Introduction to the Standard Model

Lecture 8: Quantisation and Feynman Rules

Quantisation of Gauge Fields

problem with gauge fields: Given the field equation:

$$M^{\mu\nu}A_\mu = J^\nu \quad \text{where} \quad M^{\mu\nu} \equiv \partial^\mu g^{\mu\nu} - \partial^\nu\partial^\mu$$

we see that because $M^{\mu\nu}\partial_\nu = 0$, M is not invertible. The problem can be solved by using the fact that not all degrees of freedom for A^μ are physical (observable). This can be seen by applying a gauge transformation to A^μ:

$$A'^\mu = A^\mu + \partial^\nu\Lambda \rightarrow \partial_\mu A'^\mu = \partial_\mu A^\mu + \Box\Lambda$$

We can now choose Λ such that $\partial_\mu A^\mu = 0$ which removes one degree of freedom from the vector field A^μ. The gauge function Λ is not completely determined; there is another gauge freedom

$$\Lambda \rightarrow \Lambda + \Lambda'$$

such that $\Box\Lambda' = 0$. Then we have

$$A^\mu \rightarrow A''^\mu = A^\mu + \partial^\mu\Lambda + \partial^\mu\Lambda'$$

where $\partial^\mu\Lambda$ can be used to remove $\partial_\mu A^\mu$ and the term $\partial^\mu\Lambda'$ can be used to remove another degree of freedom, *e.g.* $A^0 = 0$. Thus, A^μ now only has two degrees of freedom; the other two can be "gauged away."

The mode expansion for the 4-component gauge field is

$$A_\mu(x) = \int \frac{d^3\mathrm{k}}{(2\pi)^3} \sum_{r=0}^{3} \left(a_r(\mathrm{k})\varepsilon^r_\mu(k)e^{-ik\cdot x} + a^\dagger_r(\mathrm{k})\varepsilon^{r*}_\mu(k)e^{ik\cdot x} \right)$$

Notice that $A_\mu(x) = A^*_\mu(x)$; the gauge fields are real-valued.

Appyling the gauge condition:

$$\left.\begin{array}{r} \partial_\mu A^\mu = 0 \Longrightarrow k_\mu\varepsilon^{r\mu} = 0 \\ A^0 = 0 \Longrightarrow \varepsilon^\mu = (0, \boldsymbol{\varepsilon}) \end{array}\right\} \Longrightarrow \mathbf{k}\cdot\boldsymbol{\varepsilon} = 0$$

This is the *transversality* condition; it is consistent with the observation that EM radiation is tranversly polarised.
Note: $k_\mu\varepsilon^\mu_r = 0$ is manifestly covariant whereas $A^0 = 0$ is not.

By choosing a reference frame $k^\mu = \omega(1,\ 0,\ 0,\ 1)$, the polarisation vectors read:

$$\left.\begin{aligned} \varepsilon^{1\mu} &= (0,\ 1,\ 0,\ 0) \\ \varepsilon^{2\mu} &= (0,\ 0,\ 1,\ 0) \end{aligned}\right\} \quad \text{Linearly polarised}$$

Using a basis change, one obtains the polarisation vectors which correspond to circularly polarised light:

$$\varepsilon^{\pm\mu} = \mp\frac{1}{\sqrt{2}}(\varepsilon_1^\mu \pm \varepsilon_2^\mu) = \mp\frac{1}{\sqrt{2}}(0,\ 1,\ \pm i,\ 0)$$

$\varepsilon_\mu^\pm$ are the *helicity eigenstates* of the photon.

Note: $\varepsilon_{1,2}, \varepsilon_\mu^\pm$ correspond to the two observable degrees of freedom of the gauge field A_μ.

The quantisation of gauge fields is non-trivial because $\partial^\mu A_\mu = 0$ cannot be implemented at the operator level due to a contradiction with the canonical commutation relations. This issue is solved by the Gupta-Bleuler formalism (see Rel. QFT notes for more detail):

- only quantum states which correspond to transverse photons ($\varepsilon^\pm/\varepsilon^{1,2}$) are relevant for observables.
- unphysical degrees of freedom do not contribute in the scattering matrix ($\mathcal{S}-$matrix) elements.

The Gupta-Bleuler formalism works for any $U(1)$ gauge theory but fails for Non-Abelian theories (this was later solved by Fedeev and Popov in 1958 -see Modern QFT).

Feynman Rules and Feynman Diagrams

The dynamics of a theory are determined by the propagation of the fields and the interactions between them. To start, we shall mention some points:

- Symmetries and the particle content determine the Lagrangian
- The terms in the Lagrangian define propagation and interaction of the particles (or field quanta).
- The quantum field theory foralism leads to computational rules to evaluate the $\mathcal{S}-$matrix elements: $\mathcal{S}_{if} = \langle i|\widehat{\mathcal{S}}|f\rangle$ where $\widehat{\mathcal{S}}$ is the scattering operator (see further lectures).

Propagators:

The *propagator* is the Green's function for the inhomogeneous field equation.

i) Scalar propagator

$$(\Box + m^2)\phi(x) = J(x)$$

where $J(x)$ is the source term that creates the inhomogeneity required for this definition; this result follows from the inhomogeneous Lagrangian: $\mathcal{L} = \mathcal{L}_{\mathrm{KG}} + J(x)\phi(x)$.

$$\phi(x) = \phi_0(x) + i\int d^4y\, \widehat{G}(x-y)J(y)$$

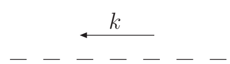
$$-(\Box + m^2)\widehat{G}(x-y) = i\delta(x-y)$$

We use a Fourier ansatz:

$$\widehat{G}(x) = \int \frac{d^4\mathrm{k}}{(2\pi)^4} G(k) e^{-ik\cdot x}$$

$$\delta(x) = \int \frac{d^4\mathrm{k}}{(2\pi)^4} e^{-ik\cdot x}$$

to find



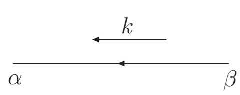
$$\Leftrightarrow \qquad G(k) = \frac{i}{k^2 - m^2 + i\epsilon}$$

ii) Fermion propagator

$$(i\partial\!\!\!/ - m)\widehat{S}(x-y) = i\delta(x-y)$$

$$\widehat{S}(x) = \int \frac{d^4\mathrm{k}}{(2\pi)^4} S(k) e^{-ik\cdot x}$$

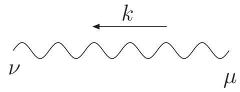
where $S(k) \equiv (S(k))_{\alpha\beta}$ is a matrix in spinor space.



$$\Leftrightarrow \qquad S(k)_{\alpha\beta} = i\left(\frac{k\!\!\!/ + m}{k^2 - m^2 + i\epsilon}\right)_{\alpha\beta} = \left(\frac{i}{k\!\!\!/ - m}\right)_{\alpha\beta}$$

iii) Gauge boson propagator

$$\Big(-g_{\mu\nu} + \big(1 - \frac{1}{\lambda}\big)\partial_\mu \partial_{nu}\Big)\widehat{D}^{\nu\rho}(x) = -g_\nu^\rho\, \delta(x)$$



$$\Leftrightarrow \qquad D^{\mu\nu}(k) = \frac{i}{k^2 + i\epsilon}\Big(-g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2 - i\epsilon}\big(1-\lambda\big)\Big)$$

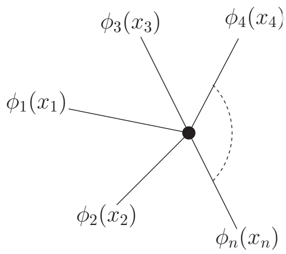
where λ is a gauge fixing term:

$$\begin{aligned} \lambda = 1 &\qquad \text{Feynman Gauge} \\ \lambda = 0 &\qquad \text{Landau Gauge} \end{aligned}$$

In the Landau gauge, $D^{\mu\nu}(k)$ obeys transversality condition, $k_\mu D^{\mu\nu}(k) = 0$.

Interaction Vertices

Derivation of Feynman Rules Each term in a Lagrangian that contains products of fields, $\varphi_1, \ldots, \varphi_N; \varphi_j \in \{\phi, \psi, A^\mu\}$, leads to an *n-point vertex*:



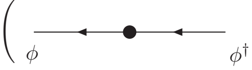
$$\sim V_{\phi_1 \ldots \phi_n}(x_1, \ldots, x_n) = \frac{\delta}{\delta\phi_1(x_1)} \cdots \frac{\delta}{\delta\phi_n(x_n)} \left(i \int d^4x \mathcal{L} \right)$$

We desire a momentum space representation $\rightarrow$ Fourier Transform
Noether's theorem $\Rightarrow$
translational invariance $\Rightarrow$
Poincaré invariance $\Rightarrow$
energy and momentum are conserved:

$$\delta(p_1 + \cdots + p_n) \quad \text{overall}$$

A propagator (up to a minus sign) is the inverse two-vertex:

$$P_{\phi\phi^\dagger} \qquad = \qquad -\left(V_{\phi\phi^\dagger}\right)^{-1}\Big|_{k^2 \to k^2 - i\epsilon}$$

 $= -\Big($  $\Big)^{-1}\Big|_{k^2 \to k^2 - i\epsilon}$

Note: The $i\epsilon$ term is called the 'Feynman prescription' (or simply the '$i\epsilon$ prescription') and it ensures causality.

Recipe for deriving Feynman rules: Rather than actually performing the Fourier transform and the functional derivative, the following rules can be used:

1.) Search for all terms in $\mathcal{L}$ which contain a certain selection of the fields, *e.g.*:

$$-g(\partial_\mu A_\nu)A^\mu B^\nu = -g(\partial_\mu A^\rho)g_{\nu\rho}A^\mu B^\nu$$

2.) Replace all derivatives by $(-i)$ times the incoming momenta of the respective fields (Fourier Transform):

$$-g(\partial_\mu A^\rho(a))g_{\nu\rho}A^\mu(a')B^\nu(b) \rightarrow igq_\mu g_{\nu\rho}A^\rho(q)A^\mu(q')B^\nu(p)$$

3.) Symmetrize indices of all identical bosonic fields:

$$igq_\mu g_{\nu\rho}A^\rho A^\mu B^\nu \rightarrow ig(q_\mu g_{\nu\rho} + q'_\rho g_{\mu\nu})A^\rho(q)A^\mu(q')B^\nu(p)$$

and omit external fields.
The Feynman rule for the example vertex is then:

$$ig(q_\mu g_{\nu\rho} + q'_\rho g_{\mu\nu}) \qquad \Leftrightarrow$$