Physical Mathematics 2010: Problems 1 (week 2)

1. Hellenic Calligraphy. Like it or not, Greek letters are very popular in physics. It will be easier to follow what is going on in the course if you know how they are pronounced.

lower case	name	upper case
α	alpha	A
β	beta	B
γ	gamma	Γ
δ	delta	Δ
ϵ, ε	epsilon	E
ζ	zeta	Z
η	eta	H
θ, ϑ	theta	Θ

lower case	name	upper case
ι	iota	I
κ	kappa	K
λ	lambda	Λ
μ	mu	M
ν	nu	N
ξ	xi	Ξ
o	omicron	O
π, ϖ	pi	Π

lower case	name	upper case
ρ, ϱ	rho	P
σ, ς	sigma	Σ
τ	tau	T
υ	upsilon	Υ
ϕ, φ	phi	Φ
χ	chi	X
ψ	psi	Ψ
ω	omega	Ω

2. **Trig identities:** Using

$$e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$$

and

$$e^{i\theta} = \cos\theta + i\sin\theta$$

prove that

(a) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

(b) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

(c) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

(d) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

(e) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

(f) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

(g) $2\sin A\sin B = -\cos(A+B) + \cos(A-B)$

(h) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(i) $\cos 2\theta = 1 - 2sin^2\theta$

(j) $\sin 2\theta = 2\sin\theta\cos\theta$

3. **Travelling and standing waves**

(a) Explain why $\cos(kx - \omega t)$ is a travelling wave

(b) Explain why $\cos(kx)\cos(\omega t)$ is a standing wave

(c) Write $\cos(kx - \omega t)$ as a sum of standing waves

(d) Write $\cos(kx)\cos(\omega t)$ as sum of travelling waves

4. **Trig differentiation** Differentiate

   (a) $\sin 3x$

   (b) $\sin(\cos 4x)$

   (c) $\sin(5\cos 4x)$

   (d) e^{ax}

   (e) e^{iax}

   (f) e^{iax^2}

   (g) By differentiating e^{ikx} and considering real and imaginary parts find the derivatives of $\cos kx$ and $\sin kx$

5. **Trig integration** Integrate

   (a) $\sin 3x$

   (b) $\cos 5x$

   (c) $\cos 5x \sin 3x$

   (d) $\cos 2x \cos 8x$

   (e) $\sin x \sin 3x$

   (f) $2x \cos x^2$

   (g) e^{ax}

   (h) e^{iax}

6. **Orthogonality**

   for $k_n = n\frac{\pi}{L}$, $k_m = m\frac{\pi}{L}$, show

   (a) $\int_{-L}^{L} \sin k_n x \sin k_m x = L\delta_{mn}$

   (b) $\int_{-L}^{L} \cos k_n x \cos k_m x = \begin{cases} L\delta_{mn} & n \neq 0 \\ 2L\delta_{mn} & n = 0 \end{cases}$

   (c) $\int_{-L}^{L} \sin k_n x \cos k_m x = 0$

   (d) For each case draw a graph explaining why.

7. **Integration by parts:**

(a) $\displaystyle\int_0^\infty dx\, x\, e^{-ax}$.

(b) $\displaystyle\int_0^\infty dx\, x^2\, e^{-ax}$.

(c) $\displaystyle\int_0^\pi dx\, x \cos x$.

(d) $\displaystyle\int_{-\pi}^\pi dx\, x \sin x$.

(e) $\displaystyle\int_a^{2a} dx\ \ln\left(\frac{x}{\pi}\right)$. [Hint: substitute $u = x/\pi$.]

(f) $\displaystyle\int_1^y dx\, x \ln x$.

(g) $\displaystyle\int_0^1 dx\ (1-x)\ln(1-x)$. [Hint: substitute $u = 1-x$.]

8. **Curve sketching**

(a) $f(x) = \dfrac{1}{x-a} - \dfrac{1}{x+a}$.

(b) $f(x) = \operatorname{sinc} x \ = \dfrac{\sin x}{x}$.

(c) $f(x) = \dfrac{\cos x}{x}$.

(d) Sketch (and label) $f_1(x) = xe^{-x}$ and $f_2(x) = xe^{-2x}$ on the same graph

(e) $f(x) = x^2\, e^{-x}$ for $x \geq 0$.

(f) $f(x) = \sin(\pi x)e^{-x}$ for $x \geq 0$.

(g) Sketch the function $f(x) = \exp\left(-\frac{x^2}{2\sigma^2}\right)$, labelling locations of any crossings of the axes. How would increasing σ change the plot?
Sketch $f(x) \times \cos(\pi x)$ for $\sigma \simeq 3$.
Label the value at $x = 0$ and the position of any nodes (zeros).

(h) $f(x) = e^{-a^2x^2} \cos bx$ with $b > 2\pi a$.

9. Calculate these integrals by integrating by parts. They will be very useful.

(a) $\displaystyle\int_{-L}^L dx\, x \sin\frac{m\pi x}{L}$.

(b) $\displaystyle\int_{-L}^L dx\, x^2 \sin\frac{m\pi x}{L}$.

(c) $\displaystyle\int_{-L}^L dx\, x \cos\frac{m\pi x}{L}$.

(d) $\displaystyle\int_{-L}^L dx\, x^2 \cos\frac{m\pi x}{L}$.

Repeat (c) when the lower limit of the integral is 0 rather than $-L$.

10. **l'Hôpital's Rule**

> *If* $f(x=c) = g(x=c) = 0$ *for two functions at some value* $x = c$*, then*
>
> $$\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \lim_{x\to c}\left[\frac{f'(x)}{g'(x)}\right] \ ,$$

   (a) Evaluate $\lim_{x\to 0} \mathrm{sinc}(ax)$ where $\mathrm{sinc}(x) \equiv \dfrac{\sin x}{x}$.

   (b) Evaluate $\lim_{x\to 0} \dfrac{\cos x}{x}$.

   (c) Prove l'Hôpital's Rule by writing $f(x)$ and $g(x)$ as Taylor series expansions around $x = c$.

11. In this question we will prove the standard result

$$I = \int_{-\infty}^{\infty} du\ e^{-u^2} = \sqrt{\pi}$$

   (a) Write I^2 as a double integral. In the first factor, call the dummy variable x and in the second call it y.

   (b) Change to circular polar coordinates $(x, y) \to (\rho, \phi)$ and evaluate the angular integral (remember if you change variables correctly, the area of a ring should enter as $2\pi r dr$).

   (c) Now do the radial integral and obtain an expression for I.

   (d) Use this to show that the "normalised Gaussian"

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/(2\sigma^2)}$$

   really is normalised.

12. Evaluate $\int_{-\infty}^{\infty} dx\ e^{-ax^2-bx}$, given the standard result $\int_{-\infty}^{\infty} dx\ e^{-x^2} = \sqrt{\pi}$.

   [Hints: begin by "completing the square" to write $ax^2 + bx$ in the form $(Ax + B)^2 + C$]

13. The transverse displacement $u(x, t)$ of a string stretched between $x = 0$ and $x = L$ and initially at rest is described by the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

   where c is the (constant) wave speed.

   Use the method of separation of variables to obtain a solution of this equation in the form

$$u(x, t) = (A_k \sin kx + B_k \cos kx) \times (C_k \sin \omega_k t + D_k \cos \omega_k t) \ .$$

   Explain clearly the meaning of all the symbols including the relationship between k and ω_k.

   Show how the imposition of the boundary and initial conditions restricts the possible modes of vibration and hence leads to a general solution

$$u(x, t) = \sum_{n=1}^{\infty} E_n \sin k_n x \cos \omega_n t$$

Again, the meanings of all symbols should be clearly explained.

The functions $\sin k_n x$ form an orthogonal set:

$$\int_0^L dx\ \sin(k_n x)\ \sin(k_m x) = a_n \delta_{nm}\ ,$$

for some constants a_n (which you need not evaluate). If the string initially has a displacement $f(x)$, show that

$$E_n = \frac{1}{a_n}\int_0^L dx\ f(x) \sin k_n x\ .$$

(Aug.07.7)

What physical role does the constant c play for: (a) travelling waves, and (b) standing waves?

14. Sketch each of the following functions and find their Fourier series expansions (i.e. components a_n and b_n) in the range $-L \le x \le L$. Add to your sketch the periodically extended function described by the Fourier series. For each function, explain why particular components turn out to be zero.

    (a) $f(x) = \sin \dfrac{3\pi x}{L}$

    (b) $f(x) = \operatorname{signum} x \equiv \dfrac{x}{|x|} \equiv \begin{cases} +1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$

    (c) $f(x) = x$

    (d) $f(x) = |x|$

    (e) $f(x) = x^2$

    [Hint: for (b), (d) split the integrals into two parts, $-L \le x < 0$ and $0 \le x \le L$.]

15. As seen in lectures, the transverse displacements of a string stretched from $x = 0$ to $x = L$ are described by a general solution:

$$u(x,t) = \sum_{n=1}^{\infty} \left(E_n \sin k_n x \sin \omega_n t + F_n \sin k_n x \cos \omega_n t\right) \tag{1}$$

    with $k_n = n\pi/L$ and $\omega_n = ck_n$.

    A guitar string is initially plucked gently from the centre such that $\dot{u}(x, t = 0) = 0$ and

$$u(x, t = 0) = \begin{cases} 2px/L & 0 \le x \le \frac{L}{2} \\ 2p(L - x)/L & \frac{L}{2} \le x \le L \end{cases}$$

    Sketch $u(x, t = 0)$ (labelling the maximum value). Why is it important that the string is plucked "gently"? Find E_n, F_n. For which n are E_n and F_n both zero (i.e. this frequency is not present)? Give a physical explanation. Which frequencies dominate? Again, give a physical explanation.

16. Sketch the following function and express it as a real Fourier series, finding its Fourier components, a_n and b_n:

$$f(x) = \begin{cases} (x + \pi)h & -\pi \le x \le 0 \\ (\pi - x)h & 0 \le x \le \pi \end{cases} \tag{2}$$

where h is a constant. Why are $b_n = 0$? Comment on the relative sizes of the non-zero components. Use the series to find a series expression for π (hint: note that $f(0) = \pi$ if $h = 1$). Add the first few terms of the series to see how well it does.

17. Reduce the 1-dimensional Schrödinger Equation to separated form

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\,\psi(x,t) = i\hbar\frac{\partial\psi(x,t)}{\partial t}$$

Also:

(a) What is the physical interpretation of the separation constant?

(b) What physical significance is attached to the normal modes?

(c) What changes (if anything) if the potential is time dependent i.e. $V(x) \to V(x,t)$?

(d) For the infinite square well, explain why the normal modes look (at least spatially) like Fourier basis functions

18. Reduce the 2-dimensional Schrödinger Equation to separated form when the potential has the form $V(x,y) = V(x) + V(y)$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} - \frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial y^2} + [V(x) + V(y)]\,\psi(x,t) = i\hbar\frac{\partial\psi(x,t)}{\partial t}$$

How could you proceed if $V(x,y) = V(x+y) + V(x-y)$?

19. **Rectangular drumskin**

The wave equation of a square drumskin (defined by $x \in [0,L]$, $y \in [0,L]$) is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Substitute a separable solution of the form $X(x)Y(y)T(t)$ and derive the separate ODE's for X, Y, and (t).

Apply the boundary conditions

$$u(x,L,t) = u(L,y,t) = u(x,0,t) = u(0,y,t) = 0$$

to x and y and find the allowed values for the separation constants governing the X, Y and T differential equations.

Suppose that initially the velocity

$$\frac{\partial u(x,y,t=0)}{\partial t} = 0$$

and

$$u(x,y,t=0) = x(L-x)y(L-y).$$

Find the motion for all later times.

20. The Fourier modes are defined (for integer n) as

$$\psi_n(x) = \begin{cases} 1 & \text{for } n = 0\ , \\ \cos\dfrac{n\pi x}{L} & \text{for } n \geq 1\ , \end{cases}$$
$$\text{and} \quad \phi_n(x) = \sin\frac{n\pi x}{L} \qquad \text{for } n \geq 1\ .$$

(a) Explain what it means to say $\{\psi_{n\geq 0}(x), \phi_{n>0}(x)\}$ form an orthogonal set of functions on the interval $-L \leq x \leq L$.

(b) Prove (by evaluating appropriate integrals) that they do indeed form an orthogonal set.

(c) Explain what we mean by an orthonormal set of functions, and how we convert an orthogonal set into an orthonormal one.

(d) Derive an orthonormal set of functions $\{\hat{\psi}_{n\geq 0}(x), \hat{\phi}_{n>0}(x)\}$ from the Fourier modes defined above.

(e) By analogy with vectors, explain what is meant by a basis set in function space. Does such a basis set have to be complete?

(f) Express the function $f(x) = \sin^2 \frac{2\pi x}{L}$ as a Fourier series, preferably (for your sake) without doing any integrals.

21. Write down the Fourier series for a general function $f(x)$ between $[-L, L]$, and then both differentiate it and integrate with respect to x.

Fourier Transforms

22. Define the Fourier transform of a function and the inverse relation that specifies how that function is represented in terms of its Fourier transform.

23. **Complex Fourier series** Write down the complex Fourier series for a function $f(x)$ which is defined on the interval $-L \leq x \leq L$ and give an expression for the coefficients.

    Consider the complex Fourier series for a *real* function $f(x)$. How are the coefficient c_n and c_{-n} related.

    Explain briefly how the Fourier transform $F(k)$ is obtained from the Fourier series and state the equations which relate $F(k)$ and $f(x)$, along with an expression for the wavenumber k.

    If $f(x)$ is real, how are $F(k)$ and $F(-k)$ related?

24. The top-hat function is defined to be zero everywhere, save in a region of width a centred at $x = 0$. Within this region, the function takes a constant value such that the total area under the function is unity.

    Obtain the Fourier transform of the top hat function, and comment on the relative widths of the function (in x-space) and its Fourier transform (in k-space). Make sure you provide some justification for your comments.

25. Write down the Fourier transform for a general function $f(x)$ between $[-\infty, \infty]$, and then both differentiate it and integrate with respect to x.

26. Find the Fourier transforms

$$F(k) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x)\, e^{ikx}$$

of the following functions

(a) $f(x) = e^{-a|x|}$ for positive a. [Hint: consider $x \geq 0$ (with $f(x) = e^{-ax}$) and $x < 0$ (with $f(x) = e^{ax}$) separately.]

(b) The normalised Gaussian:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) .$$

Dirac δ

27. (a) Explain how the Fourier transform is used to obtain the far field diffraction pattern for collimated light normally incident on a single slit of width a. Your explanation should include a labelled diagram, and you should carefully define all mathematical symbols.

    (b) In the limit $a \to 0$, the top hat function approaches the Dirac delta function. Discuss briefly the properties that make this true. By considering the integral of the product of an arbitrary function with the top-hat in this limit, derive the "sifting property" of the Dirac delta function. (You may find it convenient to exploit the smallness of a and Taylor expand around $x = 0$.)

    (c) What is the convolution of an arbitary function $h(x)$ with $\delta(x-b)$? Explain, with a sketch graph, the geometric significance of this particular convolution operation.

28. Use the sifting property of the Dirac delta function

$$\int_{-\infty}^{\infty} dx\ f(x)\delta(x) = f(0) . \tag{3}$$

to prove the properties of the Dirac delta function given in the lecture notes:

(a) $\delta(ax) = \delta(x)/|a|$ with special case $a = -1$: $\delta(-x) = \delta(x)$. [Hint: consider

$$\int_{-\infty}^{\infty} dx\ f(x)\delta(ax) ,$$

substitute $y = ax$ and compare with Eqn. (3). You will need to consider positive and negative a separately — in the latter case use $y = -|a|x$.]

(b) $x\ \delta(x) = 0$. [Hint: consider Eqn. (3) for $f(x) = x$. By looking at which points could contribute to the integral you can then say something about the integrand.]

(c) $\delta(x^2 - a^2) = [\delta(x-a) + \delta(x+a)]/(2|a|)$. Use the result that

$$\delta(g(x)) = \sum_n \frac{\delta(x - x_n)}{|g'(x_n)|} \tag{4}$$

where $x = x_n$ are the zeros of the function $g(x)$.

(d) A bit harder. Prove Eqn. (4) using the following method:

   i. Consider

$$I = \int_{-\infty}^{\infty} dx\ \delta(g(x)) \ :$$

   the only contributions are going to come when $g(x) = 0$. Why?

   ii. Split the integration up into small regions around the zeros of width ε. Why do we get

$$I = \sum_n \int_{x_n-\varepsilon}^{x_n+\varepsilon} dx\ \delta(g(x))\ ?$$

iii. Define $y = x - x_n$ and Taylor expand $g(x)$ for small y. If ε is small enough that we can ignore terms of order ε^2 and above, show that

$$I = \sum_n \int_{-\varepsilon}^{\varepsilon} dy\ \delta(yg'(x_n))\ .$$

iv. Make a substitution $z = yg'(x_n)$ and explain why

$$I = \sum_n \int_{-\infty}^{\infty} \frac{dz}{|g'(x_n)|}\ \delta(z)\ .$$

In particular, make sure you explain the modulus sign, and why the integration range has been expanded.

v. Evaluate I, and compare with Eqn. (3) to obtain the result.

29. Prove that $x\frac{d}{dx}\delta(x) = -\delta(x)$. [Hint: integrate the LHS over all x, and then integrate by parts.]

30. Find the Fourier transforms of the following functions[1]

    (a) $f(x) = \delta(x - d)$ for some fixed d
    (b) $f(x) = \delta(x + d) + \delta(x - d)$
    (c) $f(x) = \sum_{n=-N}^{N} \delta(x - nd)$ for integer N.
    (d) $f(x) = e^{iqx}$
    (e) $f(x) = \cos(ax)$. Comment on your answer.
    (f) $f(x) = \sin(ax)$. Comment on your answer.

    For what optical systems would (a), (b) and (c) be transmission functions?
    For what optical systems would the transmission functions be given by the convolutions of a top hat with (a), with (b) or with (c)?

Convolution

31. (a) Define the convolution of $f(x)$ and $g(x)$.
    (b) State and prove the convolution theorem for $f(x) * g(x)$
    (c) The normalised Gaussian is:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\ \exp\left(-\frac{x^2}{2\sigma^2}\right)\ .$$

    Evaluate the Fourier transform of the Gaussian.
    Evaluate the convolution of two Gaussians of width σ_1 and σ_2 using the convolution theorem
    Interpret your result

[1] If you look back through old exam papers (`http://www.lib.ed.ac.uk/resources/collections/exams.shtml`), this and last week's lists cover almost every Fourier transform that has been recently asked.

32. Show that the convolution of $f(x)$, the top-hat function of width $2a$ centred at the origin, with itself is a triangle-shaped function of width $4a$. [Hint: if y is the dummy variable in the convolution, consider in what range of x functions $f(y)$ and $f(x-y)$ are non-zero. Split the integral into 4 regions: $x < -2a$, $-2a \le x \le 0$, $0 \le x \le 2a$ and $x > 2a$.]

    The convolution operation is *commutative*: $f * g = g * f$. Use this property to sketch $f(x) * [\delta(x+b) + \delta(x-b)] * f(x)$ for $b > 2a$.

33. The following questions verify some expressions from the lecture notes

    (a) Slightly harder: Show that the convolution operation is *commutative* i.e. the result does not depend on the order: $f * g = g * f$. [Hint: change variable $y \to z = x - y$.]
    Show the convolution operation is *associative* i.e. we can combine convolutions in any order: $(f*g)*h = f*(g*h)$. [Hint: $(f*g)*h = \int dy \int dx\ f(x)\ g(y-x)\ h(z-y)$ and change variables $(x, y) \to (u = y - x, v = z - x)$. Recognise that we have $(g * h)$ convolved with f.]

PDE's

34. In this question, we'll see how to use Fourier transforms to solve the wave equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u(x,t)}{\partial t^2}$$

for an **infinitely long** string.

(a) First Fourier transform both sides to show that

$$-(ck)^2\, U(k,t) = \frac{\partial^2 U(k,t)}{\partial t^2}$$

where $U(k,t)$ is the (spatial) F.T. of $u(x,t)$.

(b) For fixed k, solve this ODE and show that the general solution is

$$U(k,t) = U_0(k)\cos(ckt) + \frac{V_0(k)}{ck}\sin(ckt)$$

where $U_0(k) \equiv U(k, t=0)$ and $V_0(k) \equiv \left.\frac{\partial U(k,t)}{\partial t}\right|_{t=0}$.

(c) The string is initially at rest and bent into the shape $u(x, t = 0) = \sin(\frac{3\pi x}{L})$ where L is just a parameter. Find the solution for $U(k,t)$ and hence $u(x,t)$, given these initial conditions. How does this compare to a similar problem of a string stretched between $x = 0$ and $x = L$? Why?

(d) A similar string is also initially at rest, but is instead bent into the shape

$$u(x, t=0) = \begin{cases} \sin(\frac{3\pi x}{L}) & \text{for } -L \le x \le L\,, \\ 0 & \text{otherwise.} \end{cases}$$

Find the solution for $U(k,t)$ and comment on the difference between this and the previous case.