Physical Mathematics 2010: Problems 3 (week 6)

Wave Equation in Spherical polars

It is *very* beneficial to work through the procedure in the notes by yourself.

1. Follow the procedure in the notes to separate the wave equation

$$\nabla^2 u = \underbrace{\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}}_{\text{radial part}} + \underbrace{\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}}_{\text{angular part}} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

   into ODEs for $R(r)$, $\Theta(\theta)$, $\Phi(\phi)$ and $T(t)$.

2. Substitute $w = \cos\theta$ to reduce the Θ-equation to the Legendre equation for $m = 0$.

3. Use the Method of Frobenius to find the first three Legendre polynomials $P_0(\cos\theta)$, $P_1(\cos\theta)$ and $P_2(\cos\theta)$.

   Combine these with your solution for the ϕ equation to form the $Y_l^0(\theta, \phi) = \Theta(\theta)\Phi(\phi)$ in the following table (up to normalisation factors).

Spherical Harmonics

The low (unnormalized) spherical harmonics are

	l=0	l=1	l=2
m=2			$Y_2^2(\theta,\phi) = \sin^2\theta e^{i2\phi}$
m=1		$Y_1^1(\theta,\phi) = \sin\theta e^{i\phi}$	$Y_2^1(\theta,\phi) = \sin\theta\cos\theta e^{i\phi}$
m=0	$Y_0^0(\theta,\phi) = 1$	$Y_1^0(\theta,\phi) = \cos\theta$	$Y_2^0(\theta,\phi) = 3\cos^2\theta - 1$
m=-1		$Y_1^{-1}(\theta,\phi) = \sin\theta e^{-i\phi}$	$Y_2^{-1}(\theta,\phi) = \sin\theta\cos\theta e^{-i\phi}$
m=-2			$Y_2^{-2}(\theta,\phi) = \sin^2\theta e^{-i2\phi}$

4. Show that each of the above Y_l^m are eigenmodes of the operator

$$-i\frac{\partial}{\partial \phi}$$

   with eigenvalue m. This is closely related to the z-component of orbital angular momentum in quantum mechanics.

5. Show that each of the above Y_l^m are eigenmodes of the operator

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}$$

   with eigenvalue $-l(l+1)$. This is closely related to the squared orbital angular momentum in quantum mechanics.

6. Show that, when represented in terms of Cartesian coordinates on the unit sphere $x^2 + y^2 + z^2 = r^2 = 1$,

$$Y_1^m = \{x + iy, z, x - iy\}$$

   for $m = 1, 0, -1$ and that

$$Y_2^m = \{(x + iy)^2, z(x + iy), 3z^2 - 1, z(x - iy), (x - iy)^2\}$$

   for $m = 2, 1, 0, -1, -2$.

7. Consider cartesian operators

$$L_x = -i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$
$$L_y = -i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$
$$L_z = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

(a) Express x,y, and z in spherical polars and find expressions for $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ in terms of of cartesian components $\hat{x}$, $\hat{y}$, and $\hat{z}$.
Use your result and the relation

$$\frac{\partial}{\partial x}f = \hat{x}\cdot\nabla f,$$

and the spherical polar representation of ∇ to show that

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$
$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$
$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

Hence, show that

$$L_x = i\left(\sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\theta}{\sin\theta}\cos\phi\frac{\partial}{\partial\phi}\right)$$
$$L_y = i\left(-\cos\phi\frac{\partial}{\partial\theta} + \frac{\cos\theta}{\sin\theta}\sin\phi\frac{\partial}{\partial\phi}\right)$$
$$L_z = -i\frac{\partial}{\partial\phi}$$

(b) Show that

$$-(L_x^2 + L_y^2 + L_z^2) = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}$$

(c) Show that Y_1^1 is *not* an eigenfunction of L_x or of L_y. [1]

(d) Show that

$$y + iz, \quad x, \quad y - iz$$

are equal to

$$\frac{1}{2i}(Y_1^1 - Y_1^{-1}) + iY_1^0, \quad \frac{1}{2}(Y_1^1 + Y_1^{-1}), \quad \frac{1}{2i}(Y_1^1 - Y_1^{-1}) - iY_1^0.$$

and that they *are* eigenfunctions of both L_x and of L^2 (and what are their eigenvalues)?
Conclude that this is wavefunction basis we should use if we observe angular momentum in the x-direction instead of the z-direction, and that L_x will have magnetic quantum numbers $-1, 0, 1$.

[1] This can also be shown for Y_1^0, and Y_1^{-1} if you are keen

8. **Rodrigues formula for Legendre polynomials**

The $m = 0$ spherical harmonics are given by

$$Y_l^0(\theta, \phi) = N P_l(\cos\theta)$$

(a) Use the Rodrigues formula for Legendre polynomials

$$P_l(w) = \frac{1}{2^l l!} \frac{d^l}{dw^l}(w^2 - 1)^l \propto \frac{d^l}{dw^l}(w^2 - 1)^l$$

to reproduce (ignoring normalisation) Y_0^0, Y_1^0 and Y_2^0 above.

(b) The associated Legendre polynomials are then given by

$$P_l^m(w) = \frac{(-1)^m}{2^l l!}(1 - w^2)^{m/2} \frac{d^{l+m}}{dw^{l+m}}(w^2 - 1)^l$$

Use this to reproduce (ignoring normalisation) all the above spherical harmonics.

9. **Raising and lowering**

Using the spherical polar representation, define

$$L_+ = L_x + iL_y \qquad L_- = L_x - iL_y.$$

Apply L_+ (L_-) to each of the above Y_1^m and show that this raises (lowers) the magnetic quantum number.

Note also that

$$L_+ Y_1^1 = 0$$

and that

$$L_- Y_1^{-1} = 0.$$