Physical Mathematics 2010: Problems 2 (week 4)

Curvilinear coordinate systems

1. Find the scale factors for changing volume integrals from Cartesian coordinates (x, y, z) to the following coordinate sets:

   (a) Cylindrical polar coordinates (ρ, ϕ, z): $x = \rho \cos\phi$; $y = \rho \sin\phi$; $z = z$.
   ANSWER:

   $$dxdydz \equiv \rho d\rho d\phi dz$$

   (b) Spherical polar coordinates (r, θ, ϕ): $x = r \sin\theta \cos\phi$; $y = r \sin\theta \sin\phi$; $z = r \cos\theta$.
   ANSWER:

   $$dxdydz \equiv r^2 \sin\theta d\theta d\phi$$

2. **Cylindrical polars**

   (a) Determine scale factors for cylindrical polar coordinates and the gradient operator in the $\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z$ basis.

   (b) Combine this with the divergence formula:

   $$\nabla \cdot \mathbf{v} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial h_2 h_3 v_1}{\partial \xi_1} + \frac{\partial h_3 h_1 v_2}{\partial \xi_2} + \frac{\partial h_1 h_2 v_3}{\partial \xi_3} \right]$$

   to find the Laplacian in cylindrical polar coordinates.

   (c) Seek separable solutions in the cylindrical coordinates:

   $$f(x, y, z) = R(r)\Phi(\phi)Z(z)$$

   and separate the equation into three ODE's and separation constants.

   (d) Suppose f has boundary conditions in the z-direction such that

   $$f(x, y, 0) = f(x, y, L) = 0$$

   and that

   $$\Phi(2\pi) = \Phi(0).$$

   What constraints does this introduce on the allowed separation constants?

Wave equation for light

3. If $\nabla \cdot \mathbf{E} = 0$, $\nabla \cdot \mathbf{B} = 0$, $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$, and $\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}$ use the "GDMCC" rule for the vector Laplacian show that $\mathbf{E}$ and $\mathbf{B}$ satisfy a wave equation. **ANSWER:**

$$\begin{aligned}
\nabla^2 \mathbf{E} &= \nabla(\nabla \cdot \mathbf{E}) - \nabla \times (\nabla \times \mathbf{E}) \\
&= 0 - \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t}\right) \\
&= \frac{\partial}{\partial t} \nabla \times \mathbf{B} \\
&= \frac{\partial^2}{\partial t^2} \mathbf{E}
\end{aligned}$$

Also,

$$\begin{aligned}
\nabla^2 \mathbf{B} &= \nabla(\nabla \cdot \mathbf{B}) - \nabla \times (\nabla \times \mathbf{B}) \\
&= 0 - \nabla \times \left(\frac{\partial \mathbf{E}}{\partial t}\right) \\
&= -\frac{\partial}{\partial t} \nabla \times \mathbf{E} \\
&= +\frac{\partial^2}{\partial t^2} \mathbf{B}
\end{aligned}$$

Up to factors of ϵ_0 and μ_0 this derives the electromagnetic wave equation in free space from Maxwell's equations.

Method of Froebenius & Special functions

4. **Bill and Ted's excellent misadventure**

   Bill and Ted have brought Pythagoras to the future and lost him in a night club. We now live in a world without sin and cos.

   To rectify this it is up to you to use the method of Froebenius to rediscover these precious functions

   (a) Substitute the infinite series $y(x) = \sum_{n=0}^{\infty} C_n x^n$ to the differential equation

   $$y'' + y = 0$$

   You should end up with two sums.
   ANSWER:

   $$\sum_{n=0}^{\infty} c_n n(n-1) x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0$$

   (b) Relabel the summation using $m = n - 2$ on the y'' term
   ANSWER:

   $$\sum_{m=-2}^{\infty} c_{m+2}(m+2)(m+1) x^m + \sum_{n=0}^{\infty} c_n x^n = 0$$

(c) Use a notation where $C_i = 0$ for $i < 0$ to sum the y term over the range $\sum_{n=-2}^{\infty}$

ANSWER:

$$\sum_{m=-2}^{\infty} c_{m+2}(m+2)(m+1)x^m + \sum_{m=-2}^{\infty} c_m x^m = 0$$

(d) Hence obtain the *indicial equation*

$$C_{m+2}(m+1)(m+2) = -C_m$$

This relates every other coefficient in a *recurrence relation*.

(e) Deduce that C_0 can be non-zero even though $C_{-2} = 0$, and that C_1 can be non-zero even though $C_{-1} = 0$.

We have two independent series, and so two free parameters C_0 and C_1 as should be the case for a 2nd order ODE.

(f) Find the series with (a) $C_0 = 1$, $C_1 = 0$ and, (b) $C_0 = 0$, $C_1 = 1$ **ANSWER:**

$$\sum_{n=0}^{\infty}(-1)^{(n)}\frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2.1} + \frac{x^4}{4.3.2.1}\cdots$$

$$\sum_{n=0;\text{nodd}}^{\infty}(-1)^{(n)}\frac{x^{(2n+1)}}{(2n+1)!} = x - \frac{x^3}{3.2} + \frac{x^5}{5.4.3.2}\cdots$$

(g) How would you make up the world's first table of sinusoids? **ANSWER:**
Sum the series to high order, use a computer these days.

(h) Give these two independent series their names to save the world from Bill and Ted's misadventure. **ANSWER:**

$$C_0 = 1, C_1 = 0 \equiv \cos x$$
$$C_0 = 0, C_1 = 1 \equiv \sin x$$

5. **Exponential:** apply the above method to find the exponential function solving the 1st order ODE

$$y' = y$$

fixing the single free parameter via $y(0) = 1$

ANSWER:
Substitute power series

$$\begin{aligned}
\sum_{n=0}^{\infty} C_n n x^{n-1} - \sum_{n=0}^{\infty} C_n x^n &= 0 \\
\sum_{m=-1}^{\infty} C_{m+1} m + 1 x^m - C_m x^m &= 0 \\
C_{m+1}(m+1) - C_m &= 0
\end{aligned}$$

Series switches on when $C_{-1} = 0$ and C_0 is non-zero. Take $C_0 = 1$ and series is

$$y(x) = \exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3.2} \ldots$$

6. **Bessel bookwork**: use the method in the notes to solve Bessel's equation in the case $n = 0$, and $k = 1$:

$$r^2 R'' + rR' + r^2 R = 0$$

Solve for the first four terms of the series solution that remains finite at $r = 0$, and sketch the behaviour of the function.

7. **Cauchy's Equation**:

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{n(n+1)}{r^2} R = 0$$

Verify (by substitution) that it is solved by solutions of the form $R(r) = r^p$ for $p = n$ or $p = -(n+1)$. Explain why the general solution is simply a linear combination of these. [i.e. Why is it a linear combination? Why are there no more functions?]

ANSWER:
Substitute in a solution of the form $R(r) = r^p$ to obtain

$$(p(p-1) + 2p - n(n+1))\, r^{p-2} = 0$$

so $p^2 + p - n(n+1) = (p-n)(p+n-1) = 0$.
Cauchy's equation is linear in R and its derivatives, so the general solution is a linear superposition of the two solutions, with 2 unknown constants of integration. We only expect 2 constants, as it is a second order ODE, so we have the full solution.

8. By making the substitutions $x = pr$, show that the following radial equation for a function $R(r)$:

$$r^2 R'' + rR' + (p^2 r^2 - n^2) R = 0$$

can be written in the standard form of Bessel's Equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2) y(x) = 0 \,. \tag{1}$$

ANSWER:
If $x = pr$, then the chain rule tells us that

$$\frac{d}{dr} = \frac{dx}{dr}\frac{d}{dx} = p\,\frac{d}{dx} \,.$$

so $r\,\frac{dR}{dr} = x\,\frac{dR}{dx}$ and $r^2\,\frac{d^2R}{dr^2} = x^2\,\frac{d^2R}{dx^2}$. Substituting these in gives the standard form, where we identify $R = y(x)$. This means that if $y(x) = J_n(x)$ is a solution to Bessel's equation, then $R(r) = J_n(pr)$ is a solution to the radial equation. So, in our solution for R we will never see r on its own, it will always be multiplied by p.

The solutions to Bessel's equation for give (fixed) n are $y(x) = J_n(x)$, which are the Bessel functions of order n.

By substituting into Bessel's equation, show the following are Bessel functions and find their order:

$$(i) \quad \frac{\sin x}{\sqrt{x}} \qquad (ii) \quad \frac{\cos x}{\sqrt{x}}$$

ANSWER:
Consider part (i):

$$
\begin{aligned}
\frac{d}{dx}\frac{\sin x}{\sqrt{x}} &= \frac{\cos x}{\sqrt{x}} - \frac{\sin x}{2x\sqrt{x}} \\
\frac{d^2}{dx^2}\frac{\sin x}{\sqrt{x}} &= -\frac{\sin x}{\sqrt{x}} - \frac{\cos x}{x\sqrt{x}} + \frac{3\sin x}{4x^2\sqrt{x}}
\end{aligned}
\tag{2}
$$

Substitute this into the standard form to get $(\frac{1}{4} - n^2)\frac{\sin x}{\sqrt{x}} = 0$, *so this is a solution if* $n = \pm\frac{1}{2}$.
A similar analysis for (ii) gives the same result.

Explain why these solutions will never be seen in when solving the radial equation in problems with circular or cylindrical symmetry.

ANSWER:
The radial equation for circular/cylindrical polar coordinates is Bessel's equation with integer n, *because the angular separated solutions look like* $\Phi(\phi) = \cos(n\phi + \delta)$ *and we require single valued solutions and hence periodicity* $\Phi(\phi+2\pi) = \Phi(\phi)$, *so* n *is integer.*

9. A circular membrane has its edge attached to a fixed circular ring centred on the origin and of radius unity. For small displacements from its equilibrium position $u(\rho, \phi, t)$ perpendicular to the plane of the ring

$$
\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},
$$

for times $t > 0$ where $0 \le \rho \le 1$ and $0 \le \phi < 2\pi$ are radial and angular plane-polar coordinates respectively.

If the motion is axially symmetric, explain why we can solve this differential equation by considering solutions of separated form $u(\rho, \phi, t) = R(\rho)\,T(t)$.

Show that the radial equation reduces to

$$
\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + k^2R = 0
$$

where k is a constant.

By making the substitution $x = k\rho$, explain why the solution is $R(\rho) = J_0(k\rho)$, given that $J_0(x)$ is the only solution of the differential equation

$$
\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0
$$

that is finite at the origin.

Apply the boundary conditions to show that the general solution is

$$
u(\rho, \phi, t) = \sum_{n=1}^{\infty}\{E_n\, J_0(\lambda_n\rho)\, \cos(\lambda_n ct) + F_n\, J_0(\lambda_n\rho)\, \sin(\lambda_n ct)\},
$$

where E_n and F_n are unknown constants and the zeros of J_0 are located at λ_n for $n = 1, 2, ...$, i.e. $J_0(\lambda_n) = 0$.

Initially the membrane is held at rest in the shape of a cone with its centre distance h above its equilibrium position. If the membrane is then released, show that the displacement of the point (ρ, ϕ) of the membrane at a later time is given by

$$u(\rho, \phi, t) = \sum_{n=1}^{\infty} A_n \cos(\lambda_n ct)\, J_0(\lambda_n \rho)\ ,$$

where

$$A_n = \frac{2h}{[J_0'(\lambda_n)]^2} \int_0^1 d\rho\ \rho(1-\rho)\ J_0(\lambda_n \rho)\ .$$

[The relation

$$\int_0^1 ds\ s\ J_0(\lambda_n s)\ J_0(\lambda_m s) = \tfrac{1}{2}\delta_{mn}\,[J_0'(\lambda_n)]^2$$

may be assumed without proof, where $\delta_{mn} = 1$ for $m = n$ and zero otherwise.]

(Maths Skills, Dec.08)

10. **Quantum square well**

The Schroedinger equation in a two dimensional infinite square well defined by

$$V(x,y) = \begin{cases} 0 & : x, y \in [0, L] \\ \infty & \text{otherwise} \end{cases}$$

is

$$i\hbar\frac{\partial}{\partial t}\psi(x,y,t) = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi(x,y,t) + V(x,y)\psi(x,y,t).$$

(a) Substitute a separable solution of the form $X(x)Y(y)T(t)$ and derive the separate ODE's for $X(x)$, $Y(y)$, and $T(t)$.

(b) Apply the boundary conditions

$$\psi(x,L,t) = \psi(L,y,t) = \psi(x,0,t) = \psi(0,y,t) = 0$$

to x and y and find the allowed values for the separation constants governing the X, Y and T differential equations.

(c) Suppose

$$\psi(x,y,t=0) = x(L-x)y(L-y).$$

Find the wavefunction ψ for all later times.

Maths Skills Nov 2010

11. Drumskin

a) The displacement $u(x,y,t)$ of a circular drumskin of radius L is described in Cartesian coordinates by the two dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

In circular polar coordinates (r, θ) the Laplacian is

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

Rewrite the wave equation in circular coordinates, and use the method of separation of variables to determine separated differential equations in r, θ, and t and define the separation constants.

b) Solve the θ equation and apply boundary conditions to show that the radial equation reduces to Bessel's equation of integral order

$$r^2R'' + rR' + (k^2r^2 - n^2)R = 0$$

Where $R(r)$ is the separated radial dependence, k is a separation constant, and n is an integer.

c) The solutions are Bessel functions of the form $R(r) = J_n(kr)$.

Apply boundary conditions in the r variable to determine the frequency of the corresponding normal mode in terms of α_{nm}, where α_{nm}, is the m-th zero of the Bessel function J_n.

d) The zeroes of the first two Bessel functions are

$$\begin{array}{cc} J_0 & J_1 \\ \alpha_{01} = 2.405 & \alpha_{11} = 3.832 \\ \alpha_{02} = 5.520 & \alpha_{12} = 7.016 \end{array}.$$

A drummer strikes the drum somewhere with $\theta = 0$. Considering only these four modes, where should he hold his other drumstick to excite:
i) only the $n = 0, m = 2$ mode?
ii) both the $n = 1$ modes with no $n = 0$ contributions?
iii) only the $n = 1, m = 2$ mode?

Physical Mathematics May 2011

12. **Hermite Polynomials & and Harmonic Oscillator**

The time independent Schrödinger equation for a particle of mass m in a harmonic oscillator potential $V(x) = \frac{1}{2}kx^2$ can be placed in the form

$$\frac{d^2}{dx^2}\psi(x) + (\lambda - x^2)\psi(x) = 0,$$

where $\lambda = \frac{2E}{\hbar\omega}$, E is the energy, and the angular frequency $\omega = \sqrt{\frac{k}{m}}$.

(a) Show that after a substitution $\psi(x) = e^{-\frac{x^2}{2}}H(x)$ the equation reduces to

$$H'' - 2xH' + (\lambda - 1)H = 0.$$

(b) Substitute a power series for $H(x)$:

$$H(x) = \sum_{n=-\infty}^{\infty} c_n x^n.$$

Use the method of Fröbenius to determine a recurrence relation for c_n.

(c) Analyse this recurrence relation:
For what values of n can the series begin and terminate?
What is the form of $\psi(x)$ at large x, and how does normalisability constrain λ?

(d) Deduce (unnormalised) eigenfunctions $\psi_1(x)$, $\psi_2(x)$ $\psi_3(x)$ and associated energies E_1, E_2, E_3 for the first, second and third states.

(e) Consider operators $a_+ = \left(x - \frac{d}{dx}\right)$, and $a_- = \left(x + \frac{d}{dx}\right)$.
Apply a_+ and a_- to your solution ψ_1.
Show that if ψ_n is a solution of the equation for $\lambda = \lambda_n$, then $\psi_{n+1} = a_+\psi_n$ is a solution for $\lambda = \lambda_n + 2$.

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