Electromagnetism - Lecture 16

Waveguides & Cavities

- Waves between Parallel Plates
- TE and TM Modes
- Rectangular Waveguides
- The TE01 Mode
- Power Transmission
- RF Cavities
Waves between Parallel Plates

Two infinite parallel conducting plates in the $x/z$ plane are separated by a distance $y = b$

A wave incident on a plate at an angle $\theta$ in the $y/z$ plane is reflected and bounces back and forth between the plates

*Waves propagate between the plates in the $z$ direction*

**T**ransverse **E**lectric modes have $E_x$ and $B$ in the $y/z$ plane

**T**ransverse **M**agnetic modes have $B_x$ and $E$ in the $y/z$ plane

**TEM** modes have $B_x$ and $E_y$, and propagate along $z$ only at $\theta = 90^\circ$, i.e. they don’t bounce of the walls
TE modes between Plates

Electric field is parallel to the plates in the \( x \) direction!

Incident plane wave: \( E_I = E_0 e^{i(\omega t - k z \cos \theta + k y \sin \theta)} \)

Reflected plane wave: \( E_R = -E_0 e^{i(\omega t - k z \cos \theta - k y \sin \theta)} \)

\[
E_x = E_I + E_R = E_0 e^{i(\omega t - k z \cos \theta)} (e^{iky \sin \theta} - e^{-iky \sin \theta}) \\
E_x = 2i \sin(ky \sin \theta) E_0 e^{i(\omega t - k z \cos \theta)}
\]

Boundary condition at plates is \( E_{||} = 0 \)

\( E_x = 0 \) at \( y = 0 \) and \( y = b \) is satisfied if \( kb \sin \theta = n\pi \)

where \( n \) is an integer known as the mode number
Notes:

Diagrams:
Fields of TE modes

We write the fields in terms of the mode number $n$ and the effective wavenumber $k' = k \cos \theta$ where $kb \sin \theta = n\pi$

$$E_x = 2i \sin\left(\frac{n\pi y}{b}\right)E_0 e^{i(\omega t - k'z)}$$

From M3:

$$\frac{\partial B_y}{\partial t} = -\frac{\partial E_x}{\partial z} \quad \frac{\partial B_z}{\partial t} = \frac{\partial E_x}{\partial y}$$

$$B_y = 2i \frac{k'}{\omega} \sin\left(\frac{n\pi y}{b}\right)E_0 e^{i(\omega t - k'z)}$$

$$B_z = 2\frac{\sqrt{k^2 - k'^2}}{\omega} \cos\left(\frac{n\pi y}{b}\right)E_0 e^{i(\omega t - k'z)}$$

Boundary conditions on magnetic field at plates:

$B_\parallel = 0$ is satisfied by $B_y$ for $y = 0$ and $y = b$

$H_\parallel = \mathbf{J}$ from $B_z$ requires oscillating surface currents at the plates
Fields of TM modes

Magnetic field is parallel to the plates in the $x$ direction!

$$B_x = B_0 \cos\left(\frac{n\pi y}{b}\right)e^{i(\omega t-k'z)} \quad kb \sin \theta = n\pi$$

The mode numbers are the same as for the TE modes!

From M4 the electric field components can be derived:

$$E_y = B_0 \frac{\omega k'}{k^2} \cos\left(\frac{n\pi y}{b}\right)e^{i(\omega t-k'z)}$$

$$E_z = iB_0 \frac{\omega \sqrt{(k^2-k'^2)}}{k^2} \sin\left(\frac{n\pi y}{b}\right)e^{i(\omega t-k'z)}$$

Boundary conditions on fields at plates:

$H_{||} = J$ from $B_x$ requires oscillating surface currents at the plates

$E_{||} = 0$ is satisfied by $E_z$ for $y = 0$ and $y = b$

$D_\perp = \rho$ from $E_y$ requires oscillating surface charges at the plates
Rectangular Waveguides

Formed by pairs of parallel conducting plates in the $y/z$ and $x/z$ planes with separations $x = a$ and $y = b$ ($a < b$ by convention)

Waves still propagate in the $z$ direction by bouncing off plates

**TE** modes have $E_x$ and $E_y$ but no $E_z$

**TM** modes have $B_x$ and $B_y$ but no $B_z$

**TEM modes with no $E_z$ or $B_z$ do not satisfy boundary conditions!**

**TE** and **TM** modes have *two* mode numbers $m$ and $n$ and an effective wavenumber $k'$:

$$k' = \sqrt{(k^2 - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2})}$$

There is a cutoff frequency $\nu_{min}$ for each mode:

$$k^2 > \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right) \quad \nu_{min} = \frac{c}{2} \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$
Fields of TE modes in Waveguides

Extrapolating from what we found for one set of parallel plates:

\[
E_x = -\frac{C n\pi}{b} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)e^{i(\omega t-k'z)}
\]

\[
E_y = \frac{C m\pi}{a} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)e^{i(\omega t-k'z)}
\]

\[
B_x = -\frac{k'}{\omega} \frac{C m\pi}{a} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)e^{i(\omega t-k'z)}
\]

\[
B_y = -\frac{k'}{\omega} \frac{C n\pi}{b} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)e^{i(\omega t-k'z)}
\]

\[
B_z = i \frac{(k^2 - k'^2)}{\omega} C \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)e^{i(\omega t-k'z)}
\]

where \(C\) is a constant related to the amplitude
The TE01 mode of a Waveguide

The highest effective wavenumber has $m = 0, n = 1$ where $a < b$

The TE01 mode with $m = 0, n = 1$ has the lowest cutoff $\nu_{\text{min}}$

There is no TM01 mode!
$m = 0$ or $n = 0$ do not satisfy boundary conditions.

The lowest TM mode is TM11.

Can choose $\nu$ and $b$ so that TE01 is the only mode allowed.

The TE01 mode of rectangular waveguides is widely used for routing electromagnetic waves

Optical fibres can be treated as circular waveguides
For a discussion see Jackson Pp.378-389
Fields of TE01 mode

Setting $m = 0$, $n = 1$ in the field expressions:

$$E_x = -\frac{C\pi}{b} \sin\left(\frac{\pi y}{b}\right)e^{i(\omega t-k'z)}$$

$$B_y = -\frac{k'}{\omega} \frac{C\pi}{b} \sin\left(\frac{\pi y}{b}\right)e^{i(\omega t-k'z)}$$

$$B_z = i \frac{(k^2 - k'^2)}{\omega} C \cos\left(\frac{\pi y}{b}\right)e^{i(\omega t-k'z)}$$

The other field components $B_x$, $E_y$, $E_z$ are zero.

Boundary conditions on fields at plates:

$E_{||} = 0$ is satisfied by $E_x$ for $y = 0$ and $y = b$

$B_{\perp} = 0$ is satisfied by $B_y$ for $y = 0$ and $y = b$

$H_{||} = J$ surface currents from $B_z$ and $B_y$ at the $y/z$ plates

For pictures of these see Pp.422-423 of Grant & Philips
Notes:

Diagrams:
Power transmission in a Waveguide

Power transmission is given by time-averaged Poynting vector

For the TE01 mode:

\[< N_z >= < E_x H_y > = \frac{k'}{2 \mu_0 \omega} E_0^2 \sin^2 \left( \frac{\pi y}{b} \right)\]

The other time-averaged components \(< N_x > \) and \(< N_y > \) are zero

The total power over the waveguide cross-section is:

\[P_z = \int_{0}^{b} < N_z > dy \int_{0}^{a} dx = \frac{abk'}{2\pi \mu_0 \omega} E_0^2 \int_{0}^{\pi} \sin^2 \theta d\theta\]

\[P_z = \frac{abk'}{4 \mu_0 \omega} E_0^2 = \frac{A}{4 \mu_0 \nu_P} E_0^2\]

The total power is proportional to \(A = ab\), the cross-sectional area and inversely proportional to \(\nu_P = \omega/k'\), the effective phase velocity
RF Cavities

A rectangular box is formed by three pairs of parallel conducting plates with separations $x = a$, $y = b$ and $z = d$ ($d < a < b$)

**Standing waves** can be excited in the box

*The waves do not propagate in any direction!*

**TE** modes have $E_x$ and $E_y$ but no $E_z$ (lowest mode TE011)

**TM** modes have $B_x$ and $B_y$ but no $B_z$ (lowest mode TM111)

**TE** and **TM** modes have *three* mode numbers $l$, $m$ and $n$

The resonant frequencies of the cavity $\omega$ are:

$$\frac{\omega}{c} = k = \sqrt{\frac{l^2\pi^2}{d^2} + \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}}$$

*Oscillating fields in RF cavities are used to accelerate charges*