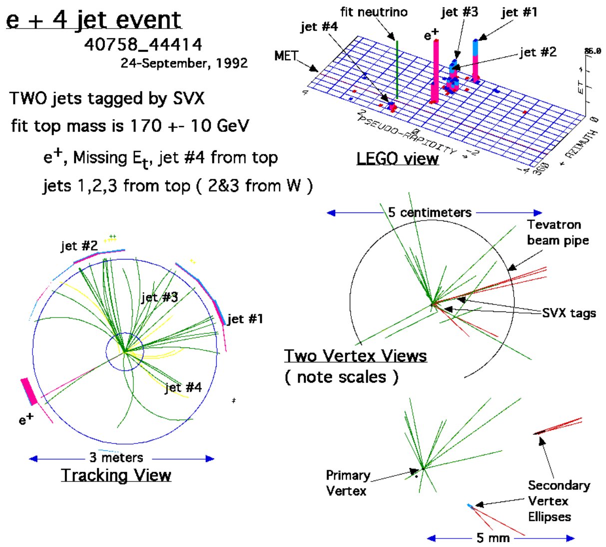


Figure 33: Top quark event taken from the CDF webpage: http://www-cdf.fnal.gov/experiment/events/events.html. If this doesn't convince you, nothing will !