Electromagnetism - Lecture 16

Waveguides & Cavities

- Waves between Parallel Plates
- TE and TM Modes
- Rectangular Waveguides
- The TE01 Mode
- Power Transmission
- RF Cavities

Waves between Parallel Plates

Two infinite parallel conducting plates in the x/z plane are separated by a distance $y = b$

A wave incident on a plate at an angle θ in the y/z plane is reflected and bounces back and forth between the plates

Waves propagate between the plates in the z direction

T(ransverse) **E**(lectric) modes have $\mathbf{E}_x$ and $\mathbf{B}$ in the y/z plane

T(ransverse) **M**(agnetic) modes have $\mathbf{B}_x$ and $\mathbf{E}$ in the y/z plane

TEM modes have $\mathbf{B}_x$ and $\mathbf{E}_y$, and propagate along z only at $\theta = 90°$, i.e. they don't bounce of the walls

TE modes between Plates

Electric field is *parallel* to the plates in the **x** direction!

Incident plane wave: $E_I = E_0 e^{i(\omega t - kz\cos\theta + ky\sin\theta)}$

Reflected plane wave: $E_R = -E_0 e^{i(\omega t - kz\cos\theta - ky\sin\theta)}$

$$E_x = E_I + E_R = E_0 e^{i(\omega t - kz\cos\theta)}(e^{iky\sin\theta} - e^{-iky\sin\theta})$$

$$E_x = 2i\sin(ky\sin\theta)E_0 e^{i(\omega t - kz\cos\theta)}$$

Boundary condition at plates is $E_{||} = 0$

$E_x = 0$ at $y = 0$ and $y = b$ is satisfied if $kb\sin\theta = n\pi$

where n is an integer known as the **mode number**

Notes:

Diagrams:

Fields of TE modes

We write the fields in terms of the mode number n and the *effective wavenumber* $k' = k\cos\theta$ where $kb\sin\theta = n\pi$

$$E_x = 2i\sin(\frac{n\pi y}{b})E_0 e^{i(\omega t - k'z)}$$

From M3:

$$\frac{\partial B_y}{\partial t} = -\frac{\partial E_x}{\partial z} \qquad \frac{\partial B_z}{\partial t} = \frac{\partial E_x}{\partial y}$$

$$B_y = 2i\frac{k'}{\omega}\sin(\frac{n\pi y}{b})E_0 e^{i(\omega t - k'z)}$$

$$B_z = 2\frac{\sqrt{k^2 - k'^2}}{\omega}\cos(\frac{n\pi y}{b})E_0 e^{i(\omega t - k'z)}$$

Boundary conditions on magnetic field at plates:

$B_\perp = 0$ is satisfied by B_y for $y = 0$ and $y = b$

$H_{||} = \mathbf{J}$ from B_z requires oscillating surface currents at the plates

Fields of TM modes

Magnetic field is *parallel* to the plates in the $\mathbf{x}$ direction!

$$B_x = B_0 \cos(\frac{n\pi y}{b}) e^{i(\omega t - k' z)} \qquad kb \sin\theta = n\pi$$

The mode numbers are the same as for the TE modes!

From M4 the electric field components can be derived:

$$E_y = B_0 \frac{\omega k'}{k^2} \cos(\frac{n\pi y}{b}) e^{i(\omega t - k' z)}$$

$$E_z = iB_0 \frac{\omega \sqrt{(k^2 - k'^2)}}{k^2} \sin(\frac{n\pi y}{b}) e^{i(\omega t - k' z)}$$

Boundary conditions on fields at plates:

$H_{||} = \mathbf{J}$ from B_x requires oscillating surface currents at the plates

$E_{||} = 0$ is satisfied by E_z for $y = 0$ and $y = b$

$D_\perp = \rho$ from E_y requires oscillating surface charges at the plates

Notes:

Diagrams:

Notes:

Diagrams:

Rectangular Waveguides

Formed by pairs of parallel conducting plates in the y/z and x/z planes with separations $x = a$ and $y = b$ ($a < b$ by convention)

Waves still propagate in the z direction by bouncing off plates

TE modes have E_x and E_y but no E_z

TM modes have B_x and B_y but no B_z

TEM modes with no E_z or B_z do not satisfy boundary conditions!

TE and **TM** modes have *two* mode numbers m and n and an effective wavenumber k':

$$k' = \sqrt{(k^2 - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2})}$$

There is a cutoff frequency ν_{min} for each mode:

$$k^2 > (\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}) \qquad \nu_{min} = \frac{c}{2}\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

Fields of TE modes in Waveguides

Extrapolating from what we found for one set of parallel plates:

$$E_x = -\frac{Cn\pi}{b}\cos(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})e^{i(\omega t - k' z)}$$

$$E_y = \frac{Cm\pi}{a}\sin(\frac{m\pi x}{a})\cos(\frac{n\pi y}{b})e^{i(\omega t - k' z)}$$

$$B_x = -\frac{k'}{\omega}\frac{Cm\pi}{a}\sin(\frac{m\pi x}{a})\cos(\frac{n\pi y}{b})e^{i(\omega t - k' z)}$$

$$B_y = -\frac{k'}{\omega}\frac{Cn\pi}{b}\cos(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})e^{i(\omega t - k' z)}$$

$$B_z = i\frac{(k^2 - k'^2)}{\omega}C\cos(\frac{m\pi x}{a})\cos(\frac{n\pi y}{b})e^{i(\omega t - k' z)}$$

where C is a constant related to the amplitude

The TE01 mode of a Waveguide

The highest effective wavenumber has $m = 0, n = 1$ where $a < b$

The TE01 mode with $m = 0, n = 1$ has the lowest cutoff ν_{min}

There is no TM01 mode!

$m = 0$ or $n = 0$ do not satisfy boundary conditions.

The lowest TM mode is TM11.

Can choose ν and b so that TE01 is the only mode allowed.

The TE01 mode of rectangular waveguides is widely used for routing electromagnetic waves

Optical fibres can be treated as circular waveguides

For a discussion see Jackson Pp.378-389

Fields of TE01 mode

Setting $m = 0$, $n = 1$ in the field expressions:

$$E_x = -\frac{C\pi}{b}\sin(\frac{\pi y}{b})e^{i(\omega t - k'z)}$$

$$B_y = -\frac{k'}{\omega}\frac{C\pi}{b}\sin(\frac{\pi y}{b})e^{i(\omega t - k'z)}$$

$$B_z = i\frac{(k^2 - k'^2)}{\omega}C\cos(\frac{\pi y}{b})e^{i(\omega t - k'z)}$$

The other field components B_x, E_y, E_z are zero

Boundary conditions on fields at plates:

$E_{||} = 0$ is satisfied by E_x for $y = 0$ and $y = b$

$B_\perp = 0$ is satisfied by B_y for $y = 0$ and $y = b$

$H_{||} = \mathbf{J}$ surface currents from B_z and B_y at the y/z plates

For pictures of these see Pp.422-423 of Grant & Philips

Notes:

Diagrams:

Power transmission in a Waveguide

Power transmission is given by time-averaged Poynting vector

For the TE01 mode:

$$< N_z > = < E_x H_y > = \frac{k'}{2\mu_0 \omega} E_0^2 \sin^2(\frac{\pi y}{b})$$

The other time-averaged components $< N_x >$ *and* $< N_y >$ *are zero*

The total power over the waveguide cross-section is:

$$\mathrm{P_z} = \int_0^b < N_z > dy \int_0^a dx = \frac{abk'}{2\pi \mu_0 \omega} E_0^2 \int_0^\pi \sin^2 \theta d\theta$$

$$\mathrm{P_z} = \frac{abk'}{4\mu_0 \omega} E_0^2 = \frac{A}{4\mu_0 v_P} E_0^2$$

The total power is proportional to $A = ab$, the cross-sectional area and inversely proportional to $v_P = \omega/k'$, the effective phase velocity

RF Cavities

A rectangular box is formed by three pairs of parallel conducting plates with separations $x = a$, $y = b$ and $z = d$ ($d < a < b$)

Standing waves can be excited in the box
The waves do not propagate in any direction!

TE modes have E_x and E_y but no E_z (lowest mode TE011)

TM modes have B_x and B_y but no B_z (lowest mode TM111)

TE and **TM** modes have *three* mode numbers l, m and n

The resonant frequencies of the cavity ω are:

$$\frac{\omega}{c} = k = \sqrt{\frac{l^2\pi^2}{d^2} + \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}}$$

Oscillating fields in RF cavities are used to accelerate charges