Electromagnetism - Lecture 15

Waves in Conductors

- Absorption in Conductors
- Skin Depth
- Reflection at Conducting Surfaces
- Radiation Pressure
- Power Dissipation
- Conservation of Electromagnetic Energy

Maxwell's Equations in Conductors

Maxwell's Equations M1-3 are as for insulators:

$$\nabla.\mathbf{D} = \rho_C \qquad \nabla.\mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

but M4 in conductors includes a free current density $\mathbf{J}_C = \sigma \mathbf{E}$:

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J_C}$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_r \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \sigma \mathbf{E}$$

where we have again assumed that $\mu_r = 1$

Solution of M1-4 in Conductors

Taking the curl of M3:

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = \nabla(\nabla.\mathbf{E}) - \nabla^2\mathbf{E}$$

using M1 with the assumption that the free charge density is zero:

$$\nabla \rho_c = \epsilon_r \epsilon_0 \nabla(\nabla.\mathbf{E}) = 0$$

Subtituting for $\mathbf{B}$ from M4 leads to a modified wave equation:

$$\nabla^2\mathbf{E} = \frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = \mu_0 \epsilon_r \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu_0 \sigma \frac{\partial \mathbf{E}}{\partial t}$$

The first order time derivative is proportional to the conductivity σ

This acts as a damping term for waves in conductors

Notes:

Diagrams:

Plane Waves in Conductors

The solution can be written as an attenuated plane wave:

$$E_x = E_0 e^{i(\omega t - \beta z)} e^{-\alpha z}$$

Substituting this back into the modified wave equation:

$$(-i\beta - \alpha)^2 E_0 = \mu_0 \epsilon_r \epsilon_0 (i\omega)^2 E_0 + \mu_0 \sigma (i\omega) E_0$$

Equating the real and imaginary parts:

$$-\beta^2 + \alpha^2 = -\mu_0 \epsilon_r \epsilon_0 \omega^2 \qquad 2\beta\alpha = \mu_0 \sigma \omega$$

For a **good conductor** with $\sigma \gg \epsilon_r \epsilon_0 \omega$:

$$\alpha = \beta = \sqrt{\frac{\mu_0 \sigma \omega}{2}}$$

For a perfect conductor $\sigma \to \infty$ *there are no waves and* $\mathbf{E} = 0$

Skin Depth

The attenuation length in a conductor is known as the **skin depth**:

$$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\mu_0 \sigma \omega}}$$

Skin depths for a good conductor (metal):

- $\delta \approx 10cm$ at $\nu = 50Hz$ (mains frequency)
- $\delta \approx 10\mu m$ at $\nu = 50MHz$ (radio waves)

High frequency waves are rapidly attenuated in good conductors

Practical application of this for RF shielding of sensitive equipment against external sources of EM waves.

Magnetic Field of Plane Wave

The magnetic field of a plane wave in a good conductor can be found using M4:

$$\frac{\partial H_y}{\partial z} = -\sigma E_x = -\sigma E_0 e^{i(\omega t - \alpha z)} e^{-\alpha z}$$

where we neglect the term in the wave equation proportional to ϵ_r

$$H_y = \frac{\sigma E_0}{(1+i)\alpha} e^{i(\omega t - \alpha z)} e^{-\alpha z} = H_0 e^{i(\omega t - \alpha z - \pi/4)} e^{-\alpha z}$$

Note that there is a *phase shift* of $\pi/4$ between **E** and **H**

The amplitude ratio depends on the ratio of σ and ω:

$$\frac{H_0}{E_0} = \sqrt{\frac{\sigma}{\mu_0 \omega}}$$

In a good conductor $H_0 \gg E_0$ *and they are no longer related by c!*

Notes:

Diagrams:

Reflection at a Conducting Surface

We just consider the case of **normal incidence**

The incident wave has:

$$\mathbf{E}_I = E_{0I} e^{i(\omega t - k_1 z)} \hat{\mathbf{x}} \qquad \mathbf{B}_I = \frac{E_{0I}}{c} e^{i(\omega t - k_1 z)} \hat{\mathbf{y}}$$

The reflected wave has:

$$\mathbf{E}_R = E_{0R} e^{i(\omega t + k_1 z)} \hat{\mathbf{x}} \qquad \mathbf{B}_R = \frac{E_{0R}}{c} e^{i(\omega t + k_1 z)} \hat{\mathbf{y}}$$

The transmitted wave inside the conductor has:

$$\mathbf{E}_T = E_{0T} e^{i(\omega t - \alpha z)} e^{-\alpha z} \hat{\mathbf{x}} \qquad \mathbf{B}_T = \frac{\alpha}{\omega}(1 - i) E_{0T} e^{i(\omega t - \alpha z)} \hat{\mathbf{y}}$$

For a perfect conductor $\mathbf{E}_T = 0$ *and the wave is completely reflected*

Boundary Conditions at Conducting Surface

From **H** tangential:

$$\frac{1}{\mu_0}(B_{0I} - B_{0R}) = \frac{1}{\mu_0} B_{0T}$$

$$\sqrt{\epsilon_0}(E_{0I} - E_{0R}) = \sqrt{\frac{\sigma}{2\omega}}(1-i)E_{0T}$$

From **E** tangential:

$$E_{0I} + E_{0R} = E_{0T}$$

The reflected amplitude is:

$$\frac{E_{0R}}{E_{0I}} = -\frac{(\sqrt{\sigma/2\omega\epsilon_0}(1-i) - 1)}{(\sqrt{\sigma/2\omega\epsilon_0}(1-i) + 1)}$$

The $-$ sign gives a phase change π on reflection from a conductor

Radiation Pressure

The reflection coefficient is:

$$\mathcal{R} = \frac{E_{0R}^2}{E_{0I}^2} \approx 1 - 2\sqrt{\frac{2\omega\epsilon_0}{\sigma}}$$

For a good conductor $\sigma >> 2\omega\epsilon_0$, $\mathcal{R} \rightarrow 1$.

A metallic surface is a good reflector of electromagnetic waves.

The reflection reverses the direction of the Poynting vector $\mathbf{N} = \mathbf{E} \times \mathbf{H}$ which measures energy flux

There is a **radiation pressure** on a conducting surface:

$$\mathrm{P} = \frac{2 < \mathbf{N} >}{c} = \epsilon_0 E_0^2$$

Notes:

Diagrams:

Power Dissipation in Skin Depth

The transmission coefficient is initially:

$$\mathcal{T} = \frac{E_{0T}^2}{E_{0I}^2} \approx 2\sqrt{\frac{2\omega\epsilon_0}{\sigma}}$$

but this rapidly attenuates away within the skin depth

Power is dissipated in the conduction currents

$$\frac{dP}{d\tau} = \mathbf{J}.\mathbf{E} = \sigma|E|^2$$

Time-averaging and integrating over the skin depth:

$$< \frac{dP}{dA} >= -\frac{\delta}{2}\sqrt{2\sigma\omega\epsilon_0}E_{0I}^2 = \epsilon_0 E_{0I}^2 c$$

This result balances the radiation pressure

Conservation of Electromagnetic Energy

The power transferred to a charge moving with velocity $\mathbf{v}$ is:

$$P = \mathbf{F}.\mathbf{v} = q\mathbf{E}.\mathbf{v}$$

In terms of current density $\mathbf{J} = Ne\mathbf{v}$:

$$P = \int_V \mathbf{J}.\mathbf{E} d\tau$$

We can use M4 to replace $\mathbf{J}$:

$$\int_V \mathbf{J}.\mathbf{E} d\tau = \int_V [\mathbf{E}.(\nabla \times \mathbf{H}) - \mathbf{E}.\frac{\partial \mathbf{D}}{\partial t}] d\tau$$

The first term can be rearranged using:

$$\nabla.(\mathbf{E} \times \mathbf{H}) = \mathbf{H}.(\nabla \times \mathbf{E}) - \mathbf{E}.(\nabla \times \mathbf{H})$$

Notes:

Diagrams:

and M3 can be used to replace $\nabla \times \mathbf{E}$:

$$\int_V \mathbf{J}.\mathbf{E} d\tau = -\int_V [\nabla.(\mathbf{E} \times \mathbf{H}) + \mathbf{E}.\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H}.\frac{\partial \mathbf{B}}{\partial t}] d\tau$$

The integral over the volume $d\tau$ can be removed to give local energy conservation:

$$\mathbf{J}.\mathbf{E} = -[\nabla.\mathbf{N} + \frac{\partial U_E}{\partial t} + \frac{\partial U_M}{\partial t}]$$

- Power dissipated in currents is **J.E**
- Change in energy flux is div of Poynting vector $\nabla.(\mathbf{E} \times \mathbf{H})$
- Time variation of electric field energy density is $\partial(\mathbf{D}.\mathbf{E})/\partial t$
- Time variation of magnetic field energy density is $\partial(\mathbf{B}.\mathbf{H})/\partial t$