5 Symmetries

"One of the most profound insights in theoretical physics is that interactions are dictated by **symmetry** principles. ... All interactions may be dictated by so-called local gauge symmetries ... This is intimately connected with the idea that the conserved physical quantities (such as electric charged, color, etc.) are conserved in local regions of space and not just globally." (Halzen and Martin, *Quarks and Leptons*, p.311.)

5.1 Overview

In particle physics we are mainly interested in the dynamical symmetries of motion. **Noether's theorem** states:

- Every symmetry in nature yields a conservation law; conversely, every conservation law reveals an underlying symmetry.

As an example, the laws of physics are symmetrical under rotations. Noether's theorem relates this to the conservation of angular momentum. Translation invariance and time invariance correspond to momentum and energy conservation, respectively. Similarly, the fact that electrodynamics is invariant under gauge transformations leads to the conservation of charge. Table 6 shows a list of symmetries and corresponding conservation laws which are used in particle physics. *Continous* symmetries are associated with *additive* quantum numbers whereas *discrete* symmetries yield *multiplicative* quantum numbers. We'll have a closer look at some of these now.

Symmetry	Conservation Law
Continous:	
Translations in space	Momentum
Translations in time	Energy
Rotations	Angular momentum
p - n mass degeneracy	Strong Isospin
Discrete:	
Gauge (phase) transformation	Charge
Space inversion	Parity

Table 6: Symmetries in particle physics with their associated conservation laws.

5.2 Spin $\frac{1}{2}$ Rehearsal

A particle can be characterised by two different types of angular momentum. For an example look at the electron in the hydrogen atom. It can be put into any *orbital* angular momentum states l from 0 up to $n-1$, where n is the principal quantum number. The **spin** s of the electron, however, is an *intrinsic* property and is fixed. All quarks and leptons, the proton and the neutron carry spin $s = 1/2$. Now we have a close look at this system. Once you understand the formalism for $s = 1/2$, any other case is rather similar to work out. A particle with spin 1/2 can have its projection along, say, the z-axis "up" ($m = 1/2$) or "down" (m = -1/2) A useful notation is to represent these two states by *spinors*

$$\left|\frac{1}{2}\frac{1}{2}\right\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \left|\frac{1}{2} - \frac{1}{2}\right\rangle = \begin{pmatrix}0\\1\end{pmatrix} \tag{64}$$

The most general state of a spin 1/2 particle is a linear combination of Equation 64, denoted by $\begin{pmatrix}\alpha\\\beta\end{pmatrix}$. The spin operator s can be represented by 2x2 matrices

$$s = \frac{\hbar}{2}\sigma \tag{65}$$

$$\tag{66}$$

where the *Pauli spin matrices* $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are given by

$$\sigma_x = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \qquad \sigma_y = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}, \qquad \sigma_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \tag{67}$$

The Pauli spin matrices have many useful properties which are derived in lectures or books about quantum mechanics. They

- obviously satisfy the commutation relation (Equ. 76);
- each have two eigenvalues ± 1;
- obey the anticommutation relation $\{\sigma_i, \sigma_k\} = 2\delta_{ik}$;
- and are thus unitary *traceless* matrices.

5.3 Symmetries and Groups

A symmetry is an operation on a system which leaves the system *invariant*. As an example rotations leave a spherical potential intact. To continue, it is useful to introduce some of the concepts and of the terminology of **group theory**. A set of symmetry operation on a system form a *group* if the following properties are fulfilled.

- If R_i and R_k are in the set, then the *product* R_iR_k is also in the set.
- There is an *identity* element such that $IR_i = R_iI = R_i$ for all elements R_i.
- For every element R_i there exist an *inverse* R_i^{-1} such that $R_iR_i^{-1} = R_i^{-1}R_i = I$.

- *Associativity* holds: $R_i(R_j R_k) = (R_i R_j)R_k$.

The rotation group is a *continous* group[12]. Each rotation depends on one or more continously varying parameters (for rotations in a plane, angle α). Note that the product is not necessarily *commutative*, and in general $R_i R_k \neq R_k R_i$. If the product is commutative the group is called *Abelian.*

Groups of interest in particle physics are often groups of *matrices.* The most common is of the type $U(n)$, the group of all *unitary* $n \times n$ *matrices*, see Table 7. A unitary matrix is one whose

Group	Matrix representation
U(n)	$n \times n$ unitary ($U^\dagger U = 1$)
SU(n)	$n \times n$ unitary with determinant 1
O(n)	$n \times n$ orthogonal ($O^\dagger O = 1$)
SO(n)	$n \times n$ orthogonal with determinant 1

Table 7: Important groups in particle physics.

inverse is equal to the transpose conjugate: $U^{-1} = U^{*T} = U^\dagger$. Unitary matrices with determinant equal to one form the special unitary group $SU(n)$. If we restrict the elements of a unitary matrix to be *real* the group is called $O(n)$. The group $SO(3)$ describes the rotational symmetry in our world.

Every group can be *represented* by a group of matrices. For every group element a there is a corresponding matrix M_a, which also respects the group product rule, ie. if $ab = c$ then $M_a M_b = M_c$. If the group is a group of matrices, such as $SO(3)$, it represents itself. We call this the *fundamental representation.* But in there will be other representations by matrices of various dimensions. For example, SU(2) has representations of dimensions 1, 2, 3, ... Representations are called *irreducible* if they cannot be decomposed into block-diagonal form.

5.4 Rotations in Quantum Mechanics

Using rotations as an example we discuss now the effect of symmetries on a quantum mechanical system. Most of the results can be directly applied to other continous transformation groups. Rotations is a symmetry operation which leaves the physics unchanged. In particular, the transition probabilities of the system must remain invariant. Suppose that under a rotation the states of a system, described by $|\psi\rangle$, transform as

$$|\psi\rangle \longmapsto |\psi'\rangle = U\,|\psi\rangle \tag{68}$$

[12]There are also *discrete* groups. These can be *finite* - like the triangle group defined by all symmetry operations which leave an equilateral triangle intact - or *infinite* - like the set of integers with addition as group product.

The transformed probability must be left unchanged by this rotation,

$$\begin{aligned} |\langle\psi \mid \psi\rangle|^2 = |\langle\psi' \mid \psi'\rangle|^2 &= \left|\langle\psi| \, U^\dagger U \, |\psi\rangle\right|^2 \\ \Longrightarrow \qquad U^\dagger U &= UU^\dagger = 1 \end{aligned} \tag{69}$$

and so U must be a unitary operator. Moreover the Hamiltonian H is unchanged by this rotation, so that the matrix element also must be unchanged.

$$\begin{aligned} \langle\psi| \, H \, |\psi\rangle &= \langle\psi| \, U^\dagger H U \, |\psi\rangle \ , \\ H = U^\dagger H U \qquad &\text{or} \qquad [U, H] = UH - HU = 0 \end{aligned} \tag{70}$$

The transformation U has no explicit time dependence and the equation of motion,

$$i\frac{d}{dt} \, |\psi(t)\rangle = H \, |\psi(t)\rangle \tag{71}$$

remains unchanged.

In the case of continous transformations, the properties of the group can be studied by looking at *infinitesimal* transformations. In our example, a rotation through an infinitesimal angle ε about the z-axis may be written up to first order in ε as

$$U = 1 - i\varepsilon J_z \tag{72}$$

where the operator J_z generates the rotation about the z-axis. Applying the unitarity condition (Equ. 69) gives

$$\begin{aligned} 1 = U^\dagger U &= (1 + i\varepsilon J_z^\dagger)(1 - i\varepsilon J_z) \\ &= 1 + i\varepsilon(J_z^\dagger - J_z) + \mathcal{O}(\varepsilon^2) \end{aligned} \tag{73}$$

and J_z must satisfy $J_z^\dagger - J_z = 0$. Hence J_z is a *Hermitian* operator which is associated with an observable. Using that for rotations R we have $\psi'(\vec{r}) = \psi(R^{-1}\vec{r})$ and recalling that the momentum operator is given by $\vec{p} = -i\hbar\nabla$ we can write

$$\begin{aligned} U\psi(x, y, z) &\backsimeq \psi(x + \varepsilon y, y - \varepsilon x, z) \\ &\backsimeq \psi(x, y, z) + \varepsilon \left(y\frac{\delta\psi}{\delta x} - x\frac{\delta\psi}{\delta y} \right) \\ &= (1 - i\varepsilon(xp_y - yp_x)) \, \psi \end{aligned} \tag{74}$$

Comparing Equations 74 and 72 we identify J_z with the third component of the *angular momentum operator* $\vec{r} \times \vec{p}$.

A finite transformation may be built up from a succession of n inifinitesimal rotations

$$U(\theta) = \lim_{n\to\infty} \left(1 - i\frac{\theta}{n} J_z \right)^n \longrightarrow e^{-i\theta J_z} \tag{75}$$

Analogue Hermitian operators describe rotations about the x and y-axis. The set of all rotations form a *Lie* group, and the operators J_x, J_y, and J_z are known as the *generators* of the group. The commutator relation of the generators is

$$[J_j, J_k] = i\varepsilon_{ikl} J_l \tag{76}$$

where ε_{ikl} is the Levi-Civita tensor[13]. The ε_{ikl} are known as the *structure constants* of the group. This relation completely describes the group structure and the J's are said to form a Lie algebra. Since J_x, J_y, and J_z do not commute with each other, only the eigenvalues of one generator, say J_z are useful quantum numbers. Note that the operator J^2 commutes with all the generators of the group.

$$J^2 = J_x^2 + J_y^2 + J_z^2 \tag{77}$$

$$\left[J^2, J_i\right] = 0 \qquad i = x, y, z \tag{78}$$

Thus we can construct simultaneous eigenstates $|jm\rangle$ of J^2 and one of the generators, say J_z. From Equ. 76 it follows that

$$\begin{aligned} J^2 \left|jm\right\rangle &= j(j+1) \left|jm\right\rangle \\ J_z \left|jm\right\rangle &= m \left|jm\right\rangle \end{aligned} \tag{79}$$

where, in units of $\hbar$, j can take the values 0, 1/2, 1, 3/2, ... and $m = -j, -j+1, ..., j$.

From Equation 71 we realise that the eigenvalues of the observable J_z are conserved quantum numbers. The rotation symmetry has yielded a conservation law (Noether's theorem). So the following statements are equivalent:

- Angular momentum about an axis is conserved in a closed system.
- The Hamiltonian is invariant under rotations about an axis.
- The angular momentum operator (J^2 or J_z) commutes with the Hamiltonian.

5.5 Spin $\frac{1}{2}$ and $SU(2)$

Above we have demonstrated that the spin operators are the generators of a Lie group (Equ. 76) Hence we can apply Equation 75 to transform a spinor under a rotation of the coordinate axes through an angle θ

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = U(\theta) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{80}$$

$$U(\vec{\theta}) = e^{-i\vec{\theta}\cdot\vec{\sigma}/2} \tag{81}$$

[13] $\varepsilon_{ikl} = +1(-1)$ if ikl is a cyclic (anticyclic) permutation of $x\ y\ z$ and $\varepsilon_{ikl} = 0$ otherwise.

Note that $U(\vec{\theta})$ is a 2×2 matrix which has an exponent which is itself a matrix. $U(\theta)$ has determinant 1. The set of transformation matrices $U(\vec{\theta})$ therefore form a $SU(2)$ group. The Pauli spin matrices constitute, of course, the 2-dimensional fundamental representation of this $SU(2)$ group. In general, the 1, 2, 3, 4, ... dimensional representation of $SU(2)$ corresponds to $s = 0, 1/2, 1, 3/2, ...$, respectively.

Now we show how to built other representations from the fundamental one. For an example, let us look at positronium (e^+e^-) where the spins of two spin 1/2 particles are added. Using the Dirac notation and following the rules of addition of angular momenta we can write the resulting triplet and singlet states as

$$\begin{aligned} |11\rangle &= \left|\frac{1}{2}\frac{1}{2}\right\rangle \left|\frac{1}{2}\frac{1}{2}\right\rangle \\ |10\rangle &= \sqrt{\frac{1}{2}}\left(\left|\frac{1}{2}\frac{1}{2}\right\rangle \left|\frac{1}{2}-\frac{1}{2}\right\rangle + \left|\frac{1}{2}-\frac{1}{2}\right\rangle \left|\frac{1}{2}\frac{1}{2}\right\rangle\right) \\ |1-1\rangle &= \left|\frac{1}{2}-\frac{1}{2}\right\rangle \left|\frac{1}{2}-\frac{1}{2}\right\rangle \\ |00\rangle &= \sqrt{\frac{1}{2}}\left(\left|\frac{1}{2}\frac{1}{2}\right\rangle \left|\frac{1}{2}-\frac{1}{2}\right\rangle - \left|\frac{1}{2}-\frac{1}{2}\right\rangle \left|\frac{1}{2}\frac{1}{2}\right\rangle\right) \end{aligned} \tag{82}$$

In general a composite system of angular momentum J_A and J_B may be described in terms of the basis $|j_A j_B m_A m_B\rangle$ .The combined operator $J = J_A + J_B$ also satisfies the Lie algebra (Equ. 76) and its eigenvalues $j(j+1)$, m of J^2, J_z are conserved quantum numbers. The "product" of the two irreducible representations of dimensions $2j_A + 1$ and $2j_B + 1$ may be decomposed into the "sum" of irreducible representations of dimension $2J + 1$ with $J = | J_A - J_B |, | J_A - J_B | + 1, ..., J_A - J_B$, in terms of the basis $|j_A j_B j m\rangle$ where $m = m_A + m_B$. One basis may be expressed in terms of the other by

$$|j_A j_B j m\rangle = \sum_{m_A, m_B} C(m_A m_B; jm)\, |j_A j_B m_A m_B\rangle \tag{83}$$

The coefficients C are called the it Clebsch-Gordan coefficients. They have been tabulated for example in the *Review of Particle Properties*, see Table 8. These coefficients can be calculated by repeatedly applying the "ladder" operator to the state where everything is lined up.

Equation 82 is a simple example of Equation 83. The composite system of two spin 1/2 particles may have spin 0 or 1. We may write the decomposition symbolically as

$$\begin{aligned} 2 \otimes 2 &= 3 \oplus 1 \qquad \text{or} \\ 1/2 \otimes 1/2 &= 1 \oplus 0 \end{aligned} \tag{84}$$

The former (latter) notation uses the dimensions (angular momenta) of the multiplet to label the irreducible representations. This procedure can be extended as necessary.

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			

1/2×1/2

m_1	m_2	$J=1$, $M=+1$
+1/2	+1/2	1

m_1	m_2	$J=1$, $M=0$	$J=0$, $M=0$
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	$J=1$, $M=-1$
−1/2	−1/2	1

1×1/2

m_1	m_2	$J=3/2$, $M=+3/2$
+1	+1/2	1

m_1	m_2	$J=3/2$, $M=+1/2$	$J=1/2$, $M=+1/2$
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	$J=3/2$, $M=-1/2$	$J=1/2$, $M=-1/2$
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	$J=3/2$, $M=-3/2$
−1	−1/2	1

2×1

m_1	m_2	$J=3$, $M=+3$
+2	+1	1

m_1	m_2	$J=3$, $M=+2$	$J=2$, $M=+2$
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	$J=3$, $M=+1$	$J=2$, $M=+1$	$J=1$, $M=+1$
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	6/15	−1/2	1/10

m_1	m_2	$J=3$, $M=0$	$J=2$, $M=0$	$J=1$, $M=0$
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	$J=3$, $M=-1$	$J=2$, $M=-1$	$J=1$, $M=-1$
0	−1	6/15	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	$J=3$, $M=-2$	$J=2$, $M=-2$
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	$J=3$, $M=-3$
−2	−1	1

1×1

m_1	m_2	$J=2$, $M=+2$
+1	+1	1

m_1	m_2	$J=2$, $M=+1$	$J=1$, $M=+1$
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	$J=2$, $M=0$	$J=1$, $M=0$	$J=0$, $M=0$
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	$J=2$, $M=-1$	$J=1$, $M=-1$
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	$J=2$, $M=-2$
−1	−1	1

3/2×1

m_1	m_2	$J=5/2$, $M=+5/2$
+3/2	+1	1

m_1	m_2	$J=5/2$, $M=+3/2$	$J=3/2$, $M=+3/2$
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	$J=5/2$, $M=+1/2$	$J=3/2$, $M=+1/2$	$J=1/2$, $M=+1/2$
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	$J=5/2$, $M=-1/2$	$J=3/2$, $M=-1/2$	$J=1/2$, $M=-1/2$
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	$J=5/2$, $M=-3/2$	$J=3/2$, $M=-3/2$
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	$J=5/2$, $M=-5/2$
−3/2	−1	1

2×1/2

m_1	m_2	$J=5/2$, $M=+5/2$
+2	1/2	1

m_1	m_2	$J=5/2$, $M=+3/2$	$J=3/2$, $M=+3/2$
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	$J=5/2$, $M=+1/2$	$J=3/2$, $M=+1/2$
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	$J=5/2$, $M=-1/2$	$J=3/2$, $M=-1/2$
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	$J=5/2$, $M=-3/2$	$J=3/2$, $M=-3/2$
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	$J=5/2$, $M=-5/2$
−2	−1/2	1

3/2×1/2

m_1	m_2	$J=2$, $M=+2$
+3/2	+1/2	1

m_1	m_2	$J=2$, $M=+1$	$J=1$, $M=+1$
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	$J=2$, $M=0$	$J=1$, $M=0$
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	$J=2$, $M=-1$	$J=1$, $M=-1$
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	$J=2$, $M=-2$
−3/2	−1/2	1

Table 8: From the PDG: Clebsch-Gordon coefficients $\langle m_1, m_2 | J, M\rangle$ using the Condon and Shortley phase convention.

6 Discrete Symmetries

We have learnt how continuous symmetries, like rotations, give rise to conserved quantum numbers, like angular momentum (spin). Now we introduce discrete symmetries which also play a large role in particle physics. The most important examples are parity P, charge conjugation C and time reversal T. Both parity and time reversal are also relevant to classical physics. Newtons law of gravitation requires conservative forces which are parity invariant and Boltzmann and others have discussed the significance of time reversal invariance in the context of the second law of thermodynamics. On the other hand, charge conjugation only became relevant when Dirac introduced his equation which required antimatter, and has no classical analogue.

Discrete symmetries are related to *finite* groups. To discuss parity and charge conservation in particle physics, the relevant group has just two elements, the identity and an element g with $g^2 = 1$. As discussed in Chapter 5, invariance under a transformation g can be represented as an operator $U(g)$ which commutes with the Hamiltonian and satisfies

$$[U, H] = 0 \qquad\qquad \text{and} \qquad \mathrm{U}^2 = 1\ . \tag{85}$$

Since U is unitary and Hermitian, it is an observable quantity. Its eigenvalues p are conserved quantum numbers. Say p is an eigenvalue of U corresponding to an eigenvector $|p\rangle$, it follows

$$U^2\,|p\rangle = p^2\,|p\rangle \qquad\qquad \Longrightarrow \qquad p = \pm 1\ . \tag{86}$$

The allowed eigenvalues of U are **multiplicative** quantum numbers which do not have an analogue in classical physics.

6.1 Parity

The **Parity** operation P was introduced in 1932 by Wigner who had studied atomic spectra. It can be written as

$$\begin{aligned} P\psi(\vec{x}) &= \psi(-\vec{x}) \\ \langle Pf|\,H\,|Pi\rangle &= \langle f|\,H\,|i\rangle \end{aligned} \tag{87}$$

Parity performs a spatial inversion $\vec{x} \longrightarrow -\vec{x}$ at the origin. It is an example of a discrete transformation which satisfies Equation 85. Applying parity twice restores the original states which implies, $P^2 = 1$. We call the eigenvalues $P = \pm 1$ the parity of the system. The parity of a wavefunction $\psi(\vec{x})$ can be even, $P = 1$, or odd, $P = -1$. We list a few trivial examples:

$$\begin{array}{ccc} \psi(x) = \cos x & \qquad\qquad & \psi(x) = \sin x \\ P = +1 & & P = -1\ . \end{array}$$

To illustrate parity let us look at the wavefunctions of the hydrogen atom in a spherically symmetric potential, $V(\vec{r})$, which has the property $V(\vec{r}) = V(-\vec{r})$. The hydrogen atom wavefunctions are a product of a radial function $f(r)$ multiplied by a spherical harmonics $Y_l^m(\theta, \phi)$

$$\psi(\vec{x}) = f(r)Y_l^m(\theta, \phi), \qquad Y_l^m(\theta, \phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos\theta)e^{im\phi} \ .$$

where θ and ϕ are the polar and azimuthal angles, and l and m are the angular momentum and its component along the chosen axis, respectively. As shown in Figure 13, the parity operation

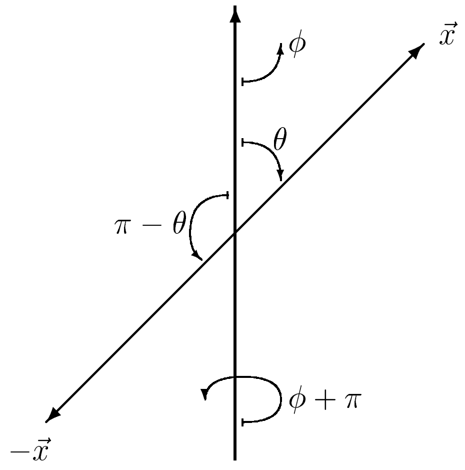

Figure 13: The effect of the parity operator on $\vec{x}$, showing the change in polar coordinates.

changes $\{r, \theta, \phi\} \longrightarrow \{r, \pi - \theta, \pi + \phi\}$. Using the relations

$$P_l^m(\cos\theta) = (-1)^{l+m} P_l^m(\cos\theta) \qquad e^{im(\phi+\pi)} = (-1)^m e^{im\phi}$$

we find the result

$$\begin{aligned} Y_l^m(\theta, \phi) &\longrightarrow Y_l^m(\pi - \theta, \pi + \phi) \\ &= (-1)^{l+m} \times (-1)^m \times Y_l^m(\theta, \phi) \\ &= (-1)^l \times Y_l^m(\theta, \phi) \end{aligned} \tag{88}$$

Thus, the spherical harmonics have parity $(-1)^l$. Electrons in atomic states with angular momentum L = 0, 2, ... have even parity while states with L =1, 3, ... have odd parity. Spectral lines from atoms correspond to the emission or absorption of photons. These electric dipole transition between atomic states obey the selection rule $|\Delta L = \pm 1|$ and we deduce that the parity of the photon must be odd:

$$(-1)^L = (-1)^{L\pm1} \times P_\gamma \qquad \Longrightarrow \qquad P_\gamma = -1 \ . \tag{89}$$

In strong and in electromagnetic reactions, parity is found to be conserved. This is not the case in weak interactions as we will discuss later. It is necessary to assign an **intrinsic parity** to the wavefunction a particle at rest. This is a purely quantum mechanical phase and, like spin, has no analogue in classical physics.

Quantum field theory tells us that bosons and antibosons have the same intrinsic parity. From the Dirac equation, one can show that fermions and antifermions have opposite intrinsic parity. Only relative phases are observable quantities and the absolute parities are assigned by convention.

$$\begin{array}{cc} P_{\text{fermion}} = -P_{\text{antifermion}} & P_{\text{boson}} = P_{\text{antiboson}} \\ P_{e^-,\mu^-,\tau^-,u,d,s,c,b,t} = +1 & P_{e^+,\mu^+,\tau^+,\bar{u},\bar{d},\bar{s},\bar{c},\bar{b},\bar{t}} = -1 \ . \end{array} \tag{90}$$

Gauge theories tell us that the parity of all gauge bosons is odd.

$$P_\gamma = P_g = P_W = P_Z = -1 \tag{91}$$

Parity is a multiplicative quantum number, so the parity of a many particle system is equal to the product of the individual parities. Particles in a definite orbital angular momentum state are also eigenstates of parity, as is shown above for spherical harmonics. For example, the parity of a two particle system, (a, b), is

$$P(ab) = P_a P_b (-1)^L \tag{92}$$

where L is the relative angular momentum between a and b.

The relative parities of many hadrons were measured before the emergence of the quark model. By conventions equivalent to the one assigning intrinsic quark parities (Equation 90), the proton, the neutron, and the Λ hyperon were assigned to have the same intrinsic parity

$$P_p = P_n = P_\Lambda = 1 \ . \tag{93}$$

The parities of all other particles can derived from these assignments.

The parity of the charged pion can be determined from the capture of negative pions in deuterium: $\pi^- + \text{d} \to \text{n} + \text{n}$. The $\pi^- d$ forms an atomic S-state ($L = 0$) from which capture take place[14]. Considering angular momentum conservation we find

$$\begin{array}{lccc} & \pi^- + \text{d} & \to & \text{n} + \text{n} \\ \text{P:} & P_\pi P_{\text{d}} (-1)^L & = & P_{\text{n}}^2 (-1)^{L'} \\ \text{S:} & 0\ ,\ 1 & & 1/2\ ,\ 1/2 \\ \text{L:} & L = 0 & & L' \end{array}$$

The total angular momentum of the $\pi^- + \text{d}$ system is $J = L + S = 1$ which must equal the J_{nn} of the neutron pair. It follows that either $S_{\text{nn}} = 0$ and $L' = 1$ or $S_{\text{nn}} = 1$ and $L' = 0, 1, 2$ The total wavefunction (space and spin) of the neutron pair must be asymmetric due to the Pauli principle. When adding the spins of two spin $\frac{1}{2}$ particle (see Equation 82) the triplet $S = 1$ states

[14] The π^- replaces the e^- in a deuterium atom and cascades very fast into a $L = 0$ state, as is proven by measuring the emitted X-rays.

are symmetric and the singlet $S = 0$ states is antisymmetric under the exchange of the two spins, respectively. This gives the relation $(-1)^{L'} \times (-1)^{S_{\rm nn}+1} = -1$ which translates into $S_{\rm nn} + L'$ even. We thus obtain $S_{\rm nn} = 0$ and $L' = 1$. Remembering that the deuteron is a S-wave p-n bound state, determines its parity to be $P_{\rm d} = P_{\rm p} P_{\rm n} (-1)^0 = +1$, and we find the result

$$P_\pi P_{\rm d} = P_{\rm n}^2 (-1)^{L'} = -1 \qquad \Longrightarrow \qquad P_\pi = -1 \tag{94}$$

Thus the charged pion has negative parity. We say the pion has spin-parity $J^P(\pi) = 0^-$ which is referred to as a pseudoscalar meson.

All particles can be classified according to their spin and parity, J^P. Interactions are also classified according to their J^P. The following notations are used:

$\mathrm{J^P} =$	0^+	Scalar (S)	0^-	Pseudoscalar (P)
	1^+	Axialvector (A)	1^-	Vector (V)
	2^+	Tensor (T)		

A vector transforms under parity as $P\vec{v} = -\vec{v}$, as you would expect for a polar vector. As you already know there are also axial vectors such as angular momentum. These do not change sign under a parity transformation $P\vec{a} = \vec{a}$. This implies that parity does not affect the spin of a particle. But there are also two kinds of scalars. A pseudoscalar transforms like the dot product of a vector and a axial vector $p = \vec{a} \cdot (\vec{b} \times \vec{c})$ and changes sign under parity $Pp = -p$ whereas a scalar transforms as the dot product of two vectors or two axial vectors, $Ps = s$, respectively.

Figure 14: Combining angular momenta and spin for $q\bar{q}$ mesons and qqq baryons.

In Figure 14 we sketch a meson consisting of a quark - antiquark $q\bar{q}$ bound state. Since q and $\bar{q}$ have opposite intrinsic parity we find that the parity of mesons is

$$P(\text{meson}) = P_q P_{\bar{q}} (-1)^L = (-1)^{L+1} \ . \tag{95}$$

We also sketch how angular momentum in a baryon can be treated. Two quarks form a state with angular momentum $L1$ which is combined with the third quark and relative angular momentum $L2$. The same holds for spin $S1$ and $S2$. In the list of mesons and baryons, shown in Tables 10 and 11, we also quote the spin-parity J^P of the particles.

6.2 Charge Conjugation

Charge Conjugation is a discrete symmetry that interchanges particles and antiparticles. Specifically it changes the sign of all additive quantum numbers of a state while leaving position, momentums and spin unchanged. The charge conjugation operator C satisfies Equation 85 and, like parity, is a Hermitian observable with possible eigenvalues ± 1, which are often referred to as C-**Parities**. For example, C transforms a π^+ into a π^-

$$C\left|\pi^+\right\rangle \rightarrow C_\pi \left|\pi^-\right\rangle \neq \left|\pi^+\right\rangle \ .$$

We assign $C\pi = 1$. Clearly most mesons and baryons are not eigenstates of C, only particles which are their own antiparticles are eigenstates of C.

The electromagnetic and strong interactions conserve charge conjugation but the weak interactions do not. Electromagnetic fields are due to moving charges which change sign under C. Thus we find that the photon has negative C-parity

$$C\left|\gamma\right\rangle \rightarrow C_\gamma \left|\gamma\right\rangle = (-1)\left|\gamma\right\rangle \ .$$

Since charge conjugation is a multiplicative quantum number, a system of n photons has $C = (-1)^n$. A similar argument can be made for all neutral gauge bosons and we have $C = -1$ also for the Z^0 and for gluons. The neutral pion has a dominant decay mode $\pi^0 \rightarrow \gamma\gamma$ from which we infer that the π^0 has even C-parity:

$$C\left|\pi^0\right\rangle \rightarrow \left|\bar{\pi}^0\right\rangle = +\left|\pi^0\right\rangle \qquad\qquad C_{\pi^0} = C_\gamma^2 = 1$$

For a system consisting of a spin $\frac{1}{2}$ particle and its antiparticle, C is equivalent to a parity transformation P followed by an interchange of the spin assignments. Parity gives a factor of $(-1)^{L+1}$ (Equation 95) and the spin exchange factor is $(-1)^{S+1}$, see the discussion on the parity of the charged pion. An example are neutral mesons, $q\bar{q}$ states, and we find

$$C = (-1)^{L+1}(-1)^{S+1} = (-1)^{L+S} \tag{96}$$

Equations 95 and 96 define P and C for a meson multiplet, once its angular momentum L and spin S are known. For the pseudoscalar mesons, $L = S = 0$, and for the vector mesons $L = 0$ and $S = 1$, respectively We find ($\vec{J} = \vec{L} + \vec{S}$)

$$\begin{aligned} J^{PC} &= 0^{-+} \qquad \text{Pseudoscalar} \\ J^{PC} &= 1^{--} \qquad \text{Vector} \end{aligned} \tag{97}$$

6.3 The CPT Theorem

The three discrete symmetries charge conjugation C, parity transformation P, and time reversal T are related by a very basic theorem which requires *all* interactions must be invariant under the combined operation of C, P and T in any order.

The proof of the CPT theorem is based on very general assumptions. Pauli's spin-statistics theorem states that integer and half-integer spin particles obey Bose-Einstein and Fermi-Dirac statistics, respectively. Local interactions must be Lorentz invariant. Consequences of the CPT theorem that may be verified experimentally are that particles and antiparticles must have the same mass and lifetime and opposite electric charges and magnetic moments. All tests of the CPT theorem until today have not yielded any evidence for a violation of CPT and some of the limits are quite spectacular, see for example Perkins, *Introduction to High Energy Physics*, p.81.

Individually C, P and T are conserved by the strong and electromagnetic interactions, but violated by the weak interaction as shown in Table 9. A consequence of CP violation is that this is

Symmetry	Strong	Electromagnetic	Weak
P	$\bullet$	$\bullet$	$\times$
C	$\bullet$	$\bullet$	$\times$
T	$\bullet$	$\bullet$	$\circ$
CP	$\bullet$	$\bullet$	$\circ$
CPT	$\bullet$	$\bullet$	$\bullet$

Table 9: The invariance of the basic forces under the discrete symmetry operations. A $\bullet$ indicates no violation, a $\circ$ indicates a small violation and a $\times$ indicates a large violation of the symmetry.

not only an asymmetry between matter and antimatter but, via the CPT theorem, CP violation also implies T violation, an asymmetry between the past and the future.

In the Big Bang model of the universe matter, antimatter and gauge bosons were created in equal amounts. Annihilation of baryons with antibaryons took place and if **baryon number** were conserved we would expect that all matter would have annihilated with antimatter, and the universe of today would look rather dull[15]. Our universe is matter dominated and we observe a ratio of $n_\mathrm{b}/n_\gamma = 10^{-9\pm1}$ for the number of baryons n_b over the number of photons n_b. In 1966, Sakharov postulated three conditions that are necessary for our universe to exist. These are:

- An epoch with no thermal equilibrium
- Baryon number violation
- CP Violation

The violation of P, C and CP will be discussed in detail when studying weak interactions. We will learn that the Standard Model of particle physics gives at least a partial answer.

[15] Only the cosmic microwave background of photons with a temperature of 2.7 K would exist, but no life which could actually observe it with a radio telescope or a television set.