Physical Mathematics 2010: Problems 1 (week 2)

1. Hellenic Calligraphy. Like it or not, Greek letters are very popular in physics. It will be easier to follow what is going on in the course if you know how they are pronounced.

lower case	name	upper case
α	alpha	A
β	beta	B
γ	gamma	Γ
δ	delta	Δ
ϵ, ε	epsilon	E
ζ	zeta	Z
η	eta	H
θ, ϑ	theta	Θ

lower case	name	upper case
ι	iota	I
κ	kappa	K
λ	lambda	Λ
μ	mu	M
ν	nu	N
ξ	xi	Ξ
o	omicron	O
π, ϖ	pi	Π

lower case	name	upper case
ρ, ϱ	rho	P
σ, ς	sigma	Σ
τ	tau	T
υ	upsilon	Υ
ϕ, φ	phi	Φ
χ	chi	X
ψ	psi	Ψ
ω	omega	Ω

2. **Trig identities:** Using

$$e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$$

and

$$e^{i\theta} = \cos\theta + i\sin\theta$$

prove that

(a) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

(b) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

(c) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

(d) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

(e) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

(f) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

(g) $2\sin A\sin B = -\cos(A+B) + \cos(A-B)$

(h) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(i) $\cos 2\theta = 1 - 2sin^2\theta$

(j) $\sin 2\theta = 2\sin\theta\cos\theta$

3. **Travelling and standing waves**

(a) Explain why $\cos(kx-\omega t)$ is a travelling wave

ANSWER:

Location of fixed phase (e.g. peak) of wave travels as defined by

$$kx - \omega t = \text{const}$$

so wave travels towards positive x.

$$x = \frac{\text{const} + \omega t}{k}$$

(b) Explain why $\cos(kx)\cos(\omega t)$ is a standing wave

ANSWER:

This has a time independent profile in x, but time dependent amplitude.

(c) Write $\cos(kx-\omega t)$ as a sum of standing waves

(d) Write $\cos(kx)\cos(\omega t)$ as sum of travelling waves

4. **Trig differentiation** Differentiate

   (a) $\sin 3x$
   ANSWER:
   $3\cos 3x$

   (b) $\sin(\cos 4x)$
   ANSWER:
   $-4\cos(\cos 4x)\sin 4x$

   (c) $\sin(5\cos 4x)$
   ANSWER:
   $-20\cos(5\cos 4x)\sin 4x$

   (d) e^{ax}
   ANSWER:
   ae^{ax}

   (e) e^{iax}
   ANSWER:
   iae^{iax}

   (f) e^{iax^2}
   ANSWER:
   $2iaxe^{iax^2}$

   (g) By differentiating e^{ikx} and considering real and imaginary parts find the derivatives of $\cos kx$ and $\sin kx$
   ANSWER:
   $\frac{d}{dx}\cos kx + i\frac{d}{dx}\sin kx = ike^{ikx} = -k\sin kx + ik\cos kx$, *and equate real and imaginary parts seperately.*

5. **Trig integration** Integrate

   (a) $\sin 3x$
   ANSWER:
   $\frac{-\cos 3x}{3}$

   (b) $\cos 5x$
   ANSWER:
   $\frac{\sin 5x}{5}$

   (c) $\cos 5x \sin 3x$
   ANSWER:
   $\frac{1}{16}\cos 8x - \frac{1}{4}\cos 2x$

   (d) $\cos 2x \cos 8x$
   ANSWER:
   $\frac{1}{20}\sin 10x + \frac{1}{12}\sin 6x$

   (e) $\sin x \sin 3x$
   ANSWER:
   $\frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x$

   (f) $2x\cos x^2$
   ANSWER:
   $\sin x^2$

(g) e^{ax}

ANSWER:

$\frac{1}{a}e^{ax}$

(h) e^{iax}

ANSWER:

$\frac{1}{ia}e^{iax}$

6. **Orthogonality**

for $k_n = n\frac{\pi}{L}$, $k_m = m\frac{\pi}{L}$, show

(a) $\int_{-L}^{L} \sin k_n x \sin k_m x = L\delta_{mn}$

(b) $\int_{-L}^{L} \cos k_n x \cos k_m x = \begin{cases} L\delta_{mn} \; n \neq 0 \\ 2L\delta_{mn} \; n = 0 \end{cases}$

(c) $\int_{-L}^{L} \sin k_n x \cos k_m x = 0$

(d) For each case draw a graph explaining why.

ANSWER:

The $\sin\cos$ *integral is zero as it is an odd integrand.*
Use double angle formulae to transform integrand to simple sines and cosines e.g.

$$\begin{aligned}
\int_{-L}^{L} \cos k_n x \cos k_m x &= \frac{1}{2}\int_{-L}^{L} (\cos(k_n + k_m)x + \cos(k_n - k_m)x)\,dx \\
&= \frac{1}{2}\begin{cases} [2x]_{-L}^{L} & ; \quad m = n = 0 \\ \left[\frac{\sin(k_n+k_m)x}{k_n+k_m} + x\right]_{-L}^{L} & ; \quad m = n \neq 0 \\ \left[\frac{\sin(k_n+k_m)x}{k_n+k_m} + \frac{\sin(k_n-k_m)x}{k_n-k_n}\right]_{-L}^{L} & ; \quad m \neq n \neq 0 \end{cases} \\
&= \begin{cases} 2L & ; \quad m = n = 0 \\ L & ; \quad m = n \neq 0 \\ 0 & ; \quad m \neq n \neq 0 \end{cases}
\end{aligned}$$

Note that $\sin k_n L = sin n\pi = 0$. *The* $\sin$ *orthogonality is similar, but the integral becomes, after use of double angle formula:*

$$\frac{1}{2}\int_{-L}^{L} (\cos(k_n - k_m)x - \cos(k_n + k_m)x)\,dx$$

7. **Integration by parts:**

(a) $\displaystyle\int_0^\infty dx\, x\, e^{-ax}$.

(b) $\displaystyle\int_0^\infty dx\, x^2\, e^{-ax}$.

(c) $\displaystyle\int_0^\pi dx\, x \cos x$.

(d) $\displaystyle\int_{-\pi}^\pi dx\, x \sin x$.

(e) $\displaystyle\int_a^{2a} dx\, \ln\left(\frac{x}{\pi}\right)$. [Hint: substitute $u = x/\pi$.]

(f) $\displaystyle\int_1^y dx\, x \ln x$.

(g) $\displaystyle\int_0^1 dx\, (1-x)\ln(1-x)$. [Hint: substitute $u = 1-x$.]

8. **Curve sketching**

(a) $f(x) = \dfrac{1}{x-a} - \dfrac{1}{x+a}$.

(b) $f(x) = \operatorname{sinc} x \; = \dfrac{\sin x}{x}$.

(c) $f(x) = \dfrac{\cos x}{x}$.

(d) Sketch (and label) $f_1(x) = xe^{-x}$ and $f_2(x) = xe^{-2x}$ on the same graph

(e) $f(x) = x^2\, e^{-x}$ for $x \geq 0$.

(f) $f(x) = \sin(\pi x)e^{-x}$ for $x \geq 0$.

(g) Sketch the function $f(x) = \exp\left(-\frac{x^2}{2\sigma^2}\right)$, labelling locations of any crossings of the axes. How would increasing σ change the plot?
Sketch $f(x) \times \cos(\pi x)$ for $\sigma \simeq 3$.
Label the value at $x = 0$ and the position of any nodes (zeros).

(h) $f(x) = e^{-a^2x^2} \cos bx$ with $b > 2\pi a$.

9. Calculate these integrals by integrating by parts. They will be very useful.

(a) $\displaystyle\int_{-L}^L dx\, x \sin\frac{m\pi x}{L}$.

ANSWER:

$$\int_{-L}^L dx\, x \sin\left(\frac{m\,\pi\, x}{L}\right) = 2\,\frac{(-1)^{1+m}\,L^2}{m\,\pi}$$

(b) $\displaystyle\int_{-L}^L dx\, x^2 \sin\frac{m\pi x}{L}$.

ANSWER:

It is zero by symmetry: we are integrating an odd function over an even (i.e. symmetric) range.

(c) $\displaystyle\int_{-L}^{L} dx\, x \cos\frac{m\pi x}{L}$.

ANSWER:
It is zero by symmetry: we are integrating an odd function over an even (i.e. symmetric) range.

(d) $\displaystyle\int_{-L}^{L} dx\, x^2 \cos\frac{m\pi x}{L}$.

ANSWER:
$$\int_{-L}^{L} dx\, x^2 \cos\left(\frac{m\,\pi\,x}{L}\right) = 4\,\frac{(-1)^m\,L^3}{m^2\pi^2}$$

Repeat (c) when the lower limit of the integral is 0 rather than $-L$.

ANSWER:
$$\int_{0}^{L} dx\, x \cos\left(\frac{m\,\pi\,x}{L}\right) = \frac{L^2\,(-1+(-1)^m)}{m^2\pi^2}$$

10. **l'Hôpital's Rule**

> *If $f(x=c) = g(x=c) = 0$ for two functions at some value $x = c$, then*
> $$\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \lim_{x\to c}\left[\frac{f'(x)}{g'(x)}\right] \,,$$

(a) Evaluate $\lim_{x\to 0} \mathrm{sinc}(ax)$ where $\mathrm{sinc}(x) \equiv \dfrac{\sin x}{x}$.

ANSWER:
At $x = 0$ we have $\frac{0}{0}$ which is undefined. This means we need to look more closely using l'Hôpital's rule. Differentiating top and bottom we get $\frac{a\cos ax}{a} = \frac{\cos ax}{1}$ and the limit is 1.

(b) Evaluate $\lim_{x\to 0} \dfrac{\cos x}{x}$.

ANSWER:
A trick question. At $x = 0$ we have $\frac{1}{0}$ which is divergent. We cannot use l'Hôpital's rule: it really is divergent. Whether we get $\pm\infty$ depends on the sign of x from which we approach zero.

(c) Prove l'Hôpital's Rule by writing $f(x)$ and $g(x)$ as Taylor series expansions around $x = c$.

ANSWER:
For x near c: $f(x) = f(c) + (x-c)f'(c) + \frac{1}{2}(x-c)^2 f''(c) + \ldots$ and we are thinking about the special case $f(c) = 0$. Do the same expansion for $g(x)$, cancel a factor of $(x-c)$ top and bottom in the fraction and then take the limit, remembering that $f'(c)$ is the value of the function at a given point and therefore is a constant.

11. In this question we will prove the standard result

$$I = \int_{-\infty}^{\infty} du\; e^{-u^2} = \sqrt{\pi}$$

(a) Write I^2 as a double integral. In the first factor, call the dummy variable x and in the second call it y.

ANSWER:

$$I^2 = \int_{-\infty}^{\infty} dx\; e^{-x^2} \int_{-\infty}^{\infty} dy\; e^{-y^2} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\; \exp\left(-(x^2+y^2)\right) \; .$$

(b) Change to circular polar coordinates $(x, y) \to (\rho, \phi)$ and evaluate the angular integral (remember if you change variables correctly, the area of a ring should enter as $2\pi r dr$).

ANSWER:

$$I^2 = \int_0^{2\pi} d\phi \int_0^{\infty} d\rho\; \rho\; \exp\left(-\rho^2\right) = 2\pi \int_0^{\infty} d\rho\; \rho\; \exp\left(-\rho^2\right) \; .$$

(c) Now do the radial integral and obtain an expression for I.

ANSWER:

The differential of the exponent is -2ρ, and we already have such a factor from the scale factor, so we can do the integral:

$$I^2 = 2\pi \left[-\frac{e^{-rho^2}}{2} \right]_0^{\infty} = \pi \; .$$

Hence the standard result.

(d) Use this to show that the "normalised Gaussian"

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/(2\sigma^2)}$$

really is normalised.

ANSWER:

We want $\int_{-\infty}^{\infty} dx\;\; f(x) = 1$. Change variables $u = x/(\sigma\sqrt{2})$. The limits are unchanged, but $dx = du.\sigma\sqrt{2}$. Overall, then, the area under the curve is $1/sqrt\pi$ times the standard integral, so the area is 1 and it really is normalised.

12. Evaluate $\displaystyle\int_{-\infty}^{\infty} dx\; e^{-ax^2-bx}$, given the standard result $\displaystyle\int_{-\infty}^{\infty} dx\; e^{-x^2} = \sqrt{\pi}$.

    [Hints: begin by "completing the square" to write $ax^2 + bx$ in the form $(Ax+B)^2 + C$]

13. The transverse displacement $u(x, t)$ of a string stretched between $x = 0$ and $x = L$ and initially at rest is described by the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

where c is the (constant) wave speed.

Use the method of separation of variables to obtain a solution of this equation in the form

$$u(x, t) = (A_k \sin kx + B_k \cos kx) \times (C_k \sin \omega_k t + D_k \cos \omega_k t) \; .$$

Explain clearly the meaning of all the symbols including the relationship between k and ω_k.

Show how the imposition of the boundary and initial conditions restricts the possible modes of vibration and hence leads to a general solution

$$u(x,t) = \sum_{n=1}^{\infty} E_n \sin k_n x \cos \omega_n t$$

Again, the meanings of all symbols should be clearly explained.

The functions $\sin k_n x$ form an orthogonal set:

$$\int_0^L dx \ \sin(k_n x) \ \sin(k_m x) = a_n \delta_{nm} \ ,$$

for some constants a_n (which you need not evaluate). If the string initially has a displacement $f(x)$, show that

$$E_n = \frac{1}{a_n} \int_0^L dx \ f(x) \sin k_n x \ .$$

(Aug.07.7)

ANSWER:
No answer: this is a past exam question

What physical role does the constant c play for: (a) travelling waves, and (b) standing waves?

ANSWER:
For travelling waves, c is the speed of propagation (see later question on this sheet for more details). For standing waves (i.e. normal mode solutions) c relates the angular frequency of the wave ω to the wavenumber k via the dispersion relation $\omega = ck$.

14. Sketch each of the following functions and find their Fourier series expansions (i.e. components a_n and b_n) in the range $-L \leq x \leq L$. Add to your sketch the periodically extended function described by the Fourier series. For each function, explain why particular components turn out to be zero.

    (a) $f(x) = \sin \dfrac{3\pi x}{L}$
    ANSWER:
    $f(x)$ is odd, so will be represented just in terms of sine functions (no cosines, $a_m = 0$). Here we are trying to represent a pure sine function in terms of pure sign functions. Obviously only one term in the series will constribute, so $b_m = \delta_{3m}$. You can show all this mathematically by doing the appropriate integrals.

    (b) $f(x) = \text{signum}\, x \equiv \dfrac{x}{|x|} \equiv \begin{cases} +1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$
    ANSWER:
    This is discussed in the lecture notes.

    (c) $f(x) = x$
    ANSWER:
    The function is odd, so $a_n = 0$ and $b_n = \dfrac{-2L(-1)^n}{n\pi}$

    (d) $f(x) = |x|$
    ANSWER:
    The function is even, so $b_n = 0$ and $a_0 = L/2$ and $a_{n>0} = \dfrac{-2L[1-(-1)^n]}{n^2\pi^2}$

(e) $f(x) = x^2$

ANSWER:

This is discussed in the lecture notes.

[Hint: for (b), (d) split the integrals into two parts, $-L \le x < 0$ and $0 \le x \le L$.]

15. As seen in lectures, the transverse displacements of a string stretched from $x = 0$ to $x = L$ are described by a general solution:

$$u(x,t) = \sum_{n=1}^{\infty} \left(E_n \sin k_n x \sin \omega_n t + F_n \sin k_n x \cos \omega_n t\right) \tag{1}$$

with $k_n = n\pi/L$ and $\omega_n = ck_n$.

A guitar string is initially plucked gently from the centre such that $\dot{u}(x, t = 0) = 0$ and

$$u(x, t = 0) = \begin{cases} 2px/L & 0 \le x \le \frac{L}{2} \\ 2p(L-x)/L & \frac{L}{2} \le x \le L \end{cases}$$

Sketch $u(x, t = 0)$ (labelling the maximum value). Why is it important that the string is plucked "gently"? Find E_n, F_n. For which n are E_n and F_n both zero (i.e. this frequency is not present)? Give a physical explanation. Which frequencies dominate? Again, give a physical explanation.

ANSWER:

The sketch is a triangle function peaking at $x = L/2$ with height p.
"Gentle" plucking is needed to have a linear wave equation (as discussed in lectures).
Initially the string is at rest, so all F_n turn out to be zero.
We calculate E_n in a similar way to in lectures.

$$\begin{aligned} E_n &= \frac{2}{L}\int_0^L dx\ \sin\frac{n\pi x}{L}\ u(x, t = 0) \\ &= \frac{4p}{L^2}\int_0^{L/2} dx\, x \sin\frac{n\pi x}{L} + \frac{4p}{L^2}\int_{L/2}^{L} dx\ (L-x)\sin\frac{n\pi x}{L} \end{aligned}$$

The smart thing to do here is make substitutions $y = n\pi x/L$ in the first integral and $y = n\pi(L-x)/L$ in the second:

$$\begin{aligned} E_n &= \frac{4p}{L^2}\int_0^{n\pi/2} \frac{Ldy}{n\pi}\frac{Ly}{n\pi}\sin y + \frac{4p}{L^2}\int_{n\pi/2}^{0} -\frac{Ldy}{n\pi}\frac{Ly}{n\pi}\sin(n\pi - y) \\ &= \frac{4p}{n^2\pi^2}\int_0^{n\pi/2} dy\ y\sin y + \frac{4p}{n^2\pi^2}\int_0^{n\pi/2} dy\ y\sin(n\pi - y) \end{aligned}$$

In the second integral, we used the minus sign from the scale factor to switch the integration limits. Now we notice that if n is even (say $n = 2m$), $\sin(n\pi - y) = \sin(2m\pi - y) = \sin(-y) = -\sin(y)$ and, if n is odd (say $n = 2m+1$), $\sin(n\pi - y) = \sin(2m\pi + \pi - y) = \sin(\pi - y) = \sin y$. Overall, then $\sin(n\pi - y) = -(-1)^n \sin y$, and the second integral looks just like the first:

$$\begin{aligned} E_n &= \frac{4p[1-(-1)^n]}{n^2\pi^2}\int_0^{n\pi/2} dy\ y\sin y \\ &= \frac{4p[1-(-1)^n]}{n^2\pi^2}\left[-y\cos y + \sin y\right]_0^{n\pi/2} \\ &= \frac{4p[1-(-1)^n]}{n^2\pi^2}\left(\sin\frac{n\pi}{2} - \frac{n\pi}{2}\cos\frac{n\pi}{2}\right) \end{aligned}$$

O.K. From the $[1-(-1)^n]$ *we know that the expression is only non-zero when* n *is odd, whereas* $\cos n\pi/2$ *is only non-zero when* n *is even. So, we can ignore the second term. Similarly,* $\sin n\pi/2$ *is only non-zero when* n *is odd, in which case* $[1-(-1)^n]$ *is always 2, so we just replace it by that value to get:*

$$E_n = \frac{8p}{n^2\pi^2}\sin\frac{n\pi}{2}$$

So the only frequencies present are odd n *i.e. ones where the wave is symmetric about the midpoint of the string. Those antisymmetric about the midpoint (even* n*) are not present. This makes sense: our initial conditions were symmetric about the midpoint. The amplitudes* $E_n \propto 1/n^2$*, so low frequencies dominate. This makes sense: the initial condition was triangular in shape, which looks a lot like the* $n=1$ *sine wave, so it is natural that low frequency modes should have greater amplitudes.*

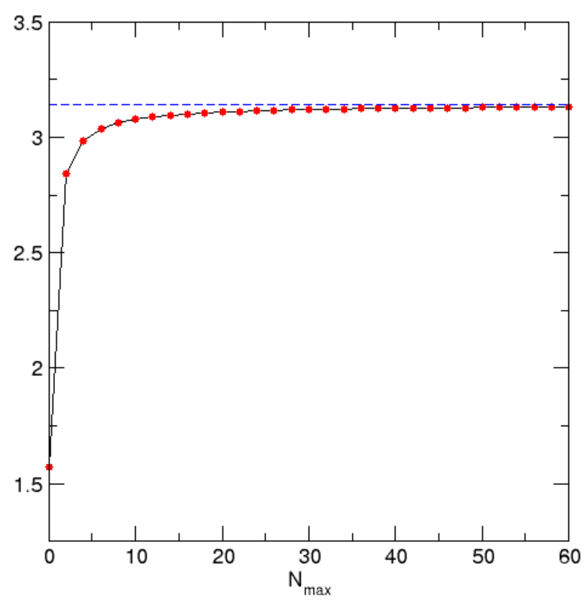
16. Sketch the following function and express it as a real Fourier series, finding its Fourier components, a_n and b_n:

$$f(x) = \begin{cases} (x+\pi)h & -\pi \le x \le 0 \\ (\pi - x)h & 0 \le x \le \pi \end{cases} \tag{2}$$

where h is a constant. Why are $b_n = 0$? Comment on the relative sizes of the non-zero components. Use the series to find a series expression for π (hint: note that $f(0) = \pi$ if $h = 1$). Add the first few terms of the series to see how well it does.

ANSWER:

The sketch is triangle with corners at $(x, y) = (-\pi, 0), (0, \pi), (\pi, 0)$.
Fourier component: $a_0 = \pi h/2$, $a_{n>0} = 2h(1-(-1)^n)/(n^2\pi)$ *and* $b_n = 0$.
b_n *are zero because we are expanding an even function, and the* $\phi_n(x)$ *basis functions are odd.*
The function is not smooth, with discontinuities in the first derivative (gradient) at $x = 0, \pm\pi$*, so we expect the Fourier components to reduce as* $1/n^2$ *in the limit of large* n*. In fact, in this case they have this dependence for all* n*.*



17. Reduce the 1-dimensional Schrödinger Equation to separated form

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}+V(x)\,\psi(x,t)=i\hbar\frac{\partial\psi(x,t)}{\partial t}$$

Also:

(a) What is the physical interpretation of the separation constant?
ANSWER:
It is the energy.

(b) What physical significance is attached to the normal modes?
ANSWER:
These are the eigenfunctions of the Schrödinger operator. They are the pure quantum states of well defined energy. If we measure the energy for a general wavefunction, the wavefunction then collapses to be one of the normal modes.

(c) What changes (if anything) if the potential is time dependent i.e. $V(x) \to V(x,t)$?
ANSWER:
In general, the equation is no longer separable, so we can't solve the equation by assuming solutions of separated form. The only exception is if $V(x,t)$ can be written as the sum of a function of x and another function of t: $V(x,t) = V_x(x) + V_t(t)$. This is still separable: we keep $V_x(x)$ on the LHS and move $V_t(t)$ to the RHS. Such a separable potential is quite artificial and rarely seen in practice.

(d) For the infinite square well, explain why the normal modes look (at least spatially) like Fourier basis functions
ANSWER:
The spatial function $X(x)$ has a harmonic solution of sines and cosines. The BCs are that the wavefunction is matched across boundaries. The wavefunction must be zero outside the infinite square well, so our BCs are that $\psi(x,t) = 0$ at $x = -L$ and $x = L$ (for instance). This quantises the allowed wavenumbers k, just as for waves on a string.

18. Reduce the 2-dimensional Schrödinger Equation to separated form when the potential has the form $V(x,y) = V(x) + V(y)$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial y^2}+[V(x)+V(y)]\,\psi(x,t)=i\hbar\frac{\partial\psi(x,t)}{\partial t}$$

How could you proceed if $V(x,y) = V(x+y) + V(x-y)$?

19. **Rectangular drumskin**

The wave equation of a square drumskin (defined by $x \in [0,L]$, $y \in [0,L]$) is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Substitute a separable solution of the form $X(x)Y(y)T(t)$ and derive the separate ODE's for X, Y, and (t).

Apply the boundary conditions

$$u(x,L,t) = u(L,y,t) = u(x,0,t) = u(0,y,t) = 0$$

to x and y and find the allowed values for the separation constants governing the X, Y and T differential equations.

Suppose that initially the velocity

$$\frac{\partial u(x,y,t=0)}{\partial t} = 0$$

and

$$u(x,y,t=0) = x(L-x)y(L-y).$$

Find the motion for all later times.

20. The Fourier modes are defined (for integer n) as

$$\psi_n(x) = \begin{cases} 1 & \text{for } n=0\ , \\ \cos\dfrac{n\pi x}{L} & \text{for } n \geq 1\ , \end{cases}$$
$$\text{and} \quad \phi_n(x) = \sin\frac{n\pi x}{L} \qquad \text{for } n \geq 1\ .$$

(a) Explain what it means to say $\{\psi_{n\geq 0}(x), \phi_{n>0}(x)\}$ form an orthogonal set of functions on the interval $-L \leq x \leq L$.

ANSWER:

Define an inner product $f \cdot g \equiv \int_{-L}^{L} dx\ f(x)^*\ g(x)$. *An orthogonal set of functions is one where the inner product of any two different functions is zero i.e.* $\psi_n \cdot \psi_m \propto \delta_{mn}$, $\phi_n \cdot \phi_m \propto \delta_{mn}$, $\psi_n \cdot \phi_m = 0$.

(b) Prove (by evaluating appropriate integrals) that they do indeed form an orthogonal set.

ANSWER:

The formulæ Eqn. (2.3) from the notes are useful here. Remember to trat the $m = n$ *and* $m \neq n$ *cases separately.*

(c) Explain what we mean by an orthonormal set of functions, and how we convert an orthogonal set into an orthonormal one.

ANSWER:

An orthonormal set of functions is an orthogonal set but where we have also arranged that $\psi_n \cdot \psi_n = \phi_n \cdot \phi_n = 1$. *Given a set of un-normalised functions, we can create a normalised set by dividing each function by its "length":* $\psi_n(x) \to \psi_n(x)/\sqrt{\psi_n \cdot \psi_n}$, $\psi_n(x) \to \phi_n(x)/\sqrt{\phi_n \cdot \phi_n}$.

(d) Derive an orthonormal set of functions $\{\hat{\psi}_{n\geq 0}(x), \hat{\phi}_{n>0}(x)\}$ from the Fourier modes defined above.

ANSWER:

$\psi_0(x) \to \psi_0(x)/\sqrt{2L}$, $\psi_{n\geq 1}(x) \to \psi_{n\geq 1}(x)/\sqrt{L}$, $\phi_n(x) \to \phi_n(x)/\sqrt{L}$.

(e) By analogy with vectors, explain what is meant by a basis set in function space. Does such a basis set have to be complete?

ANSWER:

A general function can be written as a linear combination of basis functions, if the basis set is a complete set of functions. Yes, it has to be complete, or we cannot do this.

(f) Express the function $f(x) = \sin^2 \dfrac{2\pi x}{L}$ as a Fourier series, preferably (for your sake) without doing any integrals.

ANSWER:

The aim of a Fourier series is to express a function as a sum of sines and cosines (inside a finite range). To do this, we need to find the Fourier components a_n, b_n. One way to find them is by projection: taking the inner product of a basis function with the function we want to expand. But ultimately, all we want to do is find the values of the unknown coefficients and if there is an easier way to do this, we are free to do that instead. In this case we can use a trig. identity to rewrite $f(x)$ in terms of sines and cosines directly: $f(x) = \frac{1}{2} - \frac{1}{2}\cos\left(\frac{4\pi x}{L}\right) = \frac{1}{2}\psi_0 + (-\frac{1}{2})\psi_4(x)$. Comparing with the generic Fourier expansion, all the b_n are zero (there are no sines), the constant is $a_0 = \frac{1}{2}$ and all the $a_{n>0}$ are zero except $a_4 = -\frac{1}{2}$.
These are the components using the unnormalised basis.

21. Write down the Fourier series for a general function $f(x)$ between $[-L, L]$, and then both differentiate it and integrate with respect to x.

Fourier Transforms

22. Define the Fourier transform of a function and the inverse relation that specifies how that function is represented in terms of its Fourier transform.

23. **Complex Fourier series** Write down the complex Fourier series for a function $f(x)$ which is defined on the interval $-L \leq x \leq L$ and give an expression for the coefficients.

    Consider the complex Fourier series for a *real* function $f(x)$. How are the coefficient c_n and c_{-n} related.

    Explain briefly how the Fourier transform $F(k)$ is obtained from the Fourier series and state the equations which relate $F(k)$ and $f(x)$, along with an expression for the wavenumber k.

    If $f(x)$ is real, how are $F(k)$ and $F(-k)$ related?

    ANSWER:

    The limit $L \to \infty$ is discussed in the lecture notes. Make sure you understand in particular the relation between C_n and $F(k)$.

24. The top-hat function is defined to be zero everywhere, save in a region of width a centred at $x = 0$. Within this region, the function takes a constant value such that the total area under the function is unity.

    Obtain the Fourier transform of the top hat function, and comment on the relative widths of the function (in x-space) and its Fourier transform (in k-space). Make sure you provide some justification for your comments.

    ANSWER:

    This was done in the lecture notes. It is a common exam question, so make sure you can reproduce the arguments and the conclusion.

25. Write down the Fourier transform for a general function $f(x)$ between $[-\infty, \infty]$, and then both differentiate it and integrate with respect to x.

26. Find the Fourier transforms

$$F(k) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\ f(x)\ e^{ikx}$$

of the following functions

(a) $f(x) = e^{-a|x|}$ for positive a. [Hint: consider $x \geq 0$ (with $f(x) = e^{-ax}$) and $x < 0$ (with $f(x) = e^{ax}$) separately.]
ANSWER:

$$F(k) = \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} dx\ e^{x(ik+a)} + \int_{0}^{\infty} dx\ e^{x(ik-a)} \right) = \frac{2a}{(k^2+a^2)\sqrt{2\pi}}$$

(b) The normalised Gaussian:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) .$$

ANSWER:
This was discussed in the lecture notes. Complete the square in the exponent, and then change variables to exploit the standard result
$\int_{-\infty}^{\infty} du\ e^{-u^2} = \sqrt{\pi}$.

Dirac δ

27. (a) Explain how the Fourier transform is used to obtain the far field diffraction pattern for collimated light normally incident on a single slit of width a. Your explanation should include a labelled diagram, and you should carefully define all mathematical symbols.

(b) In the limit $a \longrightarrow 0$, the top hat function approaches the Dirac delta function. Discuss briefly the properties that make this true. By considering the integral of the product of an arbitrary function with the top-hat in this limit, derive the "sifting property" of the Dirac delta function. (You may find it convenient to exploit the smallness of a and Taylor expand around $x = 0$.)
ANSWER:
This is discussed in the lecture notes.

(c) What is the convolution of an arbitary function $h(x)$ with $\delta(x-b)$? Explain, with a sketch graph, the geometric significance of this particular convolution operation.
ANSWER:
$h(x) * \delta(x-b) = h(x-b)$. *You need to work this out explicitly. Remember the trick given in the lecture notes to make it easier. The function h is displaced distance b in the direction of increasing x. Draw a sketch of a random function with some feature (e.g. a peak) at $x = 0$. Label the horizontal axis x, and the function $h(x)$. Then draw the function again, displaced to the right with the position of the feature now marked as $x = b$. Label this as $h(x-b)$.*

28. Use the sifting property of the Dirac delta function

$$\int_{-\infty}^{\infty} dx\ f(x)\delta(x) = f(0) . \tag{3}$$

to prove the properties of the Dirac delta function given in the lecture notes:

(a) $\delta(ax) = \delta(x)/|a|$ with special case $a = -1$: $\delta(-x) = \delta(x)$. [Hint: consider

$$\int_{-\infty}^{\infty} dx\ f(x)\delta(ax)\ ,$$

substitute $y = ax$ and compare with Eqn. (3). You will need to consider positive and negative a separately — in the latter case use $y = -|a|x$.]

ANSWER:

For positive a:

$$\int_{-\infty}^{\infty} dx\ f(x)\delta(ax) = \int_{-\infty}^{\infty} \frac{dy}{a}\ f\left(\frac{y}{a}\right)\delta(y) = \frac{f(0)}{a} = \frac{f(0)}{|a|}\ .$$

Comparing with Eqn. (3), $\delta(ax) = \delta(x)/|a|$.

For negative $a = -|a|$:

$$\int_{-\infty}^{\infty} dx\ f(x)\delta(ax) = \int_{\infty}^{-\infty} \frac{dy}{-|a|}\ f\left(-\frac{y}{|a|}\right)\delta(y) = \int_{-\infty}^{\infty} \frac{dy}{|a|}\ f\left(-\frac{y}{|a|}\right)\delta(y) = \frac{f(0)}{|a|}\ .$$

and again $\delta(ax) = \delta(x)/|a|$.

(b) $x\ \delta(x) = 0$. [Hint: consider Eqn. (3) for $f(x) = x$. By looking at which points could contribute to the integral you can then say something about the integrand.]

ANSWER:

$$\int_{-\infty}^{\infty} dx\ x\ \delta(x) = x|_{x=0} = 0\ .$$

The integrand is clearly zero for all $|x| > 0$, so the only way we can get zero for the integral is if the integrand is also zero at $x = 0$.

(c) $\delta(x^2 - a^2) = [\delta(x-a) + \delta(x+a)]/(2|a|)$. Use the result that

$$\delta(g(x)) = \sum_n \frac{\delta(x - x_n)}{|g'(x_n)|} \tag{4}$$

where $x = x_n$ are the zeros of the function $g(x)$.

ANSWER:

The zeros of $g(x) = x^2 - a^2$ occur at $x_1, 2 = \pm a$, where $|g'| = 2a$. Summing over $n = 1, 2$ we get the result.

(d) A bit harder. Prove Eqn. (4) using the following method:

i. Consider

$$I = \int_{-\infty}^{\infty} dx\ \delta(g(x))\ :$$

the only contributions are going to come when $g(x) = 0$. Why?

ANSWER:

Everywhere else the delta function must be zero.

ii. Split the integration up into small regions around the zeros of width ε. Why do we get

$$I = \sum_n \int_{x_n-\varepsilon}^{x_n+\varepsilon} dx\ \delta(g(x))\ ?$$

ANSWER:

In the regions in between the integrand is zero.

iii. Define $y = x - x_n$ and Taylor expand $g(x)$ for small y. If ε is small enough that we can ignore terms of order ε^2 and above, show that

$$I = \sum_n \int_{-\varepsilon}^{\varepsilon} dy\ \delta(yg'(x_n))\ .$$

ANSWER:
$g(x) = g(x_n + y) = g(x_n) + yg'(x_n) + \mathcal{O}(y^2)$ and $g(x_n) = 0$.

iv. Make a substitution $z = yg'(x_n)$ and explain why

$$I = \sum_n \int_{-\infty}^{\infty} \frac{dz}{|g'(x_n)|}\ \delta(z)\ .$$

In particular, make sure you explain the modulus sign, and why the integration range has been expanded.
ANSWER:
Use the result for $\delta(ay)$ with $a = g'(x_n)$. By linearising (dropping terms of ordery^2), we enforce that the argument of the delta function only has one zero. So we can expand the integration range.

v. Evaluate I, and compare with Eqn. (3) to obtain the result.

29. Prove that $x\frac{d}{dx}\delta(x) = -\delta(x)$. [Hint: integrate the LHS over all x, and then integrate by parts.]

ANSWER:

$$\int_{-\infty}^{\infty} dx\ x\frac{d}{dx}\delta(x) = [x\delta(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx\delta(x) = -\int_{-\infty}^{\infty} dx\delta(x)$$

which therefore tells us the identity must hold for the integrands.

30. Find the Fourier transforms of the following functions[1]

(a) $f(x) = \delta(x - d)$ for some fixed d
ANSWER:
This is discussed in the lecture notes.

(b) $f(x) = \delta(x + d) + \delta(x - d)$
ANSWER:

$$F(k) = \frac{1}{\sqrt{2\pi}}(e^{ikd} + e^{-ikd}) = \sqrt{\frac{2}{\pi}}\cos(kd)$$

(c) $f(x) = \sum_{n=-N}^{N} \delta(x - nd)$ for integer N.
ANSWER:

$$\begin{aligned} F(k) &= \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} e^{ikdn} = \frac{1}{\sqrt{2\pi}}\left[1 + \sum_{n=1}^{N}(e^{ikdn} + e^{-ikdn})\right] \\ &= \frac{1}{\sqrt{2\pi}}\left[1 + 2\sum_{n=1}^{N}\cos(kdn)\right] \end{aligned}$$

[1] If you look back through old exam papers (`http://www.lib.ed.ac.uk/resources/collections/exams.shtml`), this and last week's lists cover almost every Fourier transform that has been recently asked.

(d) $f(x) = e^{iqx}$

ANSWER:

This is discussed in the lecture notes.

(e) $f(x) = \cos(ax)$. Comment on your answer.

ANSWER:

$$F(k) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} dx\ (e^{i(k+a)x} + e^{i(k-a)x}) = \sqrt{\frac{2}{\pi}} \times 2\pi(\delta(k+a) + \delta(k-a))$$

using Eqn.(10.24) from the notes. Note only the frequencies $k = \pm a$ *contribute.*

(f) $f(x) = \sin(ax)$. Comment on your answer.

ANSWER:

$$F(k) = \sqrt{\frac{2}{\pi}} \times i \int_{-\infty}^{\infty} dx\ (e^{i(k+a)x} - e^{i(k-a)x}) = \sqrt{\frac{2}{\pi}} \times 2\pi i(\delta(k+a) - \delta(k-a))$$

as above. Again, only frequencies $k = \pm a$ *contribute.*

For what optical systems would (a), (b) and (c) be transmission functions?
For what optical systems would the transmission functions be given by the convolutions of a top hat with (a), with (b) or with (c)?

ANSWER:

(a): 1 infinitely narrow slit; (b): 2 infinitely narrow slits, separation $2d$*; (c): a grating consisting of* $2N + 1$ *infinitely narrow slits, separation* d*; (g)*(a,b,c): as before, but in each case the slits have finite width* $2a$.

Convolution

31. (a) Define the convolution of $f(x)$ and $g(x)$.

(b) State and prove the convolution theorem for $f(x) * g(x)$

(c) The normalised Gaussian is:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) .$$

Evaluate the Fourier transform of the Gaussian.
Evaluate the convolution of two Gaussians of width σ_1 and σ_2 using the convolution theorem
Interpret your result

32. Show that the convolution of $f(x)$, the top-hat function of width $2a$ centred at the origin, with itself is a triangle-shaped function of width $4a$. [Hint: if y is the dummy variable in the convolution, consider in what range of x functions $f(y)$ and $f(x-y)$ are non-zero. Split the integral into 4 regions: $x < -2a$, $-2a \le x \le 0$, $0 \le x \le 2a$ and $x > 2a$.]

ANSWER:

Sketch a graph with the dummy index y along the horizontal axis. Sketch f(y), a top-hat extending from $y = -a$ *to* $y = a$*. Choose a value of* x*. Sketch a (reflected) top-hat centred at this point. Now consider the region of y in which the two top-hats ovelap. For* $|x| > 2a$ *there is no overlap, and the convolution is zero for these values of* x*. For*

$0 \le x \le 2a$, the overlap extends from $y = x - a$ (the LHS of the shifted top-hat) to $y = a$ (the RHS of the unshifted top-hat). So, the integral over y reduces to:

$$(f * f)(x) = \int_{x-a}^{a} dy\, h^2 = h^2[y]_{x-a}^{a} = h^2(a - (x - a)) = h^2(2a - x)$$

where h is the height of the top-hat.
If $-2a \le x \le 0$, the overlap is from $y = -a$ to $y = x+a$, giving $(f*f)(x) = h^2(2a+x)$. Combining these 4 results, we get the triangle of base $4a$ and height $2ah^2$:

$$(f * f)(x) = \begin{cases} 0 & x \le -2a \\ h^2(2a + x) & -2a \le x \le 0 \\ h^2(2a - x) & 0 \le x \le 2a \\ 0 & 2a \le x \end{cases}$$

This function is continuous (i.e. joined up) but not smooth (i.e. it has kinks at $x = 0, \ \pm 2a$.

The convolution operation is *commutative*: $f * g = g * f$. Use this property to sketch $f(x) * [\delta(x + b) + \delta(x - b)] * f(x)$ for $b > 2a$.

ANSWER:
Use the commutativity to rearrange this to $(f * f)(x) * [\delta(x + b) + \delta(x - b)]$. The first convolution gives us the triangle above. When we convolve with a δ-function, it just shifts the function. When we convolve with a sum of delta-functions, we get multiple, shifted copies. So we end up with two triangles, base $4a$ centred at $x = b$ and $x = -b$. The condition $b > 2a$ tells us the triangles do not overlap.

33. The following questions verify some expressions from the lecture notes

    (a) Slightly harder: Show that the convolution operation is *commutative* i.e. the result does not depend on the order: $f * g = g * f$. [Hint: change variable $y \to z = x - y$.]
    Show the convolution operation is *associative* i.e. we can combine convolutions in any order: $(f*g)*h = f*(g*h)$. [Hint: $(f*g)*h = \int dy \int dx\, f(x)\, g(y-x)\, h(z-y)$ and change variables $(x, y) \to (u = y - x, v = z - x)$. Recognise that we have $(g * h)$ convolved with f.]

PDE's

34. In this question, we'll see how to use Fourier transforms to solve the wave equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u(x,t)}{\partial t^2}$$

for an **infinitely long** string.

    (a) First Fourier transform both sides to show that

$$-(ck)^2\, U(k,t) = \frac{\partial^2 U(k,t)}{\partial t^2}$$

    where $U(k,t)$ is the (spatial) F.T. of $u(x,t)$.
    ANSWER:
    This uses the relation that $FT(f') = (ik)FT(f)$, proved in the lecture notes.

(b) For fixed k, solve this ODE and show that the general solution is

$$U(k,t) = U_0(k)\cos(ckt) + \frac{V_0(k)}{ck}\sin(ckt)$$

where $U_0(k) \equiv U(k, t = 0)$ and $V_0(k) \equiv \left.\frac{\partial U(k,t)}{\partial t}\right|_{t=0}$.

ANSWER:

Solve the second-order ODE, then apply the initial conditions to identify the unknown constants.

(c) The string is initially at rest and bent into the shape $u(x, t = 0) = \sin(\frac{3\pi x}{L})$ where L is just a parameter. Find the solution for $U(k,t)$ and hence $u(x,t)$, given these initial conditions. How does this compare to a similar problem of a string stretched between $x = 0$ and $x = L$? Why?

ANSWER:

$u(x, t = 0) = \sin(\frac{3\pi x}{L})$, *so we can F.T. to get* $U(k, t = 0) = \left[\delta(k + \frac{3\pi}{L} - \delta(k - \frac{3\pi}{L}\right] \cdot \frac{\sqrt{2\pi}}{2i}$. *Substitute this is into the above general result for* $U(k,t)$.

So, we start with a pure sine wave, and we have just those frequencies present for all times i.e. the spectrum is just a pair of Dirac delta functions.

This is just the same as waves on a string. The period of our initial condition divides the interval L exactly, so the wave just repeats periodically the behaviour inside $0 \leq x \leq L$. *This is the same as the periodic extension that we get outside the range for Fourier series.*

(d) A similar string is also initially at rest, but is instead bent into the shape

$$u(x, t = 0) = \begin{cases} \sin(\frac{3\pi x}{L}) & \text{for } -L \leq x \leq L\,, \\ 0 & \text{otherwise.} \end{cases}$$

Find the solution for $U(k,t)$ and comment on the difference between this and the previous case.

ANSWER:

$U(k, t = 0) = \frac{L}{\sqrt{2\pi}}\left[\mathrm{sinc}(kL + 3\pi) - sinc(kL - 3\pi)\right]$. *This is more complicated, being a pair of sinc functions centred on* $k = \pm\frac{3\pi}{L}$. *These centre frequencies contribute most, but* ***all*** *frequencies are now present (aside from the zeros of the sinc function), compared to just the centre frequencies* $k = \pm\frac{3\pi}{L}$ *in the previous case.*

If we represent a periodic function using a F.T., we only need a limited number of frequencies (i.e. the F.T. simplifies to become a Fourier Series). If the function to be expanded is aperiodic, we will need to include all frequencies (apart from accidental zeros that are specific to the function we are expanding).