Introduction to the Standard Model

Lecture 2

Symmetries

As an example, consider

Classification

$\text{global:}$ spacetime symmetries $\rightarrow$ momentum, angular momentum, spin

$\text{internal symmetries} \rightarrow$ weak isospin, charge, colour

$\text{local:}$ spacetime symmetries $\rightarrow$ gravity as a gauge theory

$\text{internal symmetries} \rightarrow$ gauge theory

Basics of Group theory

(see tutorial for more details)

The Standard Model requires knowledge of the groups, $U(1)$, $SU(2)$, and $SU(3)$, along with some of their matrix representations and associated Lie-algebras.

The $U(1)$ group

Each group element of $U(1)$ can be represented by a pure phase factor, $e^{i\alpha}$. The parameter, $\alpha$, is real and continuous which indicates that $U(1)$ has an infinite set of group elements and is continuous.

Since $e^{i\alpha} = \cos \alpha + i \sin \alpha$, $U(1)$ is isomorphic to 2-by-2 rotation matrices, i.e. elements of $SO(2)$, which also form a Lie-group:
\[ e^{i\alpha} = \cos \alpha + i \sin \alpha \sim \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \in SO(2) \]

\[ \exp \left( i\alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \]

To see this consider the unit circle in the complex plane:

The \textit{SU}(N) group

i.) The \textit{SU}(N) group is defined as the collection of all unitary \( N \times N \) matrices \( U \), i.e. \( U^{-1} = U^\dagger \), with determinant equal to one.

\[ U \in SU(N) \Rightarrow UU^\dagger = \mathbb{I}, \det(U) = 1 \]

As \( U \in \mathbb{C}^{N\times N} \) has \( N^2 \) complex entries or \( 2N^2 \) real entries, we are left with \( 2N^2 - N^2 - 1 = N^2 - 1 \) independent parameters.

ii.) Consider the \( N \)-vector valued field, \( \vec{\psi} \), transforming under \textit{SU}(N) as

\[ \vec{\psi} \rightarrow \vec{\psi}' = U \vec{\psi} \quad \left( \psi'_j = U_{jl} \psi_l \text{ with } j, l = 1, \ldots, N \right) \]

\( \vec{\psi} \) is in the fundamental representation. We also note that \( \vec{\psi}^\dagger \vec{\psi} \) is invariant under an \textit{SU}(N) transformation since

\[ \vec{\psi}^\dagger \vec{\psi} \rightarrow \vec{\psi'}^\dagger \vec{\psi}' = (U \vec{\psi})^\dagger U \vec{\psi} \]
\[ = \vec{\psi}^\dagger U^\dagger U \vec{\psi} \]
\[ = \vec{\psi}^\dagger \vec{\psi} \]

iii.) A group element \( U \) can be expressed as an exponential

\[ U(\Lambda_1, \ldots, \Lambda_{N^2-1}) = \exp \left( i \sum_{a=1}^{N^2-1} \Lambda_a T_a \right) = \lim_{n \to \infty} \left( \mathbb{I} + i \frac{\Lambda_a}{n} T_a \right)^n \]

where \( \Lambda_a \) are real-valued and continuous, and \( T_{a=1,\ldots,N^2-1} \) are called the \textit{generators} of the group. The conditions imposed by i.) and ii.) above restricts the \( T \)'s:
a.)

\[ U^\dagger U = \mathbb{1} \Rightarrow \left( e^{i\varepsilon_a T_a} \right)^\dagger \left( e^{i\varepsilon_b T_b} \right) \]

\[ = \left( \mathbb{1} + i\varepsilon_a T_a + \mathcal{O}(|\varepsilon_a|^2) \right)^\dagger \left( \mathbb{1} + i\varepsilon_b T_b + \mathcal{O}(|\varepsilon_b|^2) \right) \]

\[ = \left( \mathbb{1} - i\varepsilon_a T_a^\dagger + \cdots \right) \left( \mathbb{1} + i\varepsilon_b T_b + \cdots \right) \]

\[ = \mathbb{1} + i\varepsilon_b \left( T_b - T_b^\dagger \right) + \cdots \]

We see that the generators must be Hermitian: \( T_b = T_b^\dagger \).

b.)

\[ \det U = 1 \Rightarrow \det \left( e^{i\varepsilon_a T_a} \right) \]

\[ = 1 + i\varepsilon_a \text{tr} \left( T_a \right) + \cdots \]

This implies that the generators must be traceless: \( \text{tr} \left( T_a \right) = 0 \).

**Lie Algebra**

The generators of \( SU(N) \) obey an important property. Evaluate

\[ U = U_2^{-1} U_1^{-1} U_1 U_2 \in SU(N) \quad (1) \]

by using

\[ U = \mathbb{1} + i\lambda_c T_c + \cdots \]

\[ U_1 = \mathbb{1} + i\varepsilon_a T_a - \frac{1}{2} (\varepsilon \cdot T)^2 + \cdots \leftrightarrow U_1^{-1} = U_1^\dagger = \mathbb{1} - i(\varepsilon \cdot T) - \frac{1}{2} (\varepsilon \cdot T)^2 + \cdots \]

\[ U_2 = \mathbb{1} + i\delta_b T_b - \frac{1}{2} (\delta \cdot T)^2 + \cdots \leftrightarrow U_2^{-1} = U_2^\dagger = \mathbb{1} - i(\delta \cdot T) - \frac{1}{2} (\delta \cdot T)^2 + \cdots \]

One gets

\[ R H S \text{ of } (1) = \mathbb{1} - i(\delta \cdot T + \varepsilon \cdot T - \delta \cdot T - \varepsilon \cdot T) + \varepsilon_a \delta_b \left( T_a T_b - T_b T_a \right) + \cdots \]

compared to the LHS of (1)

\[ \mathbb{1} + i\lambda_c T_c + \cdots = \mathbb{1} + \varepsilon_a \delta_b \left[ T_a , T_b \right] + \cdots \]

results into

\[ \left[ T_a , T_b \right] = i f_{abc} T_c \quad f_{abc} \in \mathbb{R} \]

The \( f_{abc} \)'s are anti-symmetric *structure constants* of the Lie group. The *generators*, \( T_a \), with such a property, form the so-called *Lie-algebra* associated to the Lie-group.

\[ U = \exp \left( i\Lambda_a T_a \right) = \mathbb{1} + i\Lambda_a T_a + \cdots \]

*whole group* *local elements define the properties of the whole group*
Note The Lie group forms a compact manifold; the algebra is defined on the tangent to the unit element.

We choose a normalisation, \( \text{tr}(T_a T_b) = T_R \delta_{ab} \) with \( T_R = \frac{1}{2} \) (convention) such that

\[
f_{abc} = -2i \text{tr}([T_a, T_b] T_c)
\]

**Important Examples**

**SU(2)**

The generators are proportional to the Pauli matrices: \( T_a = \frac{1}{2} \sigma^a \).

The algebra is given as:

\[
\left[ \frac{\sigma^a}{2}, \frac{\sigma^b}{2} \right] = i \varepsilon^{abc} \frac{\sigma^c}{2}
\]

where the structure constants are the defined by the totally anti-symmetric epsilon tensor \( \varepsilon^{abc} \). Remember that

\[
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

**SU(3)**

The algebra is:

\[
\left[ \frac{\lambda^a}{2}, \frac{\lambda^b}{2} \right] = i f^{abc} \frac{\lambda^c}{2}
\]

where the \( \lambda^a \)'s are the Gell-Mann matrices,

\[
\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},
\]

\[
\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\]

The structure constants are:

\[
f^{123} = 1, \quad f^{147} = f^{246} = f^{345} = f^{316} = f^{257} = f^{637} = \frac{1}{2}, \quad f^{458} = f^{678} = \frac{\sqrt{3}}{2}, \quad f^{\text{other}} = 0
\]

Note that we can see the SU(2) subgroup in SU(3):

\[
\lambda^1 = \begin{pmatrix} \sigma^1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & 0 \end{pmatrix}
\]

**Exercise** check all the above