Electromagnetism - Lecture 2

Electric Fields

- Review of Vector Calculus
- Differential form of Gauss's Law
- Poisson's and Laplace's Equations
- Solutions of Poisson's Equation
- Methods of Calculating Electric Fields
- Examples of Electric Fields

Vector Calculus

Gradient operator ("grad") of a *scalar field* ϕ is a **vector**

$$\nabla \phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$$

Divergence operator ("div") of a **vector field K** is a *scalar*

$$\nabla.\mathbf{K} = \frac{\partial K_x}{\partial x} + \frac{\partial K_y}{\partial y} + \frac{\partial K_z}{\partial z}$$

Curl operator ("curl") of a **vector field K** is an **axial-vector**

$$\nabla \times \mathbf{K} = \left[\frac{\partial K_z}{\partial y} - \frac{\partial K_y}{\partial z}\right]\mathbf{i} + \left[\frac{\partial K_x}{\partial z} - \frac{\partial K_z}{\partial x}\right]\mathbf{j} + \left[\frac{\partial K_y}{\partial x} - \frac{\partial K_x}{\partial y}\right]\mathbf{k}$$

Operators in cylindrical and spherical polar coordinates can be found in Riley, Hobson & Bence P.276 and P.279

Useful Identities

- Divergence theorem: $\oint_A \mathbf{K}.\mathbf{dS} = \int_V \nabla.\mathbf{K} d\tau$
- Stokes's theorem: $\oint_L \mathbf{K}.\mathbf{dl} = \int_A \nabla \times \mathbf{K}.\mathbf{dS}$
- "curl grad = 0": $\nabla \times (\nabla\phi) = 0$
- "div curl = 0": $\nabla.(\nabla \times \mathbf{K}) = 0$
- "del squared = div grad" of a *scalar*: $\nabla^2\phi = \nabla.(\nabla\phi)$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

  "del squared" of a **vector** is a vector with components from the second derivatives of K_x, K_y, K_z respectively

- "curl curl = grad div - del squared"

$$\nabla \times (\nabla \times \mathbf{K}) = \nabla(\nabla.\mathbf{K}) - \nabla^2\mathbf{K}$$

Differential Form of Gauss's Law

Applying the divergence theorem to the integral of the electric flux over a closed surface:

$$\oint_A \mathbf{E}.\mathbf{dS} = \int_V \nabla.\mathbf{E} d\tau = \int_V \frac{\rho}{\epsilon_0} d\tau$$

Removing the integral over the volume gives the *differential* form of Gauss's Law:

$$\nabla.\mathbf{E} = \frac{\rho}{\epsilon_0}$$

At any point in space the divergence of the electric field is proportional to the local charge density

For the magnetic field the divergence is always zero!

$$\oint_A \mathbf{B}.\mathbf{dS} = 0 \qquad \nabla.\mathbf{B} = 0$$

Poisson's Equation

Replacing $\mathbf{E}$ with $-\nabla V$ in the differential form of Gauss's Law:
leads to Poisson's equation:

$$\nabla^2 V = \nabla.\nabla V = -\nabla.\mathbf{E} = -\frac{\rho}{\epsilon_0}$$

At any point in space the second derivative of the electrostatic potential is proportional to the local charge density

In the absence of free charges this reduces to Laplace's equation:

$$\nabla^2 V = 0$$

the solution of which is a uniform field $\mathbf{E} = E_0$, $V = V_0 + E_0 x$

Solutions of Poisson's Equation

- If you know V or $\mathbf{E}$ everywhere you can obtain ρ everywhere by differentiation
- If you know ρ everywhere you can obtain V by numerical integration over the contributions from charge elements:

$$V = \sum_i dV_i \qquad \rho = \sum_i d\rho_i \qquad\qquad dV = \frac{\rho d\tau}{4\pi\epsilon_0 r}$$

  Principle of linear **superposition**

- If a potential obeys Poisson's equation and the boundary conditions it is the **only** solution (**uniqueness** theorem)

  "sometimes easy to find solution by mere inspection or simple trials of plausible solutions" (Duffin P.96)

Calculating Electric Fields

1. By summing the contributions to V as *scalars* and then taking the gradient:

$$V = \int dV = \int_V \frac{\rho d\tau}{4\pi\epsilon_0 r} \qquad \mathbf{E} = -\nabla V$$

The easiest method for a non-uniform charge distribution

2. By summing the contributions to $\mathbf{E}$ (or $\mathbf{F}$) as components of **vectors**

$$E_z = \int dE_z = \int_V \frac{\rho d\tau \cos\theta}{4\pi\epsilon_0 r^2}\hat{\mathbf{z}} \qquad \text{and} \qquad E_x, E_y$$

3. By using Gauss's Law:

$$\Phi_E = \oint_A \mathbf{E}.\mathbf{dS} = \int_V \frac{\rho d\tau}{\epsilon_0}$$

For symmetric problems and uniform charge distributions

Electric Field of Nucleus

Treat nucleus as insulating sphere with radius R and uniform charge density ρ

Apply Gauss's Law to spherical shells of radius $r < R$ and $r > R$

Outside the nucleus there is a point charge field:

$$E_r 4\pi r^2 = \frac{\rho \frac{4}{3}\pi R^3}{\epsilon_0} \qquad E_r(r > R) = \frac{Q}{4\pi\epsilon_0 r^2}$$

$$V(r > R) = -\int_\infty^r E_r dr = \frac{Q}{4\pi\epsilon_0 r}$$

Inside the nucleus the field increases linearly with r:

$$E_r 4\pi r^2 = \frac{\rho \frac{4}{3}\pi r^3}{\epsilon_0} \qquad E_r(r < R) = \frac{\rho r}{3\epsilon_0}$$

$$V(r < R) = V(R) - \int_R^r E_r dr = \frac{Q}{4\pi\epsilon_0 R} + \frac{\rho(R^2 - r^2)}{6\epsilon_0}$$

Electric Field of Infinite Line Charge

Apply Gauss's Law to a cylindrical volume (axis along line)

By symmetry there is no component E_z parallel to the line charge

No contribution to the surface integral from ends of cylinder
- *only true for an infinite line charge!*

An infinite uniform line charge λ has a $1/r$ field:

$$E_\perp 2\pi r L = \frac{\lambda L}{\epsilon_0} \qquad E_\perp = \frac{\lambda}{2\pi\epsilon_0 r}$$

$$V = -\int E_\perp dr = \frac{\lambda ln(r)}{2\pi\epsilon_0}$$

Electric Field of Finite Line Charge

For a line charge of finite length L calculate the sum of the $\mathbf{E}$ contributions:

$$\mathbf{dE} = \frac{\lambda dl}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

At the centre of the line only $E_\perp$ is non-zero

The projection of this component introduces a $\cos\theta$

Trick is to change the integral from dl to $d\theta$:

$$dE_\perp = \frac{\lambda}{4\pi\epsilon_0 a}\cos\theta d\theta$$

$$E_\perp = \frac{\lambda L}{4\pi\epsilon_0 a(a^2 + L^2/4)^{1/2}}$$

$L \to 0$ gives $1/r^2$ point charge field
$L \to \infty$ gives $1/r$ field

Notes:

Diagrams:

Notes:

Diagrams:

Electric Field of Infinite Surface Charge

Apply Gauss's Law to a cylindrical volume with axis $\perp$ to surface

By symmetry there is no component $E_{||}$ parallel to the surface

No contribution to the surface integral from sides of cylinder
- *only true for an infinite surface!*

An **insulating** surface has equal and opposite fields inside and outside

$$2E_{\perp}\pi R^2 = \frac{\sigma \pi R^2}{\epsilon_0} \qquad E_{\perp} = \frac{\sigma}{2\epsilon_0}$$

An infinite uniform surface charge σ *gives a uniform field*

A **conducting** surface has no field inside, so the field outside is twice as large:

$$E_{\perp}\pi R^2 = \frac{\sigma \pi R^2}{\epsilon_0} \qquad E_{\perp} = \frac{\sigma}{\epsilon_0}$$

Electric Field of Finite Disk of Charge

For a disk of charge of radius R calculate the sum of the V contributions along its axis from two-dimensional charge elements:

$$d^2V = \frac{\sigma a da d\phi}{4\pi\epsilon_0 r}$$

First integrate over $d\phi$ round ring of radius a:

$$dV = \frac{\sigma a da}{2\epsilon_0 (z^2 + a^2)^{1/2}}$$

Then integrate over da from 0 to R:

$$V = \frac{\sigma}{2\epsilon_0}\left((z^2 + R^2)^{1/2} - z\right)$$

Taking the gradient of V only $E_z = -\partial V/\partial z$ is non-zero:

$$E_z = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{(z^2 + R^2)^{1/2}}\right)$$

Notes:

Diagrams:

Notes:

Diagrams: