Electromagnetism - Lecture 5

Capacitors & Electrostatic Energy

- Examples of Capacitors
- Calculations of Capacitance
- Electrostatic Energy
- Introduction of Dielectrics
- General Result for Electrostatic Energy Density
Capacitors

A capacitor is formed from two conductors with equal and opposite surface charges $+\sigma$ and $-\sigma$ separated by an insulating gap.

**Capacitance** $C$ is the ratio of the total charge $Q$ on each conductor to the potential difference $V$ across the gap:

$$C = \frac{Q}{V} = \frac{\sigma A}{V}$$

The unit of capacitance is the Farad $F = C/V$

Practical capacitors are between pF and $\mu F$
Parallel Plate Capacitor

The electric field for infinite plates is obtained from Gauss’s Law or by using the superposition of two uniform fields. There is a uniform field in the gap of width $d$:

$$E_{gap} = \frac{\sigma}{\varepsilon_0} \hat{d}$$

The field outside the plates is zero!

The potential difference and capacitance are:

$$V = Ed = \frac{Qd}{A\varepsilon_0} \quad C = \frac{Q}{V} = \frac{A\varepsilon_0}{d}$$

Note that $C$ is a purely geometric property of the plates!
Cylindrical Capacitor

Two concentric conducting cylinders of length $l$ and radii $a$ and $b$, carry line charges $+\lambda$ and $-\lambda$

Use Gauss’s Law assuming $l \gg a, b$:

$$E_{\text{gap}} = \frac{\lambda}{2\pi\epsilon_0 r} \hat{r}$$

There is no field for $r < a$ or $r > b$!

The potential difference and capacitance are:

$$V = \int_{a}^{b} E \cdot dr = \frac{Q}{2\pi\epsilon_0 l} \ln(b/a)$$

$$C = \frac{Q}{V} = \frac{2\pi\epsilon_0 l}{\ln(b/a)}$$

Again $C$ is a purely geometric property
Energy Stored in a Capacitor

Work is done to assemble charges $\pm Q$ on capacitor plates

$$W = \int VdQ = \int \frac{Q}{C}dQ$$

This work is stored as **electrostatic energy**

$$U_E = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} QV = \frac{1}{2} CV^2$$

For a parallel plate capacitor:

$$U_E = \frac{\varepsilon_0 AV^2}{2d}$$

This can be written in terms of the electric field between the plates:

$$U_E = \frac{\varepsilon_0 (Ad)E^2}{2}$$
Electrostatic Energy Density

Electrostatic Energy is stored in a capacitor through the creation of the Electric field in the gap

The energy density of an electric field is proportional to the square of its amplitude:

$$\frac{dU_E}{d\tau} = \frac{1}{2} \varepsilon_0 |E|^2$$

A useful exercise is to prove this gives the correct electrostatic energy for a cylindrical capacitor
Electrostatic Energy of Nucleus

A Uranium nucleus has $Z = 92$ protons and $N = 146$ neutrons uniformly distributed over a radius $R \approx 10^{-15}$ m

Electric field of nucleus:

\[
E(r < R) = \frac{Zer}{4\pi\epsilon_0 R^3} \hat{r} \quad E(r > R) = \frac{Ze}{4\pi\epsilon_0 r^2} \hat{r}
\]

Total electrostatic energy by integration over energy density:

\[
dU_E = \frac{1}{2} \epsilon_0 |E|^2 d\tau
\]

\[
U_E = \int_0^R \left( \frac{Zer}{4\pi\epsilon_0 R^3} \right)^2 2\pi\epsilon_0 r^2 dr + \int_R^{\infty} \left( \frac{Ze}{4\pi\epsilon_0 r^2} \right)^2 2\pi\epsilon_0 r^2 dr
\]

\[
U_E = \frac{(Ze)^2}{40\pi\epsilon_0 R} + \frac{(Ze)^2}{8\pi\epsilon_0 R} = 1.17 \times 10^{-10} J = 730\text{MeV}
\]
Energy of Nuclear Fission

A symmetric $^{238}_{92}$U fission creates two $^{119}_{46}$Pd daughter nuclei

*Note that fission actually prefers to be asymmetric!*

Nuclear radii obey $R \propto A^{1/3}$ where $A = Z + N$

Electrostatic energy of daughter nuclei compared to $^{238}$U:

$$U'_E = 2 \left( \frac{U_E}{4(0.5)^{1/3}} \right) = 0.63U_E = 460\text{MeV}$$

Predicted release of *electrostatic* energy in fission is 270 MeV

Observed release is about 200 MeV
Dielectrics in Capacitors

Capacitance depends on the insulating material in the gap

\[ C = \varepsilon_r C_0 \] where \( C_0 \) is the result for a vacuum

\( \varepsilon_r \geq 1 \) is the dielectric constant of the material

Energy stored in capacitor is increased by dielectric material:

\[ U = \frac{1}{2} CV^2 = \varepsilon_r U_0 \]

Electrostatic energy density is proportional to \( \varepsilon_r \)

\[ \frac{dU_E}{d\tau} = \frac{1}{2} \varepsilon_r \varepsilon_0 |\mathbf{E}|^2 \]
Simple Model for Dielectrics

If an electric dipole is placed between the capacitor plates it aligns itself with the electric field in the gap \( \mathbf{p} \parallel \mathbf{E} \)

Potential energy of dipole:

\[ U = -\mathbf{p} \cdot \mathbf{E} \]

The energy stored between the plates is *increased* by this amount

In dielectric materials the atoms or molecules become **polarized** with intrinsic electric dipole moments pointing in the direction of the external field.

*This will be explained in more detail in a later lecture*
Proof of Electrostatic Energy Density

A set of \( n - 1 \) charges at positions \( \mathbf{r}_j \) gives a potential at \( \mathbf{r}_i \):

\[
V(\mathbf{r}_i) = \frac{1}{4\pi\varepsilon_0} \sum_{j=1}^{n-1} \frac{Q_j}{|\mathbf{r}_i - \mathbf{r}_j|}
\]

Work done to bring up a charge \( Q_i \) from infinity to point \( \mathbf{r}_i \):

\[
W_i = Q_i V(\mathbf{r}_i) = \frac{Q_i}{4\pi\varepsilon_0} \sum_{j=1}^{n-1} \frac{Q_j}{|\mathbf{r}_i - \mathbf{r}_j|}
\]

The total energy stored in the system of \( n \) charges is:

\[
U_E = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{j<i} Q_i Q_j = \frac{1}{8\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{Q_i Q_j}{|\mathbf{r}_i - \mathbf{r}_j|}
\]

Generalize to a double integral over the charge density \( \rho \):

\[
U_E = \frac{1}{8\pi\varepsilon_0} \int \int \frac{\rho(\mathbf{r}_i)\rho(\mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|} d\tau_i d\tau_j
\]
Perform one integral over $d\tau$ to get the potential $V$:

$$U_E = \frac{1}{2} \int \rho(r)V(r)d\tau$$

Then use Poisson’s equation to eliminate the other $\rho$:

$$U_E = -\frac{\epsilon_0}{2} \int V\nabla^2 V d\tau$$

Now the tricky bit - integrate this by parts:

$$\int_0^\infty V \nabla.(\nabla V)d\tau = [V(\nabla V)]_0^\infty - \int_0^\infty (\nabla V)(\nabla V)d\tau$$

Apply the boundary conditions $V(\infty) = 0$ and $\nabla V(0) = 0$:

$$U_E = \frac{\epsilon_0}{2} \int (\nabla V)^2 d\tau = \frac{\epsilon_0}{2} \int |E|^2 d\tau$$

*This proof comes from Jackson P.40-41*