Electromagnetism - Lecture 17

Radiation Fields

- The Lorentz Gauge
- Hertzian Dipole
- Radiation Fields
- Antennas
- Synchrotron Radiation
The Lorentz Gauge

Electromagnetism (Maxwell’s Equations) are unchanged by:

\[ V \rightarrow V - \frac{\partial \Lambda}{\partial t} \quad A \rightarrow A + \nabla \Lambda \]

The gauge transformation \( \Lambda \) is a scalar satisfying:

\[ \nabla^2 \Lambda - \frac{1}{c^2} \frac{\partial^2 \Lambda}{\partial t^2} = - \left( \nabla \cdot A + \frac{1}{c^2} \frac{\partial V}{\partial t} \right) \]

In electrostatics we use the **Coulomb gauge**:

\[ \nabla \cdot A = 0 \quad \frac{\partial V}{\partial t} = 0 \quad \nabla^2 V = -\frac{\rho}{\varepsilon_0} \quad \nabla^2 A = -\mu_0 \mathbf{J} \]

In electrodynamics we use the **Lorentz gauge**:

\[ \nabla \cdot A = -\frac{1}{c^2} \frac{\partial V}{\partial t} \quad \nabla^2 V - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_0} \quad \nabla^2 A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = -\mu_0 \mathbf{J} \]
Retarded Potentials

Variations in charge density $\rho$ or current density $\mathbf{J}$ at $(r', t')$ lead to changes in the potentials $V$ and $\mathbf{A}$ at $(r, t)$:

$$V(r, t) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(r', t - \Delta t)}{|r - r'|} d\tau'$$

$$\mathbf{A}(r, t) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(r', t - \Delta t)}{|r - r'|} d\tau'$$

These are known as retarded potentials

The propagation time of the changes is:

$$\Delta t = t - t' = \frac{|r - r'|}{c}$$

The effect of the changes in $\rho$ or $\mathbf{J}$ propagate from $r'$ to $r$ as an electromagnetic wave with speed $c$
The Hertzian Dipole

An electric dipole has its charges oscillating with frequency $\omega$:

\[ Q = Q_0 \sin \omega t \quad p = Qa = p_0 \sin \omega t \]

This is a simple model for atomic and molecular vibrations

Corresponds to oscillating current between the ends of the dipole:

\[ I = \frac{dQ}{dt} = I_0 \cos \omega t \quad I_0 = \omega Q_0 \]

The changes in $Q$ and $I$ are propagated as electromagnetic waves radiated outwards from the centre of the dipole

*EM waves are produced by oscillating charges and currents*

*Reverse process: Absorption of EM waves creates oscillating $Q, I$*
Potentials of Hertzian Dipole

There is a retarded magnetic vector potential parallel to the current, which is along the direction of the dipole:

\[
A_z(r, t) = \frac{\mu_0}{4\pi} \int \frac{I(r', t - \Delta t)}{|r - r'|} dz'
\]

We take the far-field limit \( r \gg a \) and ignore the variation in \( I \) along the dipole, which is equivalent to assuming \( \lambda \gg a \)

\[
A_z(r, t) = \frac{\mu_0 a}{4\pi r} I(t - \Delta t) = \frac{\mu_0}{4\pi r} \frac{dp}{dt}
\]

The scalar potential is obtained from the Lorentz gauge condition:

\[
\nabla \cdot \mathbf{A} = -\frac{1}{c^2} \frac{\partial V}{\partial t} = -\mu_0 \varepsilon_0 \frac{\partial V}{\partial t}
\]

\[
V(r, t) = \frac{\cos \theta}{4\pi \varepsilon_0 r^2} \left( p + \frac{r}{c} \frac{dp}{dt} \right)
\]
Fields of Hertzian Dipole

The magnetic field is obtained from $\mathbf{B} = \nabla \times \mathbf{A}$:

$$\mathbf{B} = \frac{\mu_0 \sin \theta}{4\pi} \left( \frac{1}{r^2} \frac{dp}{dt} + \frac{1}{rc} \frac{d^2 p}{dt^2} \right) \hat{\phi}$$

$\mathbf{B}$ is in the $\phi$ direction (around the dipole axis)

The electric field is obtained from M4 after some tedious manipulation using $\nabla \times \mathbf{B}$ in spherical polars:

$$\mathbf{E} = \frac{1}{4\pi \epsilon_0} \left( \frac{2p \cos \theta}{r^3} \hat{\mathbf{r}} + \frac{p \sin \theta}{r^3} \hat{\theta} + \frac{2 \cos \theta}{r^2 c} \frac{dp}{dt} \hat{\mathbf{r}} + \frac{\sin \theta}{r^2 c} \frac{dp}{dt} \hat{\theta} + \frac{\sin \theta}{rc^2} \frac{d^2 p}{dt^2} \hat{\theta} \right)$$

$\mathbf{E}$ has components in the $r$ and $\theta$ directions
Interpretations of Hertzian Fields

1. Static field is proportional to $1/r^3$ and depends on $p$

$$E = \frac{p}{4\pi\epsilon_0 r^3} \left( 2\cos\theta \ \hat{r} + \sin\theta \ \hat{\theta} \right)$$

2. Induction fields are proportional to $1/r^2$ and $dp/dt$

$$E = \frac{1}{4\pi\epsilon_0 r^2 c} \frac{dp}{dt} \left( 2\cos\theta \ \hat{r} + \sin\theta \ \hat{\theta} \right)$$

$$B = \frac{\mu_0 \sin\theta}{4\pi r^2} \frac{dp}{dt} \ \hat{\phi}$$

3. Radiation fields are proportional to $1/r$ and $d^2p/dt^2$

$$E = \frac{\sin\theta}{4\pi\epsilon_0 rc^2} \frac{d^2p}{dt^2} \ \hat{\theta}$$

$$B = \frac{\mu_0 \sin\theta}{4\pi rc} \frac{d^2p}{dt^2} \ \hat{\phi}$$
Properties of Radiation Fields

At large distances $r \gg a$ the radiation fields dominate

- $\mathbf{B}_\phi$ and $\mathbf{E}_\theta$ are perpendicular to $\mathbf{r}$ and to each other
- The Poynting vector $\mathbf{N} = \mathbf{E} \times \mathbf{H}$ points radially outwards
- The amplitudes of the fields vary with $\sin \theta$ and the Poynting vector is proportional to $\sin^2 \theta$
- The ratio of the amplitudes is $B_0 = E_0/c$, which is a characteristic of electromagnetic waves
- The fields can be written in the form of plane waves:

\[
B_\phi = \frac{\mu_0 p_0 \sin \theta}{4\pi r c} \omega^2 e^{i(\omega t - kr)} \quad \quad E_\theta = \frac{p_0 \sin \theta}{4\pi \epsilon_0 r c^2} \omega^2 e^{i(\omega t - kr)}
\]
Power radiated by Hertzian Dipoles

The Poynting vector is:

\[ N_r = \frac{\sin^2 \theta}{16\pi^2 \epsilon_0 r^2 c^3} \left( \frac{d^2 p}{dt^2} \right)^2 \]

Most power is radiated in the midplane of the dipole, and none is radiated along the dipole axis!

Integrating over a spherical surface of radius \( r \), the total power is:

\[ P = \oint_A \mathbf{N} \cdot d\mathbf{S} = \left( \frac{d^2 p}{dt^2} \right)^2 \int \frac{\sin^2 \theta}{4\pi \epsilon_0 c^3} d(\cos \theta) = \frac{1}{6\pi \epsilon_0 c^3} \left( \frac{d^2 p}{dt^2} \right)^2 \]

Radiated power is conserved since this is independent of \( r \)

The time-averaged radiated power is proportional to \( \omega^4 \):

\[ < P > = \frac{p_0^2 \omega^4}{12\pi \epsilon_0 c^3} \]
Half-Wave Antennas

Reception and transmission of EM waves by antennas is the basis of TV, mobile phones, satellite communication ...

*Practical antennas do not satisfy* \( \lambda \gg a \)
⇒ have to include variation of I along length of antenna

For a half-wave antenna \( a = \lambda/2 \) the integral over \( Idz' \) gives:

\[
E_\theta = \frac{I_0}{2\pi\epsilon_0 cr} \frac{\cos[(\pi \cos \theta)/2]}{\sin \theta} e^{i(\omega t-kr)}
\]

which is still peaked at \( \theta = 90^\circ \) and zero at \( 0^\circ \)

The power radiated can be expressed in terms of an impedance:

\[
< P > = < I >^2 R_{rad} \quad R_{rad} = 73\Omega
\]
Full Wave Antennas

For a full-wave antenna $a = \lambda$, the factor $\cos[(\pi \cos \theta)/2]$ from the integral over $Idz'$ is replaced by $\sin(\pi \cos \theta)$:

$$E_\theta = \frac{I_0}{2\pi \varepsilon_0 cr} \frac{\sin(\pi \cos \theta)}{\sin \theta} e^{i(\omega t - kr)}$$

The angular distribution has four lobes

*(see Grant & Phillips P.447 for pictures)*

The impedance of a full-wave antenna is $R_{rad} \approx 100\Omega$
Accelerated Charges

A Hertzian dipole is an example of an accelerated charge

*All accelerated charges radiate EM waves*

General form for the radiation field is:

\[ E_\theta(t) = \frac{Qa(t')}{4\pi \varepsilon_0 r c^2} \sin \theta \]

where \( a(t') \) is the acceleration at \( t' = t - r/c \)

For a *non-relativistic* charge \( Q \) in circular motion \( a = R\omega^2 \):

\[ E = \frac{QR\omega^2 \sin \theta}{4\pi \varepsilon_0 r c^2} \]

and the total power radiated from the charge is:

\[ P = \frac{2Q^2 R^2 \omega^4}{3c^3} \]
**Synchrotron Radiation**

An important application of circularly accelerated charges is to produce beams of synchrotron radiation.

For a *relativistic* charge, the angular distribution of the radiation is boosted into a narrow cone around the direction of the particle:

$$\frac{dP}{d\theta} = \frac{Q^2 a^2}{4\pi c^3} \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5}$$

This has a peak at $\theta_{max} = 1/2\gamma$ with width $\theta_{rms} = 1/\gamma$.

As $\beta \to 1$, $\gamma \gg 1$ the radiated beam becomes tangential.

*See Jackson Pp. 669-670 for pictures.*

The total power radiated is proportional to $\gamma^4$ as $\beta \to 1$:

$$P = \frac{2Q^2 c \gamma^4 \beta^4}{3R^2}$$

*These energy losses limit the maximum energy of circular accelerators.*