4 Cross Sections and Decay Rates

A lot of phenomenology in particle physics is obtained by two basic types of measurements. When carrying out an experiment, we will most likely be either observing *decays* of particles or perform a *scattering* experiment. Thus we measure particle *lifetimes* or *decay widths* (decays) or *cross sections* (scattering). These two quantities determine the *probability* or strength with which a particular process occurs. In this Section we show how both are related to the underlying theory through a *matrix element*, a quantity which is calculated by using methods from quantum mechanics or quantum field theory. Later, we will see that these calculations can be replaced by describing the process with Feynman diagrams and applying set of "rules" to these diagrams.

4.1 Decays

All but very few stable particles decay with a **lifetime**, τ, ranging from $\mathcal{O}(10^{-23}\,\mathrm{s})$ to $\mathcal{O}(10^{-6}\,\mathrm{s})$. The lifetime of particle is governed by the **decay rate** Γ which is the probability per unit time that a given particle will disintegrate. This can be written as

$$\mathrm{d}N = -\Gamma N \mathrm{d}t \qquad \Longrightarrow \qquad N(t) = N(0)\exp -\Gamma t \tag{32}$$

where $N(t)$ is the number of particles surviving at time t. The decay process is intrinsically *random* and τ is the average or "mean" lifetime[8] which is simply the inverse of the decay rate. Lifetimes greater than a few times $10^{-16}\,\mathrm{s}$ are measurable. For decays mediated by the strong interaction, τ is unmeasurably short and instead one measures the natural **width**, Γ, which is the spread in energy of the decaying state and has units of energy. Note that the decay rate Γ, in s^{-1}, and the natural decay width, Γ, in MeV, are the same physical quantity:

$$\tau = \frac{1}{\Gamma} \quad \text{(decay rate)}\,, \qquad \tau = \frac{\hbar}{\Gamma} \quad \text{(decay width)}\,. \tag{33}$$

They have different SI units which are related by $\hbar = 6.582 \times 10^{-22}$ MeV s, see Equation 2. This relation between the lifetime and the width of a decaying particle state follows from the Uncertainty principle, $\Delta E \Delta t \sim \hbar$.

Most particles have many possible decay paths, e.g. the π^+ meson has the following decay modes: $\mu^+\nu_\mu$, $\mu^+\nu_\mu\gamma$, $e^+\nu_e$, $\pi^0 e^+\nu_e$ and others. Each of these processes has its own individual decay rate. The total decay rate Γ_{tot} is the sum of these and the lifetime of the particle is the inverse of Γ_{tot}. In addition to the lifetime, we define the *branching ratio* $\mathcal{B}_i$, that is, the fraction of all particles of a given type that decay into the specific mode i:

$$\Gamma_{\mathrm{tot}} = \sum_{i=1}^{n} \Gamma_i\,, \qquad \tau = \frac{1}{\Gamma_{\mathrm{tot}}}\,, \qquad \text{and} \qquad \mathcal{B}_{\mathrm{i}} = \Gamma_i / \Gamma_{\mathrm{tot}}\,. \tag{34}$$

Clearly, $\sum_{\mathrm{i}} \mathcal{B}_{\mathrm{i}} = 1$. When studying a particle, experimentalists measure its lifetime and the branching ratios of its decay channels. The theorist tries to calculate the various decay rates so that these can be compared with the experimental results.

[8] The largest decay rate is actually at $t = 0$.

4.2 Scattering

In a scattering experiment a beam of particles of type a hits a target containing particles of type b or two beams of particles collide head on. We rewrite the most general form for this reaction:

$$\underbrace{a + b}_{i} \quad \longrightarrow \quad \underbrace{c + d + e + \cdots}_{f}\,,$$

defined in Equation 26. In Figure 12 we make a sketch of this reaction in the laboratory frame.

Figure 12: A particle beam impinges on a target and scatters into the solid angle $\mathrm{d}\Omega$ (detector).

The strength of a particular reaction between the two particles is specified by the interaction **cross section** which is defined as follows. A parallel beam of particles of type a impacts normally on a target of thickness dx containing n_b particles of type b per unit volume. If the density of particles in the incident beam is n_a the flux F of beam particles per unit area and unit time through the target is

$$F = n_a v_i \tag{35}$$

where v_i is the velocity of the incident beam relative to the target. If each of the target particles has an effective cross section σ the probability that any particle a will interact is $\sigma n_b \mathrm{d}x$. The total number of scatters N can be written as

$$N = F N_b \sigma = n_a v_i N_b \sigma \tag{36}$$

where N_b is the total number of particles in the target. Thus the cross section is equal to the reaction rate per target particle and unit incident flux.

For colliding beams, the above description is not really appropriate - which is the beam and which is the target. In a more general formalism which is valid for fixed target and colliding beam experiments the incident flux and the number of target particles are replaced by the **luminosity** $\mathcal{L}$. To obtain information about the dynamics of the interaction we need to measure differential cross sections. In general, the cross section varies as a function of the *scattering angle* θ between initial and final state particles. The number of scatters dN per unit time into the *solid angle* element $\mathrm{d}\Omega = |\sin\theta \mathrm{d}\theta \mathrm{d}\varphi| = |\mathrm{d}\cos\theta\varphi|$ is

$$\mathrm{d}N = n_a v_a N_b \mathrm{d}\sigma = F N_b \mathrm{d}\sigma = \mathcal{L}\mathrm{d}\sigma \qquad \Longrightarrow \qquad \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{\mathcal{L}}\frac{\mathrm{d}N}{\mathrm{d}\Omega} \tag{37}$$

$$\sigma = \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega \qquad \Longrightarrow \qquad N = \mathcal{L}\sigma \ .$$

The **differential cross section** $\mathrm{d}\sigma/\mathrm{d}\Omega$ is the intrinsic scattering probability which describes the underlying physics of the process. It is specific to the reaction being studied and the quantity we are really interested in. It follows that the event rate is simply the cross section times the luminosity. The total cross section is a single number. and has units of area. It can be loosely interpreted as the effective size of the target. The unit of the cross section is defined as the barn: $1b = 10^{-28}\,\mathrm{m}^2 = 10^{-24}\,\mathrm{cm}^2$ which is roughly the geometrical cross section of a nucleus of mass number $A = 100$. As an example, if you fire a beam of electrons into a tank of hydrogen the cross section corresponds to the size of the proton seen by the incident beam. However, the proton wave function has a "soft" edge, it is not a simple hit-or-miss case. In addition, there is more to the cross section than just geometry. Since the different interactions have different strengths, we expect photons to scatter more often off protons than neutrinos, but less so than pions, see Table 5. The

pp	40 mb	pn	40 mb	$\overline{p}p$	45 mb	$\overline{p}n$	45 mb
π^+p	25 mb	π^-p	25 mb	π^+n	25 mb	π^-n	25 mb
K^+p	18 mb	K^-p	20 mb	K^+n	18 mb	K^-n	20 mb
γp	0.1 mb			γn	0.1 mb		
νN	0.34 pb	$\overline{\nu}N$	0.17 pb				

Table 5: Total cross-sections for various processes at a laboratory beam momentum of 50 GeV/c.

cross section depends, too, on the final state particles. We can measure not only *elastic* scattering, $e^- + p \to e^- + p$. There are also *inelastic* processes, for example $e^- + p \to e^- + p + \pi^0$, and many more In experiments that do not examine the final state products, see Figure 10, we measure the total *inclusive* cross section, which is the sum of the cross sections of all possible reactions:

$$\sigma = \sum_{i=1}^{n} \sigma_i \ . \tag{38}$$

The luminosity is specific to a given experiment and describes the rate at which particles come together. It has units of $\mathrm{cm}^{-2}\,\mathrm{s}^{-1}$ or $\mathrm{pb}^{-1}\,\mathrm{s}^{-1}$ and depends on the size and the intensity of the beam(s) and on the density and length of the target. For two colliding beams the luminosity is proportional to the number of particles in each bunch, N_1 and N_2, the number of bunches in either beam around the ring, n, and the revolution frequency, f[9], and inversely proportional to the

[9]For a linear collider $n = 1$ and f is the repetition frequency.

transverse area of the overlapping beam, $r_x r_y$. For a fixed target experiment the luminosity can be written as the product of the number of incident beam particles per bunch, N_B, the repetition rate of bunches, f_B, and the number of target particles per unit area N_T. These relations are summarised here:

$$\mathcal{L} = \frac{N_1 N_2}{r_x r_y} f n \tag{39}$$

$$\mathcal{L} = N_B f_b N_T \qquad\qquad N_T = \frac{N_A}{A}\rho l n_T$$

where N_A, A, are Avogadro's number and the atomic mass number of the target atoms, ρ and l are the mass density and length of the target, and n_T is the number of target particles per target atom, respectively.

4.3 Transition Rates

To proceed we need an important result from quantum mechanics. According to 1^{st} order **perturbation theory** we can write the **matrix element** $\mathcal{M}$ of a given process as

$$\mathcal{M} = \langle f|\, H\, |i\rangle \tag{40}$$

where $|i\rangle$ and $|f\rangle$ are the initial and final states, and the Hamiltonian of the perturbation, H, is the interaction potential. The amplitude $\mathcal{M}$ contains the *dynamics* of the interaction and can be calculated by summing up the relevant Feynman diagrams. Effectively, $\mathcal{M}$ is the overlap integral over volume, $\int \psi_f^* H \psi_i dV$, between the initial-state and final-state wave functions, brought about by the interaction potential H. **Fermi's Golden Rule** gives the **transition rate** W for a *non-relativistic* scattering or decay process as

$$(\text{transition rate}) = \frac{2\pi}{\hbar}|\mathcal{M}\,|^2 \times (\text{phase space}) \tag{41}$$

$$W = \frac{2\pi}{\hbar}\,|\,\mathcal{M}\,|^2\,\rho(E_f)$$

where $\rho(E_f)$ describes the density $\mathrm{d}N/\mathrm{d}E$ of the final states. This density is also known as **phase space** and contains only *kinematic* information. It depends on the masses, energies and momenta of the particles and reflects the fact that a process is more likely to occur the more final state configurations are kinematically available. It can be calculated as follows: For a single particle the wave function is defined as

$$\psi = N \exp\left(-ip \cdot x\right) \qquad\qquad N = \frac{1}{\sqrt{V}} \tag{42}$$

In a cubic box with volume $V = L^3$ the number of allowed states of momentum with x component in the range p_x to $p_x + \mathrm{d}p_x$ is restricted by the boundary condition

$$k_x \equiv \frac{p_x}{\hbar} = \frac{2\pi n_x}{L} \tag{43}$$

Thus for each particle into the solid angle $\mathrm{d}\Omega$ and enclosed in the physical volume V the number of final states is

$$\mathrm{d}N = \frac{V}{(2\pi\hbar)^3} p^2 \mathrm{d}p \mathrm{d}\Omega \tag{44}$$

where p is the magnitude of the 3-momentum and $d^3p = p^2\mathrm{d}p\mathrm{d}\Omega$ is momentum space volume element in spherical coordinates. The arbitrary normalisation volume V must cancel out in the expression for the cross section. In practice, one simply sets $V = 1$. The cross section is related to the Golden rule by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{W}{F} = \frac{2\pi}{\hbar}\frac{\mid \mathcal{M} \mid^2}{v_i}\frac{1}{(2\pi\hbar)^3}p^2\frac{\mathrm{d}p}{\mathrm{d}E} \tag{45}$$

where $F = v_i$ is the flux for a normalised incoming particle.

For *relativistic* particles, the cross section and transition rate must be *Lorentz invariant* quantities. The Golden rule is derived in quantum field theory and the transition rate per unit volume can be written as

$$W_{fi} = (2\pi)^4\delta^{(4)}(p_a + p_b - \sum_{f=1}^{n} p_f) \mid \mathcal{M} \mid^2 \tag{46}$$

The 4-dimensional delta function[10] expresses energy-momentum conservation for the process. The cross section is related to the Golden rule by

$$\sigma = \frac{W_{fi}}{\text{flux}} \text{ (number of final states)} \tag{47}$$

where the *flux* is the rate at which incident particles come together. The flux represents the density of the initial state particles.

In special relativity a volume with length L is contracted to L/γ along the boost direction. Since $\gamma = E/m$ it is convenient to normalise the wave functions of relativistic particles to a density of $2E$ particles per unit volume[11]. In this way, the matrix element $\mathcal{M}$ is a Lorentz invariant. The phase space factor, which is the number of final states available per unit volume for each particle, becomes $\frac{\mathrm{d}^3p}{(2\pi)^3 2E}$. The total Lorentz invariant phase space $\mathrm{d}Q$ includes the factor due to energy-momentum conservation and becomes:

$$\mathrm{d}Q = (2\pi)^4\delta^{(4)}(p_a + p_b - \sum_{f=1}^{n} p_f)\prod_{f=1}^{n}\frac{\mathrm{d}^3p_f}{(2\pi)^3 2E_f} \ . \tag{48}$$

[10] A few useful Dirac delta function relations are:

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\mathrm{d}x = f(a) \qquad \delta^{(4)}(p^\mu) = \delta(E)\delta(\vec{p})$$

$$\delta(\vec{r}) = \delta(x)\delta(y)\delta(z) \qquad \delta(kx) = \frac{1}{\mid k \mid}\delta(x)$$

[11] The factor of 2 is a convention.

The initial flux can be calculated easiest in a *fixed target* experiment. The number of beam particles with velocity $\vec{v}_a = \frac{\vec{p}}{E_a}$ passing through a unit area per unit time is $\mid \vec{v}_a \mid 2E_a$ and the number of target particle per unit volume is $2E_b$. The incident flux in the laboratory thus becomes $F = 4 \mid \vec{p}_a \mid m_b$. The flux is also Lorentz invariant and, for a general collision, can be written as

$$\begin{aligned} F &= \mid \vec{v}_a - \vec{v}_b \mid 2E_a 2E_b \\ &= 4\,(\mid \vec{p_a} \mid E_b + \mid \vec{p_b} \mid E_a) \\ &= 4\left((p_a \cdot p_b)^2 - m_a^2 m_b^2\right)^{\frac{1}{2}} \end{aligned} \tag{49}$$

which is manifestly Lorentz invariant, i.e. it is only a function of invariants.

We can now rewrite Equation 47 and obtain a Lorentz invariant definition for the cross section

$$\begin{aligned} \mathrm{d}\sigma &= \frac{\mid\mathcal{M}\mid^2}{F}\mathrm{d}Q \\ &= \frac{\mid\mathcal{M}\mid^2}{\mid \vec{v}_a - \vec{v}_b \mid 2E_a 2E_b}(2\pi)^4\delta^{(4)}(p_a + p_b - \sum_{f=1}^{n} p_f)\prod_{f=1}^{n}\frac{\mathrm{d}^3p_f}{(2\pi)^3 2E_f} \end{aligned} \tag{50}$$

The derivation of a formula for decay rates proceeds along similar lines. The differential rate for the decay $A \to 1 + 2 + 3 + \cdots + n$ is defined as

$$\begin{aligned} \mathrm{d}\Gamma &= \frac{\mid\mathcal{M}\mid^2}{F}\mathrm{d}Q \\ &= \frac{\mid\mathcal{M}\mid^2}{2E_A}(2\pi)^4\delta^{(4)}(p_A - \sum_{i=1}^{n} p_f)\prod_{f=1}^{n}\frac{\mathrm{d}^3p_f}{(2\pi)^3 2E_f} \end{aligned} \tag{51}$$

The incident flux is now $2E_A$, the number of decay particles per unit volume. When we want to compare measured cross sections and decay rates with Equations 50 and 51 we have to integrate over all unobserved momenta.

4.4 Cross Sections and Decay Rates

Let us derive the cross section for the scattering process $a + b \to c + d$. It is always useful to perform these calculations in the centre-of-mass system (CoM). If the kinematics of all initial state particles are known, measured momenta or energies can always be boosted into the CoM. From Equation 50 we can write down the Lorentz invariant phase space element

$$\mathrm{d}Q = (2\pi)^4\frac{\mathrm{d}^3p_c}{(2\pi)^3 2E_c}\frac{\mathrm{d}^3p_d}{(2\pi)^3 2E_d}\delta^{(4)}(p_a + p_b - p_c - p_d)$$

We then use the following relations which are valid in the CoM

$$\begin{aligned} \sqrt{s} &= W = E_a + E_b \\ p_f &= \mid \vec{p}_c \mid = \mid \vec{p}_d \mid \\ \mathrm{d}^3p_f &= p_f^2\mathrm{d}p_f\mathrm{d}\Omega \end{aligned}$$

to obtain

$$dQ = \frac{1}{4\pi^2}\frac{p_f^2 dp_f d\Omega}{4E_c E_d}\delta(W - E_c - E_d)$$

Integrating out the delta function by substituting dp_f with dW we find

$$dQ = \frac{1}{4\pi^2}\frac{p_f d\Omega}{4\sqrt{s}} \tag{52}$$

Similarly, we can calculate the flux to be

$$F = 4p_i\sqrt{s} \tag{53}$$

where $p_i =| \vec{p}_a |=| \vec{p}_b |$. Hence, the differential cross section is

$$\left.\frac{d\sigma}{d\Omega}\right|_{\mathrm{CoM}} = \frac{1}{64\pi^2 s}\frac{p_f}{p_i}|\mathcal{M}|^2 \tag{54}$$

In general, $|\mathcal{M}|^2$ depends on all four momenta p_k, and on their spins. In the CoM, only two of momenta (e.g. $\vec{p_i}$ and $\vec{p_f}$) are independent. Thus for spin-less particles, $|\mathcal{M}|^2$ is a function of the Lorentz invariants, p_i^2, p_f^2, and $\vec{p_i}\cdot\vec{p_f} = p_i p_f \cos\theta$, only.

We often do not record information about the spin of the particles in an experiment. To calculate this *unpolarised* cross section we therefore allow for scattering of all possible spin configurations. The number of spin states available in each initial and final states are:

$$g_i = (2s_a + 1)(2s_b + 1) \qquad g_f = (2s_c + 1)(2s_d + 1)\ . \tag{55}$$

To obtain the unpolarised cross section, we average over the spins of the incoming particles and sum over all spins of the particles in the final state. This corresponds to multiplying Equation 54 by the factor g_f/g_i.

The integrated decay rate for a 2-body decay $A \to 1 + 2$ can be calculated by integrating Equation 51 over all possible momenta p_1 and p_2. In the rest frame of the particle A, we obtain

$$d\Gamma = \frac{p_f}{32\pi^2 m_A^2}|\mathcal{M}|^2\, d\Omega\ . \tag{56}$$

The same warning with respect to spin as for Equation 54 applies and, for spinless particles, $|\mathcal{M}|^2$ only depends on p_f. We integrate over $d\Omega$ to derive the total decay rate:

$$\Gamma = \frac{p_f}{8\pi m_A^2}|\mathcal{M}|^2\ . \tag{57}$$

So far, we have computed the cross sections for the process $a + b \to c + d$. However, there is a **crossing symmetry**. When replacing ingoing particles by outgoing antiparticles and vice versa, we obtain the so-called **crossed reactions** which are also allowed. Moreover these are described by the same matrix element but, of course, different kinematic constraints:

$$\begin{array}{ll} a + b \to c + d & a + \bar{d} \to c + \bar{b} \\ a + \bar{c} \to \bar{b} + d & a \to \bar{b} + c + d \\ c + d \to a + b & \end{array} \tag{58}$$

For example, electron-positron annihilation into a muon pair, $e^-e^+ \to \mu^-\mu^+$, and electron-muon scattering, $e^-\mu^- \to e^-\mu^-$ are crossed reactions.

4.5 Decays and Resonances

Particles that interact strongly, such as the $\Delta(1232)$ resonance, decay unmeasurably fast with lifetimes of $\mathcal{O}(10^{-23}\,\mathrm{s})$. These particles are in fact observed as **resonances** in the study of cross sections of their production mechanism, e.g. pion-nucleon scattering $\pi + p \to \Delta(1232) \to \pi + p$. A resonance is characterised by its mass and by its decay width Γ. We have already shown that a lifetime of $10^{-23}\,$s corresponds to $\Gamma = 66$MeV, see Equations 2 and 33.

The form of the line shape of the resonance is determined by the exponential time dependence, see Equation 32. The energy dependence of the cross section is simply the Fourier transform of the time pulse. The wave function of decaying state with resonance energy E_R and width Γ, as well as its Fourier transform can be written as

$$\psi(t) = \psi(0)\exp(-iE_R t - \Gamma t/2) \tag{59}$$

$$\chi(E) = \int_0^\infty dt \exp(iEt)\psi(t) = \psi(0)\cdot\frac{K}{(E - E_R) + i\Gamma/2}$$

where K is a constant. The cross section $\sigma(E)$ measuring the probability of two particles 1 and 2 forming a resonant state R will be proportional to $\chi^*(E)\chi(E)$:

$$\sigma(E) = \sigma_{\max}\cdot\frac{\Gamma^2/4}{(E - E_R)^2 + \Gamma^2/4} \tag{60}$$

and is called the **Breit-Wigner** resonance formula. The cross section falls to half its peak value when $E - E_R = \pm\Gamma/2$. We refer to Γ as **Full Width at Half Maximum (FWHM)**.

The maximum of the Breit-Wigner cross section can be obtained by arguments from wave optics (see for example Perkins, page 55). We obtain

$$\sigma_{\mathrm{elastic}}^{\max} = 4\pi\lambda\!\!\!^{-}{}^2(2l + 1) \tag{61}$$

where $\lambda\!\!\!^{-} = \hbar/p$ is the de-Broglie wavelength of the resonance and l is the orbital angular momentum. So far we have not considered the spins of the particles, J_1 and J_2, and of the resonance J_R. Using Equation 55, we obtain a general Breit-Wigner cross section

$$\sigma(E) = 4\pi\lambda\!\!\!^{-}{}^2\frac{(2J_R + 1)}{(2J_1 + 1)(2J_2 + 1)}\frac{\Gamma^2/4}{(E - E_R)^2 + \Gamma^2/4} \tag{62}$$

If a resonant state decays into different channels we have to replace Γ^2 by $\Gamma_{\mathrm{in}}\Gamma_{\mathrm{out}}$ in the numerator. Note that Equation 62 is not Lorentz invariant and is used for non-relativistic cases. By describing resonances such as the Z^0 we use a relativistic Breit-Wigner formula

$$\sigma(s) = \sigma_{\max}\cdot\frac{s\Gamma^2}{(s - M_0^2)^2 + s\Gamma^2} \tag{63}$$