Electromagnetism - Lecture 4

Dipole Fields

- Electric Dipoles
- Magnetic Dipoles
- Dipoles in External Fields
- Method of Images
- Examples of Method of Images

Electric Dipoles

An electric dipole is a $+Q$ and a $-Q$ separated by a vector $\mathbf{a}$

Very common system, e.g. in atoms and molecules

The **electric dipole moment** is $\mathbf{p} = Q\mathbf{a}$ pointing from $-Q$ to $+Q$

Potential of an electric dipole:

$$V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right) = \frac{Q(r_- - r_+)}{4\pi\epsilon_0 r_+ r_-}$$

Using cosine rule, where r is distance from *centre* of dipole:

$$r_\pm^2 = r^2 + \frac{a^2}{4} \mp ar\cos\theta$$

and taking the "far field" limit $r >> a$

$$V = \frac{Qa\cos\theta}{4\pi\epsilon_0 r^2} = \frac{\mathbf{p}.\hat{\mathbf{r}}}{4\pi\epsilon_0 r^2}$$

Electric Dipole Field

Components of the electric field are derived from $\mathbf{E} = -\nabla V$

In spherical polar coordinates:

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3}$$

In cartesian coordinates, where the dipole axis is along $\mathbf{z}$:

$$E_z = \frac{p(3\cos^2\theta - 1)}{4\pi\epsilon_0 r^3}$$

$$E_{x/y} = \frac{3p\cos\theta\sin\theta}{4\pi\epsilon_0 r^3}$$

Electric dipole field decreases like $1/r^3$ (for $r \gg a$)

Notes:

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Magnetic Dipoles

A bar magnet with N and S poles has a dipole field

A **current loop** also gives a dipole field

Example: atomic electrons act as current loops

Think of current elements $+\mathrm{I}dl$ and $-\mathrm{I}dl$ on opposite sides of the current loop as equivalent to $+Q$ and $-Q$

The **magnetic dipole moment** points along the **axis** of the loop
Direction relative to **Idl** *is given by corkscrew rule*

$$\mathbf{m} = \mathrm{I}\pi a^2 \hat{\mathbf{z}} = \mathrm{I}A\hat{\mathbf{z}}$$

For a current loop **m** is the product of current times area

Magnetic dipole field has the same shape as electric dipole field:

$$B_r = \frac{2\mu_0 m \cos\theta}{4\pi r^3} \qquad B_\theta = \frac{\mu_0 m \sin\theta}{4\pi r^3}$$

Dipoles in External Fields

An **External Field** is provided by some large and distant charge (or current) distribution

Usually assumed to be uniform and constant, i.e. it is not changed by any charges or currents that are inserted in it

An electric dipole in a uniform electric field E_0 *aligns* itself with the external field with $\mathbf{p} \,||\, \mathbf{E}$

Work needed to reverse direction of dipole:

$$\Delta U = W_+ + W_- = 2QaE_0 = 2pE_0$$

Torque and Energy of Dipoles

Consider electric dipole at angle θ to external electric field E_0

Torque acting to rotate the dipole into alignment with the field:

$$\mathbf{T} = QaE_0 \sin\theta \hat{\mathbf{n}} = \mathbf{p} \times \mathbf{E}$$

The work done during this rotation is:

$$W = \int T d\theta = \int pE_0 \sin\theta d\theta = pE_0 \cos\theta$$

Potential energy of an electric dipole in an external electric field:

$$U = -\mathbf{p}.\mathbf{E}$$

Similar results for a magnetic dipole in an external magnetic field:

$$U = -\mathbf{m}.\mathbf{B} \qquad \mathbf{T} = \mathbf{m} \times \mathbf{B}$$

Notes:

Diagrams:

Method of Images

A point charge $+Q$ is a distance a from a flat conducting surface

Boundary conditions: $E_{||} = 0$ at the conducting surface
but $E_{\perp} = \sigma/\epsilon_0$ is allowed at the surface

A simple point charge field does not satisfy these conditions!

$\Rightarrow$ but a dipole field centred on the surface does

Method of images:

Put an **image charge** $-Q$ a distance $-a$ behind the surface
Note that this charge does not actually exist!

Calculate the dipole field from the $+Q$ and $-Q$ and show that it satisfies the boundary conditions

Electric Field at Conducting Surface

Using the method of images the field at the conducting surface is the z component of a dipole field at $\theta = 90°$:

$$E_z = \frac{2Qa(3\cos^2\theta - 1)}{4\pi\epsilon_0 r^3} = \frac{-Qa}{2\pi\epsilon_0(x^2+y^2)^{3/2}}$$

This field must be produced by a surface charge density σ:

$$E_z = \frac{\sigma}{\epsilon_0} \qquad \sigma = \frac{-Qa}{2\pi(x^2+y^2)^{3/2}}$$

There is no electric field inside the conductor $E(z < 0) = 0$

The distribution of σ on the surface has exactly the same effect as a $-Q$ placed at $-a$ (can show that $\int \sigma dS = -Q$)

Note that there is an attractive force between $+Q$ *and the surface*

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Conducting Sphere in External Field

Boundary conditions: $E_\theta = 0$ at the spherical surface
but $E_r = \sigma/\epsilon_0$ is allowed at the surface

A uniform external field E_0 does not satisfy these conditions!

Think what would happen to a surface charge distribution:

$+\sigma$ will prefer to be on one side of the sphere
$-\sigma$ will prefer to be on the other side of the sphere

The sphere becomes **polarized** with a **dipole moment p**

Hypothesis: the polarization of the sphere can be represented by an electric dipole at its centre

The electric field at the surface is the sum of the uniform external field and the dipole field

Electric Field at Surface of Sphere

Potential is superposition of uniform and dipole potentials:

$$V = -E_0 r \cos\theta + \frac{p\cos\theta}{4\pi\epsilon_0 r^2} + V_0$$

The spherical surface $r = R$ must be an *equipotential* $V = V_0$

$$p = 4\pi\epsilon_0 R^3 E_0 = 3\epsilon_0 E_0 \mathrm{V}_S$$

Dipole moment depends on E_0 and volume of sphere V_S

Note that this works for all $\cos\theta$*!*

The surface charge density depends on E_0 and $\cos\theta$:

$$\sigma = \epsilon_0 E_r = \epsilon_0 (E_0 \cos\theta + \frac{2p\cos\theta}{4\pi\epsilon_0 R^3}) = 3\epsilon_0 E_0 \cos\theta$$

From $\cos\theta$ there is $+\sigma$ in the right hemisphere and $-\sigma$ in the left hemisphere.

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