3 Relativistic Kinematics

This Section summarizes the basic principles, notation and terminology of relativistic particles. The formalism of particle physics is a combination of quantum mechanics and special relativity. Observables in particle physics are not only expectation values of quantum mechanical operators but must also be *relativistically invariant*. Thus in the first half of the 20$^{\text{th}}$ century not only special relativity and quantum mechanics were invented, but the two principles were also successfully combined[7]. The kinematics of relativistic particles is discussed in detail.

3.1 Units

Particle physics is both relativistic and quantum mechanical; as a consequence $\hbar$ and c appear everywhere in equations and, when using SI units numerical values of measured quantities or constants often have large (usually negative) powers of ten. The typical scales are:

- The speed of light $c = 299\ 792\ 458\ \text{ms}^{-1}$ and Planck's constant $\hbar = 1.055 \times 10^{-34}$ J s.
- The size of the proton is ~ 1 fm $= 10^{-15}$ m.
- Lifetimes of particles can be as short as $\sim 10^{-23}$ s.
- Charge is measured in units of the charge of the electron $e = 1.602 \times 10^{-19}$ C.
- The mass of the proton is 938.3 MeV/c^2 $= 1.673 \times 10^{-27}$ kg.
- The maximum electron energy in the decay $\mu^- \to e^- \bar{\nu}_e \nu_\mu$ is 52.83 MeV $= 8.463 \times 10^{-12}$ J.

It is therefore convenient to measure energy in units of *electron volts*, defined as the increase in energy when an electron is accelerated through 1 V.

$$1\ \text{eV} = 1.602 \times 10^{-19}\ \text{J}\ . \tag{1}$$

Energies are measured in 1 keV = 10^3 eV, 1 MeV = 10^6 eV, 1 GeV = 10^9 eV or 1 TeV = 10^{12} eV. Momenta are measured in MeV/c or GeV/c and masses in MeV/c^2 or GeV/c^2, respectively. Particle physicists often use a system of **natural units** in which $\hbar = c = 1$. As a consequence, energy, momentum and mass are all expressed in units of MeV, and length ($\hbar c/E$) and time ($\hbar/E$) are in units of MeV^{-1}, respectively. The scales are set by the following relations

$$\hbar c = 197\ \text{MeV fm}, \qquad \text{and} \qquad \hbar = 6.582 \times 10^{-22}\ \text{MeV s}. \tag{2}$$

Hence a length of 1 fm corresponds to a momentum of about 200 MeV/c and a time of 10^{-23} s corresponds to an energy (width) of 66 MeV, respectively. To change from natural units into SI units and to obtain numerical results, it is necessary to restore the required factors of $\hbar$ and c at the end of a calculation using dimensional analysis. Other useful quantities are $(\hbar c)^2 = 0.389$ GeV^2 mbarn, where 1 barn $= 10^{-28}$m^2 and the coupling constant $\alpha = e^2/(4\pi\varepsilon_0\hbar c) = 1/137$.

[7]It has not been possible to combine general relativity and quantum mechanics into a relativistic gravity quantum theory yet. And that is why gravity is not included in unified theories.

3.2 Four-Vectors

Let us look at invariant quantities in the 3-dimensional Euclidean geometry. We know that the length of a vector is **invariant** under rotations; more generally the *scalar product* of two vectors $\vec{a}$ and $\vec{b}$, defined as $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$ is also invariant under rotations. Using the conventions that repeated Roman indices are implicitly summed over from 1 to 3 we can define a rotation as $a'_i = R_{ik}(\theta, \phi) a_k$. We find, using the orthonormality condition for rotations $R^{-1}_{ji} R_{ik} = \delta_{jk}$, that

$$\vec{a}' \cdot \vec{b}' = a'_i b'_i = a_j R^{-1}_{ji} R_{ik} b_k = a_j b_k \delta_{jk} = a_j b_j = \vec{a} \cdot \vec{b} \tag{3}$$

Thus the length l of a 3-vector $l^2 = \vec{x}^2 = \vec{x} \cdot \vec{x} = x_i^2 + x_2^2 + x_2^3$ is an *invariant*. This relation can also be written as a matrix equation:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} ; \qquad x_T = (x_1, x_2, x_3) ; \qquad x^2 = x^T x = x_i^2 + x_2^2 + x_2^3 . \tag{4}$$

In 4-dimensional Minkowski space, we then define a *position-time* four-vector x^μ by adding a temporal component (ct) to a three-vector $\vec{x} = (x_1, x_2, x_3) = (x_x, x_y, x_z)$:

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, \vec{x}) \tag{5}$$

Similarly we can define the length of a four-vector as

$$x^2 = x^{\mu T} x^\mu = (x^{0T}, x^{1T}, x^{2T}, x^{3T}) \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} . \tag{6}$$

In 3 dimensions, the values of the transposed vector are the same as the vector itself, however, this is not the case in 4 dimensions. We therefore need another definition, $x^{\mu T} = x_\mu$ with

$$x_\mu = (x_0, x_1, x_2, x_3) = (ct, -\vec{x}) . \tag{7}$$

x^μ is called a *contravariant* four-vector and its components have 'upper' indices, whereas x_μ is called a *covariant* four-vector and its components have 'lower' indices. These are related by the *metric tensor* $g^{\mu\nu}$

$$x_\mu = g_{\mu\nu} x^\nu ; \qquad x^\mu = g^{\mu\nu} x_\nu \tag{8}$$

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{9}$$

The convention is that repeated Greek indices are implicitly summed from 0 to 3. The scalar product of two four-vectors is

$$a \cdot b = a_\mu b^\mu = g_{\mu\nu} a^\nu b^\mu = a^0 b^0 - \vec{a} \cdot \vec{b} \,. \tag{10}$$

Note that we don't apply the vector symbol ($\vec{\ }$) to denote four-vectors. The invariant length of a space-time four-vector (Equation 6) now becomes

$$x^2 = x \cdot x = x_\mu x^\mu = g_{\mu\nu} x^\nu x^\mu = {x^0}^2 - \vec{x}^2 = (c\tau)^2 - \vec{x}^2; \tag{11}$$

where τ is called the *proper time* of an event.

Lorentz transformations are linear transformations on four-vectors which leave this scalar product invariant.

$$x'^\mu = \Lambda^\mu_\nu x^\nu \tag{12}$$

Transformations from one frame, eg. the laboratory, to a frame with constant velocity $\vec{v}$ to this frame are known as a **boost**. For a boost along the x-axis

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{where} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \,, \qquad \beta = \frac{v}{c} \,. \tag{13}$$

γ is also known as boost factor. The only real example where boosts are relevant in particle physics is the dilation of the lifetime τ of a particle in the laboratory. This can be expressed as a **decay length** $L = v\tau' = \beta\gamma c\tau$.

Let us define the difference between two space-time points with coordinates a^μ and b^μ as $\Delta x^\mu = a^\mu - b^\mu = (c\Delta t, \vec{\Delta x})$. The Lorentz invariant is $(\Delta s)^2 = \Delta x_\mu \Delta x^\mu = (c\Delta t)^2 - (\vec{\Delta x})^2$. Note that $(\Delta s)^2$ need not be positive. We classify four-vectors accordingly.

$$(\Delta s)^2 \begin{cases} > 0 & c\Delta t > |\Delta\vec{x}| \quad \text{time-like} \\ = 0 & c\Delta t = |\Delta\vec{x}| \quad \text{light-like} \\ < 0 & c\Delta t < |\Delta\vec{x}| \quad \text{space-like} \end{cases}$$

Only time-like separated events can communicate with each other as is illustrated in Figure 9.

3.3 Energy and Momentum

In particle physics, we are mostly concerned with energy, E, and 3-momentum, $\vec{p}$. Comparing relativistic definitions of the momentum $\vec{p} = \gamma m\vec{v}$ and the energy $E = \gamma mc^2$ with the proper-time derivative of the space-time four-vector $\frac{dx^\mu}{d\tau} = (\frac{dx^0}{d\tau}, \frac{d\vec{x}}{d\tau}) = (\gamma c, \gamma\vec{v})$ we realise that energy and momentum together make up a four-vector - the *energy-momentum* four-vector, or short the **four-momentum**

$$p^\mu = (E/c, p_x, p_y, p_z) \,. \tag{14}$$

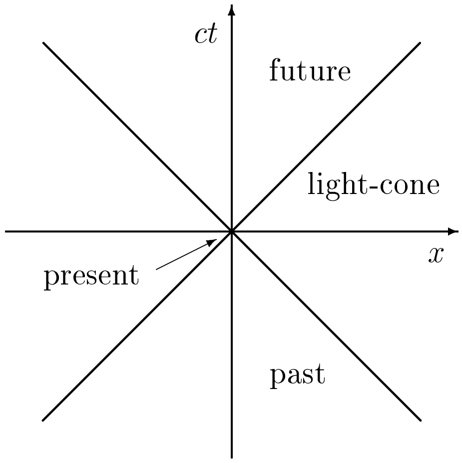

Figure 9: An schematic of an event's past, present and future. Two spatial dimensions are suppressed, in practice the lightcone is actually a spherical shell of radius ct.

The four-momentum p^μ fully describes the kinematics of a relativistic particle. It can be shown that the corresponding Lorentz invariant is equal to the square of the mass m of the particle:

$$p^2 = p_\mu p^\mu = (E/c)^2 - \vec{p}^2 = m^2c^2 \ . \tag{15}$$

Particle physicists often use natural units with $\hbar = c = 1$. In this case the velocity of a particle is simply $\vec{\beta}$. Thus the above equation can be re-written as

$$p^2 = p_\mu p^\mu = E^2 - \vec{p}^2 = m^2 \ . \tag{16}$$

Note the following useful relations for the boost factor γ and the velocity of a particle $\vec{\beta}$, and in addition the corresponding Lorentz transformations:

$$\gamma = \frac{E}{m} \ ; \qquad \beta = \frac{p}{E} \ ; \tag{17}$$

$$E' = \gamma E - \beta\gamma p_{||} \ ; \qquad p'_{||} = \beta\gamma E - \gamma p_{||} \ ; \qquad p'_\perp = p_\perp \tag{18}$$

where $p_{||}$ ($p_\perp$) is the component parallel (perpendicular) to the boost direction.

When solving problems in relativistic kinematics we make heavy use of the fact that a Lorentz invariant quantity has the same value in all reference frames. It is often convenient to work in the centre-of-mass frame (CoM) which is defined by zero net 3-momentum. Alternatively the laboratory is also a useful frame.

3.4 Two-Body Decays

We now discuss the decay of a particle with mass M into two daughters with masses m_1 and m_2. In the CoM frame, the decaying particle is at rest and hence $p^\mu = (M, \vec{0})$. By using invariant four-vector products we can calculate the energies E_1 and E_2 of the daughter particles in the CoM frame:

$$
\begin{aligned}
p_1^\mu &= p^\mu - p_2^\mu & \qquad p_2^\mu &= p^\mu - p_1^\mu \\
p_1^2 &= p^2 + p_2^2 - 2\,p\cdot p_2 & p_2^2 &= p^2 + p_1^2 - 2\,p\cdot p_1 \\
m_1^2 &= M^2 + m_2^2 - 2ME_2 & m_2^2 &= M^2 + m_1^2 - 2ME_1 \\
\Longrightarrow E_2 &= \frac{M^2 - m_1^2 + m_2^2}{2M} & \Longrightarrow E_1 &= \frac{M^2 + m_1^2 - m_2^2}{2M}
\end{aligned}
\tag{19}
$$

The corresponding CoM momenta p_1 and p_2 of the daughter particles can be calculated to be

$$
\mid \vec{p_1} \mid = \mid \vec{p_2} \mid = \frac{\left[(M^2 - (m_1+m_2)^2)\,(M^2 - (m_1-m_2)^2)\right]^{\frac{1}{2}}}{2M} \tag{20}
$$

We observe in Equation 20 that the above decay can only happen if $M > m_1 + m_2$. We define a general **threshold** condition which is valid for all decays

$$
Q \equiv M - \sum_i m_i \geq 0 \tag{21}
$$

where the sum is over the masses of all particles produced in the decay. At threshold, $Q = 0$, and the produced particles are all at rest. Above threshold the *excess energy*, $Q > 0$, goes into the kinematic energy of the final state particles.

3.5 Scattering

Let us consider the collision of two particles of masses m_1 and m_2. Since p_1^μ and p_2^μ are four-vectors, $s = (p_1 + p_2)^2$ is a Lorentz invariant. Thus the total centre-of mass energy E_{CoM} can be expressed in Lorentz-invariant form as

$$
\begin{aligned}
E_{\mathrm{CoM}} = \sqrt{s} &= \left[(E_1+E_2)^2 - (\vec{p_1}+\vec{p_2})^2\right]^{1/2} \\
&= \left[p_1^2 + p_2^2 + 2\,p_1\cdot p_2\right]^{1/2} \\
&= \left[m_1^2 + m_2^2 + 2(E_1E_2 - |\vec{p_1}||\vec{p_2}|\cos\theta)\right]^{1/2}
\end{aligned}
\tag{22}
$$

where θ is the angle between the colliding particles. In the frame where a particle with mass m_1 and energy E_{lab} impacts on a particle with mass m_2 at rest (lab frame) we have $p_1^\mu = (E_{\mathrm{lab}}, \vec{p_1})$ and $p_2^\mu = (m_2, \vec{0})$ and thus obtain

$$
E_{\mathrm{CoM}} = \sqrt{s} = \left[m_1^2 + m_2^2 + 2E_{\mathrm{lab}}m_2\right]^{1/2} \tag{23}
$$

In a scattering process $a + b \to c + d$ as shown in Figure 10 a), it is convenient to introduce the *Mandelstam variables* which are Lorentz invariant quantities:

$$
\begin{aligned}
s &= (p_a + p_b)^2 = (p_c + p_d)^2 \qquad (24) \\
t &= (p_a - p_c)^2 = (p_b - p_d)^2 \\
u &= (p_a - p_d)^2 = (p_b - p_c)^2
\end{aligned}
$$

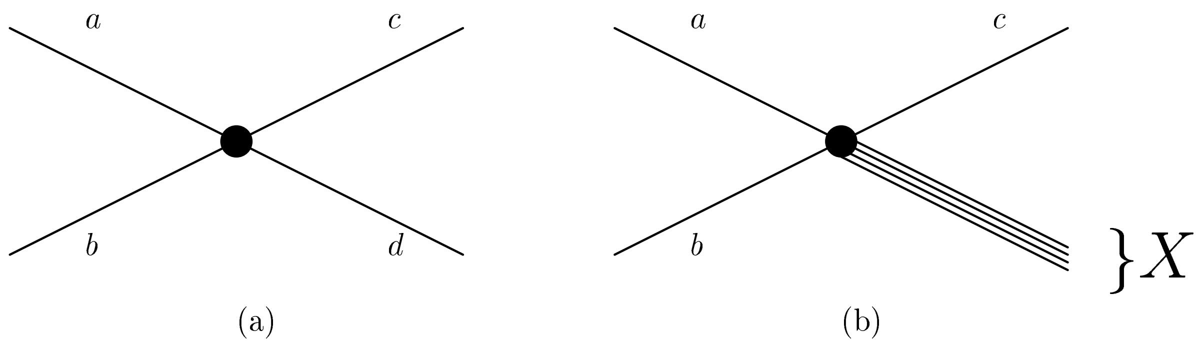

Figure 10: Scattering: — (a) exclusive processes (b) inclusive processes

Note that the three variables s, t, and u are not independent, and satisfy the relation

$$s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2 \tag{25}$$

An *exclusive* process is one in which all final state particles are observed and identified, in which case s, t and u are all well defined. However, in many experiments only one or a limited number of the final state particles are measured. These are called *inclusive* processes, illustrated in Figure 10 b). Although u cannot be determined in such processes, s and t are well defined.

In the most generic form a scattering process can be cast as follows:

$$\underbrace{a+b}_{i} \qquad \longrightarrow \qquad \underbrace{c+d+e+\cdots}_{f} \tag{26}$$

where i and f stand for *initial state* and *final state* particles, respectively. When viewed in the CoM frame the initial state particles a and b form a system which is described by $p_i^\mu = (\sqrt{s}, \vec{0})$ which subsequently decays into the final state particles c, d, $e \cdots$. By comparing with Equation 21 we note that the minimum energy (threshold) for this process is

$$\sqrt{s} \geq \sum_{j=c,d,e\cdots} m_j \ . \tag{27}$$

In many particle physics experiments accelerated particle beams are used to study scattering processes. There are two basic types of accelerator based experiments as illustrated in Figure 11.

In a **Fixed Target** experiment a beam of particles a is aimed at a stationary target comprising of particles b. The centre-of-mass energy E_{CoM} has been calculated in Equation 23 and we have

$$s = m_a^2 + m_b^2 + 2E_a m_b \qquad \Longrightarrow \qquad E_{\text{CoM}} \sim \sqrt{2E_a m_b} \qquad \text{if } E_a >> m_i \ . \tag{28}$$

Momentum conservation implies that a large fraction of the available energy must go into the kinetic energies of the final state particles.

In a **Collider** experiment two beams of accelerated particles a and b are brought into a head-on collision. Using Equation 22 we then find

$$s = m_a^2 + m_b^2 + 2(E_a E_b + |\vec{p_a}||\vec{p_b}|) \qquad \Longrightarrow \qquad E_{\text{CoM}} \sim \sqrt{4E_a E_b} \qquad \text{if } E_i >> m_i \ . \tag{29}$$

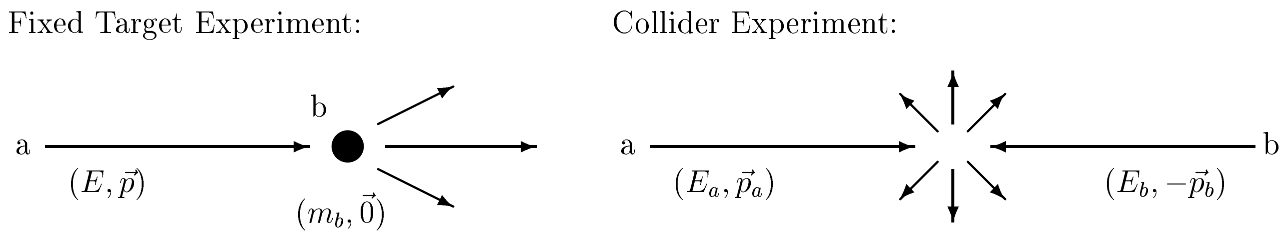

Figure 11: The kinematics for the two basic types of high energy physics experiments.

When colliding beams are used essentially all the beam energy is available for particle creation, which is im marked contrast to fixed target experiments.

As an example let us discuss the **Bevatron** accelerator which was commissioned in 1954. This machine was designed to accelerated protons up to momenta of larger than 6 GeV onto fixed targets. One of the first experiments carried out lead to the discovery of the antiproton, $\bar{p}$. The reaction chosen, was

$$\begin{array}{lcl} & p + p \longrightarrow p + p + p + \bar{p} & \\ \text{Baryon number} & 1 + 1 \;=\; 1 + 1 + 1 + (-1) & \text{conserved .} \end{array} \tag{30}$$

Since four protons or antiprotons must be produced in this process, the necessary threshold energy is $\sqrt{s} \geq 4\, m_p$. Using Equation 23 we can calculate the minimum energy of the proton beam, E_p, necessary to produce antiprotons

$$2\, m_p^2 + 2\, E_p m_p \geq (4\, m_p)^2 \qquad \Longrightarrow \qquad E_p \geq 7\, m_p \; . \tag{31}$$

The threshold kinetic energy T_p of the proton beam is $T_p = E_p - m_p = 6\, m_p = 5.63$ GeV. The experiment was carried out at $T_p = 6.6$ GeV and in 1955 Chamberlain, Segre, Wiegand and Ypsilantis, using a copper target, observed the first antiprotons.