# Electromagnetism - Lecture 6

## Induction

- Faraday's Law of Induction
- Electromotive Force
- Differential Form of Faraday's Law
- Examples of Induction
- Time-Varying Fields

# Faraday's Experiments

These involved moving a bar magnet or equivalently a current carrying coil (both have magnetic dipole fields)

Movement is towards (or away from) a conducting loop

A current is **induced** in the loop by the motion

*The motion changes the magnetic flux through the loop*

Direction of current in loop depends on direction of motion $\mathbf{v}$ and magnetic dipole moment $\mathbf{m}$

Current flow is caused by an *induced potential difference* around the loop. This is known as an *emf*.

# Faraday's Law of Induction

Faraday's Law of Induction states that:

$$\mathcal{E} = \oint_L \mathbf{E}.\mathbf{dl} = -\frac{d}{dt}\int_A \mathbf{B}.\mathbf{dS}$$

The integral of the electric field round a closed loop is related to the time-variation of the magnetic flux through the loop

$\mathcal{E}$ is known as the **induced electromotive force (emf)**
*this is confusing because* $\mathcal{E}$ *is a potential difference!*

$\mathcal{E}$ can cause a current to flow in a conducting loop: $\mathcal{E} = \mathrm{I}R$

This is first introduction to **electrodynamics** of time-varying fields

In **electrostatics** there are no time-variations

Notes:

Diagrams:

# Methods of producing an emf

$\mathcal{E}$ is produced by a time-varying magnetic flux $\frac{d\Phi_B}{dt} = \frac{d}{dt}\int_A \mathbf{B.dS}$

This can occur in many different ways:

- Moving a loop into or out of a magnetic field

$\int \mathbf{dS}$ Mechanical change to area $A$ of a loop in a magnetic field

$\mathbf{B}$ Uniform motion of a loop inside a non-uniform magnetic field
*Note that in a uniform magnetic field* $\mathcal{E} = 0$*!*

- Rotation of loop about its axis in a uniform magnetic field
*AC generators and electric motors*

$d/dt$ Time variation of the magnetic field through a static loop

# Lenz's Law

"Whenever a change in magnetic flux produces an *induced current* the direction of current flow is such as to produce effects *opposing the change in flux*"

- Force opposes movement of conducting loop into magnetic field
- Motion of conducting loop in magnetic field *decelerates*
- Torque acts against rotation of conducting loop in magnetic field

*Related to the minus sign in Faraday's Law*

# Differential Form of Faraday's Law

Differential form of Faraday's Law from Stokes's theorem:

$$\oint_L \mathbf{E}.\mathbf{dl} = \int_A (\nabla \times \mathbf{E}).\mathbf{dS} = -\frac{d}{dt}\int_A \mathbf{B}.\mathbf{dS}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

*At any point in space the curl of the induced electric field is proportional to the time derivative of the magnetic field*

Using $\mathbf{B} = \nabla \times \mathbf{A}$ and removing the curls, this can be written in terms of the magnetic vector potential:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$$

*At any point in space the induced electric field is proportional to the time derivative of the magnetic vector potential*

Notes:

Diagrams:

# Induction Examples - Rolling Wire

A conducting wire of length $l$ rolls along conducting rails with velocity $v$. The rails are connected at one end to form a circuit.

The **area** of the circuit varies with time:

$$\frac{dA}{dt} = -vl$$

A uniform magnetic field is applied perpendicular to the loop. This generates an emf:

$$\mathcal{E} = \oint_L \mathbf{E.dl} = -\frac{d}{dt}\int_A \mathbf{B.dS} = vBl$$

which causes a current I=$\mathcal{E}/R$ to flow round the circuit

The magnetic force on the current *opposes* the change and decelerates the wire:

$$F = \mathrm{I}Bl = \frac{vB^2l^2}{R} = -m\frac{dv}{dt} \qquad v = v_0 e^{-t/\tau} \qquad \tau = \frac{mR}{B^2l^2}$$

# Flux Cutting Law

A moving charge experiences a force $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$

Transform to the (primed) rest frame of the charge:
*Force is attributed to an electric field* $\mathbf{F}' = q\mathbf{E}'$

These forces must be equivalent: $\mathbf{E}' = \mathbf{v} \times \mathbf{B}$

The emf in the moving frame is:

$$\mathcal{E} = \int \mathbf{E}'.\mathbf{dl} = \int \mathbf{v} \times \mathbf{B}.\mathbf{dl}$$

Dimensionally this looks like a time-varying magnetic flux

*Emf is proportional to the rate at which a moving object cuts through magnetic flux lines*

Notes:

Diagrams:

# Rotating Faraday Disk

An insulating disk rotates with frequency $\omega$ around its axis. There is a uniform magnetic field $\mathbf{B}$ along the axis.

There is an induced emf along a radial line that sweeps round cutting through the magnetic flux lines:

$$\mathcal{E} = \int_0^a \mathbf{v} \times \mathbf{B}.\mathbf{dr} \qquad \mathbf{v} = \mathbf{r} \times \omega$$

$$\mathcal{E} = \frac{a^2 \omega B}{2}$$

Direction of emf is radially *inwards* or *outwards* depending on the sense of rotation and the direction of $\mathbf{B}$.

# Induction Examples - AC Generator

A coil of area $A$ rotates about its diameter in a uniform magnetic field with angular velocity $\omega$

The **angle** between $\mathbf{B}$ and $A$ varies with time:

$$\Phi_B = AB\cos\omega t$$

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -AB\omega\sin\omega t$$

This produces an alternating current (AC) with frequency $\omega$

*Note that the peak current is obtained when the loop is parallel to* $\mathbf{B}$ *and the magnetic flux through the loop* $\Phi_B = 0$

# Induction Examples - Betatron

A betatron consists of two iron poles shaped to give a **non-uniform** magnetic field as a function of radius $r$ from the centre of the poles.

An electron of momentum $p$ moves in a circular orbit of radius $R$ due to the magnetic force:

$$F = evB = \frac{mv^2}{R} \qquad\qquad R = \frac{p}{eB}$$

The orbits have the **cyclotron frequency**:

$$\omega = \frac{v}{R} = \frac{eB}{m}$$

Notes:

Diagrams:

If the magnetic field **B** is increased linearly with time an emf is induced by the **time-variation** of the **average** field $< B(r < R) >$ inside the orbital radius $R$:

$$\mathcal{E} = \pi R^2 \frac{d < B >}{dt} = E_\phi 2\pi R$$

The electrons are accelerated by a *tangential emf*:

$$mR\frac{d\omega}{dt} = \frac{eR}{2}\frac{d < B >}{dt} \qquad \frac{d\omega}{dt} = \frac{e}{2m}\frac{d < B >}{dt}$$

Differentiating the cyclotron frequency:

$$\omega = \frac{eB(R)}{m} \qquad \frac{d\omega}{dt} = \frac{e}{m}\frac{dB(R)}{dt}$$

The electrons remain at the same radius $R$ if:

$$\frac{dB(R)}{dt} = \frac{1}{2}\frac{d < B >}{dt} \qquad B(R) = \frac{1}{2} < B(r < R) >$$

The average field inside any radius is twice the field at that radius