Electromagnetism - Lecture 3

Magnetic Fields

- Magnetic Fields
- Integral form of Ampere’s Law
- Differential form of Ampere’s Law
- Magnetic Vector Potential
- Methods of calculating Magnetic Fields
- Examples of Magnetic Fields
Magnetic Field

The magnetic field $B$ is defined by the force on a moving charge:

$$\mathbf{F} = q \mathbf{v} \times \mathbf{B}$$

in units of Tesla, $T = NA^{-1}m^{-1}$

Force on a current element:

$$d\mathbf{F} = I dl \times \mathbf{B} = J \times \mathbf{B} d\tau$$

The directions of $\mathbf{F}$, $\mathbf{B}$ and $dl$ using the left-hand rule:

- $\mathbf{B}$ is in the direction of the thumb
- $I dl$ is in the direction of the index finger
- $\mathbf{F}$ is in the direction of motion and of the middle finger
Ampère’s Law for B

The integral of the magnetic field round a closed loop is related to the total current flowing across the surface enclosed by the loop.

\[ \oint_L \mathbf{B} \cdot d\mathbf{l} = \mu_0 I = \mu_0 \int_A \mathbf{J} \cdot d\mathbf{S} \]

*Integral must be round a closed loop!*

Lines of magnetic flux are closed loops around currents.

Direction of \( \mathbf{B} \) relative to \( \mathbf{I} \) given by **corkscrew rule**

For symmetric problems can assume that \(|\mathbf{B}|\) is the same at all points around loop
**Differential Form of Ampère’s Law**

Applying Stokes’s theorem to the integral of the magnetic field round a closed loop:

\[ \oint_{L} \mathbf{B}.d\mathbf{l} = \int_{A} \nabla \times \mathbf{B}.d\mathbf{S} = \mu_0 \int_{A} \mathbf{J}.d\mathbf{S} \]

Removing the integral over the area gives the *differential* form of Ampère’s Law:

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \]

*At any point in space the curl of the magnetic field is proportional to the local current density*

In *electrostatics* the equivalent statement for the electric field is:

\[ \nabla \times \mathbf{E} = -\nabla \times \nabla V = 0 \]
Magnetic Vector Potential

Because $\nabla \cdot \mathbf{B} = 0$, the magnetic field can always be expressed as the \textit{curl} of a magnetic vector potential $\mathbf{A}$ ("div curl =0"): 

$$
\mathbf{B} = \nabla \times \mathbf{A} \quad \nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0
$$

Using Stokes’s theorem over a closed loop gives an integral relationship:

$$
\oint_{L} \mathbf{A} \cdot d\mathbf{l} = \int_{A} \mathbf{B} \cdot d\mathbf{S} = \Phi_{B}
$$

\textit{The magnetic flux through a surface is the integral of the magnetic vector potential around the loop enclosing the surface.}

Direction of a contribution $d\mathbf{A}$ is \textit{parallel} to a current element $I dl$.
**Poisson’s Equation for A**

Replacing $\mathbf{B}$ with $\nabla \times \mathbf{A}$ in the differential form of Ampère’s Law:

$$\nabla \times \mathbf{B} = \nabla \times \nabla \times \mathbf{A} = \mu_0 \mathbf{J}$$

Using the identity “curl curl = grad div - delsquared”:

$$\nabla^2 \mathbf{A} - \nabla(\nabla \cdot \mathbf{A}) = -\mu_0 \mathbf{J}$$

The choice of $\nabla \cdot \mathbf{A} = 0$ is made for **static fields**

This is known as the **Coulomb gauge**

It leads to Poisson’s equation for $\mathbf{A}$:

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

*At any point in space the second derivatives of the magnetic vector potential are proportional to the local current density*
Calculating Magnetic Fields

1. By summing the contributions to $\mathbf{B}$ as components of vectors $\mathbf{v}$

$$\mathbf{dB} = \frac{\mu_0 I (\mathbf{dl} \times \hat{r})}{4\pi r^2} = \frac{\mu_0 (\mathbf{J} \times \hat{r}) d\tau}{4\pi r^2}$$

This is known as **Biot-Savart’s Law**

2. By summing the contributions to $\mathbf{A}$ as components of vectors $\mathbf{v}$

$$\mathbf{dA} = \frac{\mu_0 I \mathbf{dl}}{4\pi r} = \frac{\mu_0 \mathbf{J} d\tau}{4\pi r}$$

and then taking the curl $\mathbf{B} = \nabla \times \mathbf{A}$

*Note that there is no method summing a scalar potential*

3. By using Ampère’s Law

$$\oint_L \mathbf{B} \cdot d\mathbf{l} = \mu_0 I = \mu_0 \int_A \mathbf{J} \cdot d\mathbf{S}$$

*For symmetric problems and uniform charge distributions*
Notes:

Diagrams:
Magnetic Field of an Infinite Wire

Apply Ampère’s Law to a circular loop around wire
By symmetry there is only a component $B_\phi$ around the wire

Outside a wire carrying a current $I$ there is a $1/r$ field:

$$B_\phi 2\pi r = \mu_0 I \quad B_\phi = \frac{\mu_0 I}{2\pi r}$$

and the magnetic vector potential is obtained from the only non-zero contribution to the curl:

$$B_\phi = -\frac{\partial A_z}{\partial r} \quad A_z = -\frac{\mu_0 I}{2\pi} \ln(r)$$

Inside a wire with uniform current density $J$:

$$B_\phi = \frac{\mu_0 Ir}{2\pi R^2} = \frac{\mu_0 Jr}{2} \quad A_z = -\frac{\mu_0 Jr^2}{4\pi R^2} = -\frac{\mu_0 Jr^2}{4}$$
Magnetic Field of a Current Loop

Along the axis of the loop only a $B_z$ component by symmetry

Using Biot-Savart’s Law the contributions to $B_z$ are:

$$dB_z = \frac{\mu_0 I dl \sin \theta}{4\pi r^2}$$

where the $\sin \theta$ comes from $dl \times \hat{r}$

Integrating round the loop:

$$B_z = \frac{\mu_0 I 2\pi a \sin \theta}{4\pi r^2} = \frac{\mu_0 I a^2}{2(a^2 + z^2)^{3/2}}$$

At the centre of the loop: $B_z(z = 0) = \mu_0 I/2a$

At a large distance from the loop there is a $1/z^3$ field:

$$B_z(z \gg a) = \frac{\mu_0 I a^2}{2z^3}$$
Notes:

Diagrams:
Magnetic Field of a Solenoid

A solenoid consists of \( n \) circular current loops per unit length.

Calculate axial \( B_z \) from sum of individual loops:

\[
B_z = \int \frac{\mu_0 n I a^2}{4(a^2 + z^2)^{3/2}} dz
\]

Trick is to change from \( dz \) to integral over \( \theta \):

\[
B_z = \int \frac{\mu_0 n I}{2} \sin \theta d\theta
\]

For an infinite solenoid there is a uniform axial field:

\[
B_z = \mu_0 n I
\]

This result can also be shown using Ampere’s Law
Magnetic Field of a Toroid

A toroid is a solenoid bent in a large circle of radius $R$

*It looks like a doughnut with a hole in it*

There are $n$ circular loops per unit length round the large circle

Use Ampere’s Law round a loop of radius $R$:

$$B_\phi 2\pi R = \mu_0(n2\pi R)I \quad B_\phi = \mu_0 nI$$

Inside and outside the circular loops $B = 0$ from Ampere’s Law

*The magnetic field is contained inside the loops of the toroid*
Notes:

Diagrams: