## Example sheet I: Introduction to the Standard Model

Exercise 1 ( Examples for Lorentz transformations )
Show that

$$
\Lambda_{B}=\left(\begin{array}{cccc}
\gamma & 0 & 0 & v \gamma \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
v \gamma & 0 & 0 & \gamma
\end{array}\right) \quad, \quad \Lambda_{R}=\left(\begin{array}{cc}
1 & \overrightarrow{0}^{T} \\
\overrightarrow{0} & \mathcal{R}
\end{array}\right)
$$

with $\gamma=1 / \sqrt{1-v^{2}}$ and $\mathcal{R} \in S O(3)$ (i.e. $\mathcal{R}^{T} \mathcal{R}=1_{3}, \operatorname{det}(R)=1$ ) are Lorentztransformations.

## Exercise 2 ( EoM )

Derive the equations of motion for the Real Scalar Field
Show that the action is symmetric under $\phi \rightarrow-\phi$.
Exercise 3 (EoM and Noether currents )
Derive the equations of motion for each of the Complex Scalar Field
Derive the conserved current for the global $U(1)$ symmetry of the complex scalar field and the corresponding charge density operator.

Exercise 4 (EoM and Noether currents )
Derive the equations of motion for each of the Dirac Field
Derive the conserved current for the global $U(1)$ symmetry and the corresponding charge density operator.

## Exercise 5 ( EoM )

Derive the equations of motion for each of the Maxwell Field
Exercise 6 ( Free Maxwell theory )
The free Maxwell theory is defined by

$$
\begin{aligned}
\mathcal{L} & =-\frac{1}{4} F_{\mu \nu} F^{\mu \nu} \\
F^{\mu \nu} & =\partial^{\mu} A^{\nu}-\partial^{\nu} A^{\mu}=\left(\begin{array}{rrrr}
0 & -E^{1} & -E^{2} & -E^{3} \\
E^{1} & 0 & -B^{3} & B^{2} \\
E^{2} & B^{3} & 0 & -B^{1} \\
E^{3} & -B^{2} & B^{1} & 0
\end{array}\right)
\end{aligned}
$$

Show that $\mathcal{L}$ is invariant under the gauge transformation $A^{\mu} \rightarrow A^{\prime \mu}=A^{\mu}+\partial^{\mu} \chi$.
Prove the Bianchi identity

$$
\partial^{\mu} F^{\nu \rho}+\partial^{\nu} F^{\rho \mu}+\partial^{\rho} F^{\mu \nu}=0
$$

and determine the equation of motion. Show that these two equations written in terms of

$$
\begin{aligned}
\vec{E} & =-\nabla A^{0}-\partial \vec{A} / \partial t \\
\vec{B} & =\nabla \times \vec{A}
\end{aligned}
$$

define the Maxwell equations in the vacuum. (Remember that $\nabla_{j}=\partial_{j}=-\partial^{j}$ ).

## Example sheet II:

$\underline{\text { Exercise } 7}$ ( coordinate space peturbation expansion of $\phi^{3}$ theory )
Consider $\phi^{3}$ theory coupled to a source $J(x)$ :

$$
\mathcal{L}\left(\phi, \partial_{\mu} \phi\right)=-\frac{1}{2} \phi \square \phi-\frac{1}{2} m^{2} \phi^{2}-\frac{\lambda}{3} \phi^{3}+\phi J
$$

Derive the equations of motion. Using

$$
\begin{aligned}
\left(\square+m^{2}\right) G(x-y) & =-i \delta(x-y) \\
\left(\square+m^{2}\right) \phi_{0} & =0
\end{aligned}
$$

show that the e.o.m. in integal form reads

$$
\phi(x)=\phi_{0}(x)+i \int d^{4} y G(x-y)\left(J(y)-\lambda \phi^{2}(y)\right)
$$

Evaluate $\phi(x)$ by iteration for $\phi_{0}=0$ in terms of $J$ to order $\mathcal{O}\left(\lambda^{3}\right)$. Find a diagrammatic representation of your solution.

Exercise 8 ( Gauge fixing )
Introduce a so-called gauge-fixing term on the Lagrangian level

$$
\mathcal{L}=\mathcal{L}_{\mathrm{MAXWELL}}+\mathcal{L}_{\mathrm{D}}-\frac{1}{2 \lambda}\left(\partial_{\mu} A^{\mu}\right)^{2}
$$

and show that the equation of motion of the gauge field $A^{\mu}$ is of the form

$$
\left(\square g^{\mu \nu}-\left(1-\frac{1}{\lambda}\right) \partial^{\mu} \partial^{\nu}\right) A_{\nu}=j^{\mu}
$$

The photon propagator can be regarded as the Green's function of the l.h.s. operator. It is defined by

$$
\left(\square g^{\mu \nu}-\left(1-\frac{1}{\lambda}\right) \partial^{\mu} \partial^{\nu}\right) \hat{D}_{\nu \rho}(x-y)=i g_{\rho}^{\mu} \delta(x-y)
$$

Use the Fourier ansatz

$$
\hat{D}_{\nu \rho}(x)=\int \frac{d^{4} k}{(2 \pi)^{4}} D_{\nu \rho}(k) e^{-i k_{\mu} x^{\mu}}
$$

to show that

$$
D_{\mu \nu}(k)=\frac{i}{k^{2}+i \epsilon}\left(-g^{\mu \nu}+(1-\lambda) \frac{k^{\mu} k^{\nu}}{k^{2}}\right)
$$

The value $\lambda=1$ is called the Feynman gauge, the value $\lambda=0$ is called Landau gauge.

## Exercise 9: Gauge invariance of the QED Lagrangian

Given the Lagrangian of QED:

$$
\begin{equation*}
\mathcal{L}=-\frac{1}{4} F^{\mu \nu} F_{\mu \nu}+\bar{\psi}(x)(i \not D-m) \psi(x) \tag{1}
\end{equation*}
$$

where

$$
\begin{equation*}
F^{\mu \nu}=\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu} \quad D_{\mu} \equiv \partial_{\mu}+i e A_{\mu}(x) \tag{2}
\end{equation*}
$$

show that it is invariant under the gauge transformation

$$
\begin{equation*}
\psi(x) \rightarrow e^{i \Theta(x)} \psi(x), \quad A_{\mu} \rightarrow A_{\mu}-\frac{1}{e} \partial_{\mu} \Theta(x) \tag{3}
\end{equation*}
$$

## Exercise 10: Feynman rules of scalar QED

Consider the Lagrangian

$$
\begin{equation*}
\mathcal{L}=\left[\left(\partial_{\mu}+i e A_{\mu}\right) \phi^{*}\left(\partial^{\mu}-i e A^{\mu}\right) \phi\right]-\mu^{2} \phi^{*} \phi-\lambda\left(\phi^{*} \phi\right)^{2}-\frac{1}{4} F_{\mu \nu} F^{\mu \nu} \tag{4}
\end{equation*}
$$

where $F_{\mu \nu}=\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu}$, which describes the coupling of the electromagnetic field to a charged scalar field.

Using the procedure given in the lecture, derive all Feynman rules for this model.
Exercise 11 (Deriving Feynman rules by functional derivatives )
Consider the following interaction terms

$$
\begin{aligned}
L_{1} & =g A^{\mu}\left(B \partial_{\mu} C-C \partial_{\mu} B\right) \\
L_{2} & =g A^{\mu} A_{\mu} B C
\end{aligned}
$$

where $A^{\mu}$ is a gauge field and $B, C$ are real scalars. Perform the Fourier transformation and apply the adequate functional derivative to determine the two vertex expressions defining the Feynman rules. Compare your result with the one obtained by the recipe given in the lecture.
Exercise 12 ( QED amplitudes )
Write down the amplitudes for the following QED processes:

$$
\begin{aligned}
e^{-}\left(p_{1}, s_{1}\right)+e^{+}\left(p_{2}, s_{2}\right) & \rightarrow \mu^{-}\left(k_{1}, r_{1}\right)+\mu^{+}\left(k_{2}, r_{2}\right) \\
e^{-}\left(p_{1}, s_{1}\right)+\mu^{-}\left(p_{2}, s_{2}\right) & \rightarrow e^{-}\left(k_{1}, r_{1}\right)+\mu^{-}\left(k_{2}, r_{2}\right) \\
\gamma\left(p_{1}, \sigma_{1}\right)+\gamma\left(p_{2}, \sigma_{2}\right) & \rightarrow \gamma\left(k_{1}, \sigma_{1}^{\prime}\right)+\gamma\left(k_{2}, \sigma_{2}^{\prime}\right) .
\end{aligned}
$$

The first two are actually related by so-called crossing rules. How could you get the second from the first by simple relabeling?

## Exercise 13: Phase-space Integrals

The Lorentz Invariant Phase Space for a $2 \rightarrow n$ process is given by

$$
\begin{equation*}
(2 \pi)^{4} \cdot \delta^{(4)}\left(p_{1}+p_{2}-\sum_{i=1}^{n} k_{i}\right) \cdot \prod_{j=1}^{n} \frac{\mathrm{~d}^{3} k_{j}}{(2 \pi)^{3} k_{j}^{0}} \tag{5}
\end{equation*}
$$

Show that this expression is a Lorentz invariant quantity.
Hint: Show first that $\frac{\mathrm{d}^{3} k}{(2 \pi)^{3} k^{0}}$ is a Lorentz invariant measure.

Exercise 14 (The decay $\pi_{0} \rightarrow \gamma \gamma$ )
The quark substructure of a neutral pion gives rise to an effective $\pi_{0}-\gamma-\gamma$ vertex which can be parametrised as

$$
L_{\pi_{0} \gamma \gamma}=i \frac{\alpha}{8 \pi} \frac{1}{f_{\pi}} F^{\mu \nu} \tilde{F}_{\mu \nu} \phi
$$

with $\tilde{F}_{\mu \nu}=\epsilon_{\mu \nu \rho \sigma} F^{\rho \sigma}$ and $\epsilon_{\mu \nu \rho \sigma}$ the totally antisymmetric tensor. $\phi$ is the wave function of the pion and $f_{\pi} \sim 93 \mathrm{MeV}$ is the so-called pion decay constant. Write down the Feynman rule and evaluate the decay width of the pion. Compare you result to the one in the literature $\Gamma=\frac{\alpha^{2} m_{\pi}^{3}}{64 \pi^{3} f_{\pi}^{2}}=7.63 \mathrm{eV}$.
[Note: $\left.\epsilon^{\mu \nu \rho \sigma} \epsilon_{\mu \nu \tau \eta}=-2\left(g_{\tau}^{\rho} g_{\eta}^{\sigma}-g_{\eta}^{\rho} g_{\tau}^{\sigma}\right)\right]$

## Example sheet IIa:

## Introduction to the Standard Model

## Exercise 1: Spin and angular momentum conservation

Consider radiation of a collinear photon by an relativistic electron in QED.

$$
e(p, s) \rightarrow e\left(p^{\prime}, s^{\prime}\right)+\gamma(k, \sigma)
$$

Show that the Feynman amplitude for this process is

$$
-i e \bar{u}\left(p^{\prime}, s^{\prime}\right) \gamma^{\mu} u(p, s) \epsilon_{\mu}^{*}(k, \sigma)
$$

Show that the coupling to a temporally polarised photon preserves spin at leading order in the relativistic expansion $\left(\frac{p}{m} \ll 1\right)$. (This term generates the electrostatic potential).
We will choose axes momenta in such a way that photon is emitted in the z-direction. The polarisation vectors are then $\epsilon_{\mu}^{\sigma}=[0,1, \pm i, 0]$, corresponding to circularly polarised light travelling along the Z axis. We will also further restrict the kinematics by boosting to the frame where $p_{x}=p_{x}^{\prime}=0$ and $p_{y}=p_{y}^{\prime}=0$. In this frame we may take $p_{\mu}=p(1, \hat{z}), p_{\mu}^{\prime}=p^{\prime}(1, \hat{z})$, and so $k_{\mu}=k(1, \hat{z})$ where $k=p-p^{\prime}$.
We may now use the Gordon decomposition

$$
\bar{u}\left(p^{\prime}, s^{\prime}\right) \gamma_{\mu} u(p, s)=\frac{1}{2 m} \bar{u}\left(p^{\prime}, s^{\prime}\right)\left[\left(p^{\prime}+p\right)_{\mu}+i \sigma_{\mu \nu}\left(p^{\prime}-p\right)^{\nu}\right] u(p, s)
$$

to show that the amplitude is

$$
\bar{u}^{\prime}\left(i \sigma_{x z} \mp \sigma_{y z}\right) k^{z} u=\bar{u}^{\prime}\left(-i \operatorname{diag}\left(\sigma_{y}, \sigma_{y}\right) \mp \operatorname{diag}\left(\sigma_{x}, \sigma_{x}\right)\right) u k^{z}
$$

We can recognise the $S_{z}$ spin raising and lowering operators $\sigma_{x}+i \sigma_{y}=\left(\begin{array}{cc}0 & 1 \\ 0 & 0\end{array}\right) \sigma_{x}-i \sigma_{y}=\left(\begin{array}{cc}0 & 0 \\ 1 & 0\end{array}\right)$, and see that the coupling to a circularly polarised $J_{z}= \pm 1$ photon arises only through spin transitions in the electron.

In this frame, a $S_{z}=-\frac{1}{2}$ to $S_{z}=+\frac{1}{2}$ transition couples to a $J_{z}=-1$ photon moving in the z direction. The coupling to photon of opposite angular momentum vanishes for this spin transition.
The reverse transition $S_{z}=+\frac{1}{2}$ to $S_{z}=-\frac{1}{2}$ transition couples to a $J_{z}=+1$ photon (opposite sign in polarisation vector) moving in the z direction. The coupling to photon of opposite angular momentum vanishes.

In this way angular momentum is conserved. Note, however, that the kinematics were chosen carefully to align helicities and polarisation vectors in the Z-direction and general Lorentz transformations of this amplitude follow the normal vector transformation laws.

## Exercise 2: Compton Scattering

Reproduce the Compton scattering amplitude and cross-section caculation in the notes.

## Exercise 3: Gauge invariance of amplitudes in momentum space

Show that $k_{\mu} \bar{u}\left(p^{\prime}\right) \gamma^{\mu} u(p)=0$.
Consider the electron-photon amplitude above. Use this result to show that the scattering amplitude is invariant under a general gauge transformation. Show that the combined amplitude for Compton scattering is gauge invariant, although the two graphs individually are not.

## Example sheet III:

## Introduction to the Standard Model

Exercise 15a (Relation between the groups $U(1)$ and $S O(2)$ )
Multiplication of a complex number by a phase corresponds to a rotation of the corresponding vector in the complex plane. Show that $U(1)$ is a group and prove the group isomorphism

$$
e^{i \alpha}=\cos \alpha+i \sin \alpha \quad \sim \quad \sim \quad\left(\begin{array}{rr}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{array}\right) \in S O(2)
$$

Show that each element of $S O(2)$ can be generated by exponentiation:

$$
\exp (i \alpha \mathbf{T})=\left(\begin{array}{rr}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{array}\right) \quad \text { with the hermitian and traceless generator } \quad \mathbf{T}=\left(\begin{array}{rr}
0 & i \\
-i & 0
\end{array}\right)
$$

Exercise 15b: $S U(2)$ Lie-Algebra

The $S U(2)$ matrices $T^{j=1,2,3}$ satisfy the algebra

$$
\begin{equation*}
\left[T^{j}, T^{k}\right]=i \epsilon^{j k l} T^{l} \tag{6}
\end{equation*}
$$

where $\epsilon^{j k l}$ is the totally antisymmetric Levi-Civita tensor. Given their explicit form in the $\mathrm{SU}(2)$ doublet representation

$$
T^{1}=\frac{1}{2}\left(\begin{array}{ll}
0 & 1  \tag{7}\\
1 & 0
\end{array}\right), \quad T^{2}=\frac{1}{2}\left(\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right), \quad T^{3}=\frac{1}{2}\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)
$$

build the raising and lowering operators $T^{ \pm}=T^{1} \pm i T^{2}$ and show that they satisfy the commutation relations

$$
\begin{equation*}
\left[T^{+}, T^{-}\right]=2 T^{3}, \quad\left[T^{3}, T^{ \pm}\right]= \pm T^{ \pm} \tag{8}
\end{equation*}
$$

Exercise 16 (Fun with $S U(N)$ )
Given the $S U(N)$ Lie-Algebra evaluate the structure constants defined by

$$
\begin{aligned}
f^{a b c} & =-2 i \operatorname{tr}\left(\left[T^{a}, T^{b}\right] T^{c}\right) \\
d^{a b c} & =2 \operatorname{tr}\left(\left\{T^{a}, T^{b}\right\} T^{c}\right)
\end{aligned}
$$

for the cases $N=2,3$. The generators in the case $N=2$ are up to a factor of 2 the Pauli matrices $T^{a}=1 / 2 \sigma^{a}$.

$$
\sigma^{1}=\left(\begin{array}{cc}
0 & 1 \\
1 & 0
\end{array}\right) \quad, \quad \sigma^{2}=\left(\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right) \quad, \quad \sigma^{3}=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)
$$

The generators for the case $N=3$ are up to a factor of 2 the Gell-Mann matrices $T^{a}=1 / 2 \lambda^{a}$ :

$$
\begin{array}{ll}
\lambda^{1}=\left(\begin{array}{lll}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \quad \lambda^{2}=\left(\begin{array}{ccc}
0 & -i & 0 \\
i & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \quad \lambda^{3}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{array}\right) \\
\lambda^{4}=\left(\begin{array}{lll}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{array}\right), \quad \lambda^{5}=\left(\begin{array}{ccc}
0 & 0 & -i \\
0 & 0 & 0 \\
i & 0 & 0
\end{array}\right), \quad \lambda^{6}=\left(\begin{array}{lll}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{array}\right) \\
\lambda^{7}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & -i \\
0 & i & 0
\end{array}\right), \quad \lambda^{8}=\frac{1}{\sqrt{3}}\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -2
\end{array}\right),
\end{array}
$$

Further verify the relation

$$
\left\{T^{a}, T^{b}\right\}=\frac{1}{N} \delta^{a b}+d^{a b c} T^{c}
$$

Exercise 17 ( Noether currents for free Dirac field )
Consider the following Lagrangian

$$
\mathcal{L}=\bar{\psi}_{j}(i \not \partial-m) \psi_{j}
$$

where the Dirac field $\psi$ is in the fundamental representation of $S U(N)$.
Show that the Lagrangian is invariant under (global) $S U(N)$ and $U(1)$ transformations.
Derive the Noether currents for these symmetries:

$$
\begin{array}{ll}
J_{\mu}^{a}=\bar{\psi}_{j} \gamma_{\mu} \mathrm{T}_{j l}^{a} \psi_{j} & \text { for } S U(N) \\
J_{\mu}=\bar{\psi}_{j} \gamma_{\mu} \psi_{j} & \text { for } U(1)
\end{array}
$$

Exercise 18 (Quark Model and Meson/Baryon multiplets )
Give an argument for the dimensions of the irreducible representations contained in the following tensor representations of $S U(3)_{F}$ :

$$
\begin{aligned}
3 \otimes \overline{3} & =8 \oplus 1 \\
3 \otimes 3 \otimes 3 & =1 \oplus 8 \oplus 8 \oplus 10
\end{aligned}
$$

Find the quark content of the Meson octet and Baryon octet and decuplet discussed in the lecture.
Discuss qualitatively the quantum mechanical problem of a bound state of three identical fermions confined in a $d=1$ dimensional potential

$$
V(x)=\left\{\begin{array}{cc}
0 \quad, & 0<x<1 \\
\infty & , \quad \text { else }
\end{array}\right.
$$

What do you conclude for the Spin $3 / 2$ Baryon states like $\Omega^{-}$and $\Lambda^{++}$?
Exercise 19 (Transformation of the $S U(N)$ gauge field)

The transformation formula of a gauge field was derived in the lecture as

$$
A^{\prime \mu}=U(x) A^{\mu} U^{\dagger}(x)-\frac{i}{g_{N}} U(x) \partial^{\mu} U^{\dagger}(x)
$$

How does it look for an infinitesimally transformation. Discuss the difference to the $U(1)$ case .
Exercise 20 (Feynman rules of Yang-Mills Theory )
Derive the Feynman rules for Yang-Mills Theory

$$
L_{\mathrm{YM}}=-\frac{1}{4} F_{a}^{\mu \nu} F_{a \mu \nu}-\frac{1}{2 \lambda}\left(\partial_{\mu} A_{a}^{\mu}\right)^{2}
$$

by applying the recipe given in the lecture.

## Example sheet IV: Introduction to the Standard Model

Exercise 21 ( Comparison of scalar $S O(4)$ and $S U(2)$ model )
Write down the Lagrangian density of a scalar theory with
a) a global $S U(2)$ symmetry and the scalar in the fundamental representation.
b) a global $S 0(4)$ symmetry and the scalar in the fundamental representation.
where the scalar is in both cases in the fundamental representation:

$$
\phi_{S O(4)}=\left(\begin{array}{c}
\pi_{1} \\
\pi_{2} \\
\pi_{3} \\
\sigma
\end{array}\right) \quad, \quad \phi_{S U(2)}=\frac{1}{\sqrt{2}}\binom{\pi_{1}+i \pi_{2}}{\sigma+i \pi_{3}}
$$

Conclude that the ungauged Higgs sector of the Standard Model can be viewed as a $S O(4)$ linear $\sigma$-model.
Exercise 22 (Unitary gauge for $S U(2)$ gauge theory )
Consider a spontaneously broken $S U(2)$ gauge theory with a Higgs sector as in the Standard Model. Show that the Higgs doublet can be written as

$$
\binom{\phi^{+}}{\phi^{0}}=\frac{1}{\sqrt{2}} \exp \left(-i \sum_{j=1,2,3} \theta^{j} T^{j} / v\right)\binom{0}{H+v}
$$

The exponential factor defines a gauge transformation in terms of the Goldstone bosons $\theta_{j=1,2,3}$. Show that in the "unitary gauge" the Goldsone bosons are absent and the vector bosons are massive.

Exercise 23 ( Massive gauge boson propagator and polarisation vectors )
a) Derive the gauge boson propagator for the spontaneously broken $U(1)$ gauge theory in the unitary gauge.
b) In the rest frame of the massive vector boson the 4 -momentum and polarisation vectors are given by

$$
p^{\mu}=(M, \overrightarrow{0}), \epsilon_{ \pm}^{\mu}=\frac{1}{\sqrt{2}}(0, \pm i, 1,0), \epsilon_{0}^{\mu}=(0,0,0,1)
$$

Verify the following relation for the polarisation tensor of the gauge boson,

$$
\sum_{\sigma \in\{+,-, 0\}}\left(\epsilon_{\sigma}^{\mu}\right)^{*} \epsilon_{\sigma}^{\nu}=-g^{\mu \nu}+\frac{p^{\mu} p^{\nu}}{M^{2}}
$$

Exercise 24 ( vector boson pair production at LEP II )
At the LEP II experiment maximal center of mass energies of about 210 GeV have been reached which allowed to study the production of charged vector boson pairs.
Draw all Feynman diagrams for $e^{+} e^{-} \rightarrow W^{+} W^{-} / Z Z$. Discuss the different contributions.
Find the polarisation vectors of a pair of charged vector bosons scattered by an angle $\theta$ relative to the beam axis.

## Example sheet V: Introduction to the Standard Model

## Exercise 25 ( Fermions electroweak gauge boson interactions )

Derive an expression for the Lagrangian of the fermionic contribution of the electroweak Standard Model in terms of the mass eigenstates of the electroweak gauge bosons $\left(A_{\mu}, Z_{\mu}, W_{\mu}^{ \pm}\right)$. Start with the Lagrangian in terms of the $S U(2)_{L} \otimes U(1)_{Y}$ bosons $W_{\mu}^{1,2,3}, B_{\mu}$ as derived in the lecture. Especially show that

- the photon couples to a vector-current (V) only.
- the massive charged vector-boson couples to left handed fermions only, i.e. it couples to vector and axial vector current with the same strength (V-A structure).
- the neutral vector boson has a (V-A) structure where the vector and axial vector couplings are different and flavour dependent.

In the latter case write down the expressions for the flavour dependent coupling parameters $V_{f}, A_{f}$ as defined in the lecture.

Exercise 26 ( Yukawa couplings for one generation )
The Yukawa interaction terms for one generation of fermions are

$$
L=\lambda_{e} \bar{e}_{R} \phi^{\dagger} \cdot\binom{\nu_{e}}{e}_{L}+\lambda_{d} \bar{d}_{R} \phi^{\dagger} \cdot\binom{u}{d}_{L}+\lambda_{u} \bar{u}_{R} \phi^{T} \cdot \epsilon \cdot\binom{u}{d}_{L}+\text { h.c. } .
$$

By using the proper hypercharge and isospin quantum numbers show explicitly that they are invariant under local $S U(2)_{L} \otimes U(1)_{Y}$ transformations. What are the mass terms and Higgs fremion interactions in the unitary gauge?

## Exercise 27 ( Z-boson decay )

The $Z-f-\bar{f}$ vertex has (up to a trivial colour factor) the Feynman rule $-i g /\left(2 \cos \theta_{W}\right)\left[\gamma_{\mu}\left(V_{f}-V_{a} \gamma_{5}\right)\right]$. Make a list of all particles a (on-shell) Z-boson can decay into. Show that the partial decay width $\Gamma_{Z \rightarrow f \bar{f}}$ is given by

$$
\Gamma_{Z \rightarrow f \bar{f}}=\frac{\alpha m_{Z}}{12 \sin ^{2} \theta_{W} \cos ^{2} \theta_{W}} \sqrt{1-\frac{4 m_{f}^{2}}{m_{Z}^{2}}} N_{C}(f)\left[V_{f}^{2}\left(1+\frac{2 m_{f}^{2}}{m_{Z}^{2}}\right)+A_{f}^{2}\left(1-\frac{4 m_{f}^{2}}{m_{Z}^{2}}\right)\right]
$$

where $N_{C}\left(l_{j}, \nu_{j}\right)=1$ and $N_{C}\left(q_{j}\right)=3$. Show further that the branching ratios for $Z$ to hadrons, charged leptons and neutrinos are approximately $B(Z \rightarrow q \bar{q})=0.7, B\left(Z \rightarrow l^{+} l^{-}\right)=0.1$ and $B(Z \rightarrow \nu \bar{\nu})=0.2$. As numerical input use $\alpha\left(m_{Z}\right)=1 / 128, \sin \theta_{W}=0.23$ and $m_{Z}=91.7 \mathrm{GeV}$. Neglect all fermion masses apart from the b-quark which has a mass of about $m_{b}=5 \mathrm{GeV}$.

Exercise 28 (The Drell-Yan process )
Write down the cross section for the process proton proton $\rightarrow \mathrm{W} \rightarrow \mu \bar{\nu}_{\mu}$ in the parton model.

## Exercise $29\left(e^{+} e^{-} \rightarrow q \bar{q}\right)$

Consider $e^{-}(p, s)+e^{+}\left(p^{\prime}, s^{\prime}\right) \rightarrow q_{j}(k, r)+\overline{q_{j}}\left(k^{\prime}, r^{\prime}\right)$.
a $q_{j}$ represents a quark of color $j$ and charge $Q_{q} \times e$. Draw the Feynman diagram mediated by a photon at tree level, labelling particle flow, momenta and external states.
Write the amplitude $\mathcal{M}$ as a mathematical expression and determine the squared amplitude $|\mathcal{M}|^{2}$.
b Using the (high energy) spinor relation $\sum_{s} u(p, s) \bar{u}(p, s)=\not p$, and $\operatorname{Tr} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}=4\left(g^{\mu \nu} g^{\rho \sigma}-g^{\mu \rho} g^{\nu \sigma}+\right.$ $g^{\mu \sigma} g^{\nu \rho}$ ), show that the spin averaged square amplitude, with photon momentum $q$, is

$$
\frac{1}{4} \sum_{\text {colors spins }} \sum_{|\mathcal{M}|^{2}=8 \frac{e^{4} Q_{q}^{2}}{q^{4}}\left[(p \cdot k)\left(p^{\prime} \cdot k^{\prime}\right)+\left(p \cdot k^{\prime}\right)\left(p^{\prime} \cdot k\right)\right] . . . . ~ . ~}^{\text {a }}
$$

c Take collider frame with $\cos \theta=\hat{\mathbf{n}} \cdot \hat{\mathbf{z}}$ :

$$
p=E(1,0,0,1) \quad p^{\prime}=E(1,0,0,-1) \quad k=E(1, \hat{\mathbf{n}}) \quad k^{\prime}=E(1,-\hat{\mathbf{n}})
$$

Show that
d Using $\frac{d \sigma}{d \Omega}=\frac{1}{32 \pi^{2} \cdot 32 E^{2}} \sum_{\text {colors }} \sum_{\text {spins }}|\mathcal{M}|^{2}$, show that the differential cross section is

$$
\frac{d \sigma}{d \Omega}=N_{C} Q_{q}^{2} \frac{\alpha^{2}}{16 E^{2}}\left[1+\cos ^{2} \theta\right]
$$

and hence,

$$
\sigma_{\text {total }}=N_{C} Q_{q}^{2} \frac{\pi \alpha^{2}}{3 E^{2}}
$$

e Without calculation infer $\sigma_{e^{+} e^{-} \rightarrow \mu^{+} \mu^{-}}$, and determine

$$
R^{e^{+} e^{-}}=\frac{\sigma_{e^{+} e^{-} \rightarrow \text { hadrons }}}{\sigma_{e^{+} e^{-} \rightarrow \mu^{+} \mu^{-}}}
$$

at $E \simeq 3 \mathrm{GeV}, 10 \mathrm{GeV}, 200 \mathrm{GeV}$, and sketch the beam energy dependence
f Draw a Feynman graph for a three jet event and explain why the jets are planar in the lab frame How does the fraction of three jet events change as a function of the center of mass energy at high energies.

$$
\begin{array}{cccccc}
m_{d} \simeq 5 \mathrm{MeV} & m_{u} \simeq 2 \mathrm{MeV} & m_{s} \simeq 100 \mathrm{MeV} & m_{c} \simeq 1.2 \mathrm{GeV} & m_{b} \simeq 4.2 \mathrm{GeV} & m_{t} \simeq 175 \mathrm{GeV} \\
Q_{d}=-\frac{1}{3} & Q_{u}=\frac{2}{3} & Q_{s}=-\frac{1}{3} & Q_{c}=\frac{2}{3} & Q_{b}=-\frac{1}{3} & Q_{t}=\frac{2}{3}
\end{array}
$$

Exercise 30: $W^{ \pm}$and $Z^{0}$ decays

Consider the decays

$$
\begin{align*}
Z^{0}(q) & \rightarrow f(p) \bar{f}\left(p^{\prime}\right)  \tag{9}\\
W^{ \pm}(q) & \rightarrow f(p) \bar{f}^{\prime}\left(p^{\prime}\right) \tag{10}
\end{align*}
$$

- Write the amplitude for the two processes at leading order in the electroweak Standard Model.
- Ignoring the mass of the leptons, compute the following branching ratios

$$
\begin{aligned}
& -B R\left(Z^{0} \rightarrow{ }^{\prime} \text { invisible }^{\prime}\right) \equiv \frac{\Gamma\left(Z^{0} \rightarrow \bar{\nu} \nu\right)}{\Gamma\left(Z^{0} \rightarrow \text { all }\right)} \\
& -B R\left(W^{+} \rightarrow \text { leptons }\right) \equiv \frac{\Gamma\left(W^{+} \rightarrow l^{+} \nu_{l}\right)}{\Gamma\left(W^{+} \rightarrow \text { all }\right)}
\end{aligned}
$$

and compare the results with the values reported in the Particle Data Book (http://pdg.lbl.gov).

