Physical Mathematics : MOCK EXAM

Exam conditions: recommend 2h to answer
ALL part A questions
TWO part B questions.

Section A Answer ALL questions

A.1.

A quantity is measured N times with values x_j.

Define the mean and variance of this quantity. What can you say about the distribution of the average ?

[5]

A.2. Prove that the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

[5]

A.3. Suppose $Y_1(x)$ and $Y_2(x)$ are eigenfunctions of $\frac{d^2}{dx^2}$ with eigenvalues λ_1 and λ_2 respectively. Further, suppose they also satisfy periodic boundary conditions $Y_1(x) = Y_1(x+L)$, and $Y_2(x) = Y_2(x+L)$. By considering

$$\int_0^L \left(Y_1'' Y_2 - Y_1 Y_2''\right) dx$$

show that if $\lambda_1 \neq \lambda_2$ the functions must be orthogonal.

[5]

A.4. Show that

$$Y_2^2(\theta, \phi) = \sin^2 \theta e^{i2\phi}$$

is an eigenmode of the operator

$$L_z = -i\frac{\partial}{\partial \phi}$$

with eigenvalue 2, and that it is an eigenmode of the operator

$$L^2 = -\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}\right)$$

with eigenvalue 6.

[5]

Section B Answer TWO questions

B.1. **Central limiting theorem:** if X is a zero mean, unit variance random variable with otherwise arbitrary distribution $P_X(x)$, then the average of N of these

$$A_N = \frac{1}{N}(X_1 + X_2 + \ldots X_N)$$

will, in the limit of large N, be Gaussian distributed.

a) Represent the distribution of the sum of N random variables $X_1 \ldots X_N$ as a convolution. **[4]**

b) Apply the convolution theorem to represent this distribution in terms of the Fourier transform of $P_X(x)$, and change variables in k to show that

$$P_{A_N}(u)\frac{1}{\sqrt{2\pi}}(\sqrt{2\pi})^{N-1}\int\limits_{-\infty}^{\infty} dk' e^{-ik'u}\left(\tilde{P}_X(\frac{k'}{N})\right)^N dk'$$

[6]

[Hint: you may assume that the Fourier Transform is $\tilde{P}_X(k) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} P_X(x)e^{ikx}dx$, the corresponding convolution theorem is $f(x)*g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{2\pi}F(k)G(k)e^{-ikx}dk$.]

c) Write $\tilde{P}_X(\frac{k'}{N})$ as the Fourier transform of $P_X(x)$, Taylor expand this to second order in $\frac{k'}{N}$, and simplify using the given properties of the distribution. **[4]**

d) Use this result to show

$$P_{A_N}(u) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} dk' e^{-ik'u}\left(1 - \frac{k'^2}{2N}\frac{1}{N}\right)^N dk'$$

[3]

e) You may assume that $\lim_{N\to\infty}(1+b/N)^N = e^b$. Use this to conclude that

$$P_{A_N}(u) \to \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} dk' e^{-ik'u}\frac{1}{\sqrt{2\pi}}e^{-\frac{k'^2}{2N}}dk'$$

[3]

We can now recognise this as the Fourier transform of a Gaussian of width $\sigma_A = \frac{1}{\sqrt{N}}$, and conclude (no proof required)

$$P_{A_N}(u) \to \frac{1}{\sqrt{2\pi}\sigma_A}e^{-\frac{u^2}{2\sigma_A^2}}$$

B.2. **Harmonic Oscillator** The time independent Schroedinger equation for the one dimensional harmonic oscillator with potential $V(y) = \frac{1}{2}ky^2$ is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(y)}{dy^2} + \frac{1}{2}ky^2\psi(y) = E\psi(y).$$

Make a change of variables to $x = \alpha y$ and choose α to reduce this equation to the form

$$\frac{d^2\psi(x)}{dx^2} + (\lambda - x^2)\psi(x) = 0$$

defining λ in terms of E. **[4]**

Substitute the form $\psi(x) = H(x)e^{-\frac{x^2}{2}}$ to reduce this to the Hermite differential equation: **[4]**

$$H'' - 2xH' + (\lambda - 1)H = 0$$

Take a power series $H(x) = \sum_{n=0}^{\infty} c_n x^n$ and apply the method of Froebenius to obtain the recurrence relation

$$c_{m+2} = \frac{2m + 1 - \lambda}{(m + 2)(m + 1)}c_m$$

[4]

Deduce the large x asypmtotic form of H(x) if the series does not terminate **[2]**

Given the series must terminate determine the allowed values for λ and hence for E. **[2]**

From the recurrence relation deduce the values of m for which c_m can first become non-zero **[1]**

Write down the first three Hermite polynomials $H(x)$ taking the first term to have coefficient 1 for each. **[3]**

B.3. Spherical polar separation of variables

(a) Define the scale factors h_i for orthogonal curvilinear coordinate system parametrised by coordinates c_1, c_2 and c_3. **[2]**

Define the gradient of a scalar function $f(c_1, c_2, c_3)$ in terms of the unit vectors and scale factors for this general coordinate system. **[2]**

By applying the divergence theorem to the infinitessimal volume generated by infinitessimal changes δc_1, δc_2 and δc_3, derive expression for the divergence of a vector function $\mathbf{v}(c_1, c_2, c_3)$. **[6]**

Determine the scale factors for spherical polar coordinates,

$$\mathbf{x} = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta)$$

and write down the corresponding gradient operator. **[2]**

Hence show that the Laplacian operator for spherical polar coordinates is **[3]**

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\sin\theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

(b) The wave equation in three dimensions is

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

Find separated ordinary differential equations for the separated form

$$u(r, \phi, t) = R(r)\Phi(\phi)\Theta(\theta)T(t)$$

and clearly identify your constants of separation.

[5]