Lecture 4 - Dirac Spinors

- Schrödinger & Klein-Gordon Equations
- Dirac Equation
- Gamma & Pauli spin matrices
- Solutions of Dirac Equation
- Fermion & Antifermion states
- Left and Right-handedness

Non-Relativistic Schrödinger Equation

Classical non-relativistic energy-momentum relation for a particle of mass m in potential U:

$$E = \frac{p^2}{2m} + U$$

Quantum mechanics substitutes the differential operators:

$$E \rightarrow i\hbar \frac{\delta}{\delta t} \qquad\qquad p \rightarrow -i\hbar \nabla$$

Gives non-relativistic Schrödinger Equation (with $\hbar = 1$):

$$i \frac{\delta \psi}{\delta t} = \left(-\frac{1}{2m} \nabla^2 + U \right) \psi$$

Solutions of Schrödinger Equation

Free particle solutions for $U = 0$ are plane waves:

$$\psi(\vec{x}, t) \propto e^{-iEt}\psi(\vec{x}) \qquad\qquad \psi(\vec{x}) = e^{i\vec{p}.\vec{x}}$$

Probability density:

$$\rho = \psi^*\psi = |\psi|^2$$

Probability current:

$$\vec{j} = -\frac{i}{2m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)$$

Conservation of probability gives the continuity equation:

$$\frac{\delta\rho}{\delta t} + \nabla\cdot\vec{j} = 0$$

Klein-Gordon Equation

Relativistic energy-momentum relation for a particle of mass m:

$$p_\mu p^\mu = E^2 - |\vec{p}|^2 = m^2$$

Again substituting the differential operators:

$$p_\mu \longrightarrow i\hbar\delta_\mu$$

Gives the relativistic Klein-Gordon Equation (with $\hbar = 1$):

$$\left(-\frac{\delta^2}{\delta t^2} + \nabla^2\right)\psi = m^2\psi$$

Solutions of Klein-Gordon Equation

Free particle solutions for $U = 0$:

$$\psi(x^\mu) \propto e^{-ip_\mu x^\mu} = e^{-i(Et - \vec{p}\cdot\vec{x})}$$

There are positive and negative energy solutions:

$$E = \pm\sqrt{p^2 + m^2}$$

The -ve solutions have -ve probability density ρ.
Not sure how to interpret these!

The Klein-Gordon equation is used to describe **spin 0 bosons** in relativistic quantum field theory.

Dirac Equation

In 1928 Dirac tried to understand negative energy solutions by taking the "square-root" of the Klein-Gordon equation.

$$\left(i\gamma^0\frac{\delta}{\delta t} + i\vec{\gamma}\cdot\vec{\nabla} - m\right)\psi = 0$$

or in covariant form:

$$(i\gamma^\mu\delta_\mu - m)\,\psi = 0$$

The γ "coefficients" are required when taking the "square-root" of the Klein-Gordon equation

Most general solution for ψ has four components

The γ are a set of four 4×4 matrices $\gamma^0, \gamma^1, \gamma^2, \gamma^3$

Dirac equation is actually four first order differential equations

Properties of Gamma Matrices

Multiplying the Dirac equation by its complex conjugate should give back the Klein-Gordon equation:

$$\left(-i\gamma^0\frac{\delta}{\delta t} - i\vec{\gamma}\cdot\vec{\nabla} - m\right)\left(i\gamma^0\frac{\delta}{\delta t} + i\vec{\gamma}\cdot\vec{\nabla} - m\right)\psi = 0$$

The gamma matrices are **unitary**:

$$(\gamma^0)^2 = 1 \qquad (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1$$

The gamma matrices **anticommute**:

$$\gamma^i\gamma^j + \gamma^j\gamma^i = 0 \qquad i \neq j$$

These conditions can be written as:

$$\gamma^\mu\gamma^\nu = g^{\mu\nu}$$

Representation of gamma matrices

The simplest representation of the 4×4 gamma matrices that satisfies the unitarity and anticommutation relations:

$$\gamma^0 = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix} \qquad i = 1, 2, 3$$

The $\mathbf{I}$ and $\mathbf{0}$ are the 2×2 identity and null matrices

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The σ^i are the 2×2 Pauli spin matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Solutions of Dirac equation

The wavefunctions can be written as:

$$\psi \propto u(p) e^{-ip_\mu \cdot x^\mu}$$

This is a plane wave multiplied by a **four component spinor** $u(p)$
Note that the spinor depends on four momentum p^μ

For a particle at rest $\vec{p} = 0$ the Dirac equation becomes:

$$\left(i\gamma^0 \frac{\delta}{\delta t} - m \right) \psi = \left(i\gamma^0 (-iE) - m \right) \psi = 0$$

$$Eu = \begin{pmatrix} m\mathbf{I} & 0 \\ 0 & -m\mathbf{I} \end{pmatrix} u$$

There are **four** eigenstates, two with $E = m$ and two with $E = -m$.
What is the interpretation of the $-m$ *states?*

Spinors for particle at rest

The spinors associated with the four eigenstates are:

$$u^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad u^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad u^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad u^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and the wavefunctions are:

$$\psi^1 = e^{-imt} u^1 \quad \psi^2 = e^{-imt} u^2 \quad \psi^3 = e^{+imt} u^3 \quad \psi^4 = e^{+imt} u^4$$

Note the reversal of the sign of the time exponent in ψ^3, ψ^4!

Interpretation of eigenstates

ψ^1 describes an S=1/2 fermion of mass m with spin $\uparrow$

ψ^2 describes an S=1/2 fermion of mass m with spin $\downarrow$

ψ^3 describes an S=1/2 antifermion of mass m with spin $\uparrow$

ψ^4 describes an S=1/2 antifermion of mass m with spin $\downarrow$

Fermions have exponents $-imt$, antifermions have $+imt$

Negative energy solutions $E = -m$ are either:

Fermions travelling backwards in time

Antifermions travelling forwards in time

Reminder that vacuum energy can create fermion/antifermion pairs

Spinors of moving particles

Fermions:

$$u^1 = \begin{pmatrix} 1 \\ 0 \\ p_z/(E+m) \\ (p_x+ip_y)/(E+m) \end{pmatrix} \qquad u^2 = \begin{pmatrix} 0 \\ 1 \\ (p_x-ip_y)/(E+m) \\ -p_z/(E+m) \end{pmatrix}$$

Antifermions:

$$v^2 = \begin{pmatrix} p_z/(E+m) \\ (p_x+ip_y)/(E+m) \\ 1 \\ 0 \end{pmatrix} \qquad v^1 = \begin{pmatrix} (p_x-ip_y)/(E+m) \\ -p_z/(E+m) \\ 0 \\ 1 \end{pmatrix}$$

Note we have changed from $u^3(p) \to v^2(-p)$ and $u^4(p) \to v^1(-p)$

Wavefunctions of electron and positron

Electron with energy E and momentum $\vec{p}$

$$\psi = u^1(p)e^{-ip\cdot x} \qquad \uparrow$$

$$\psi = u^2(p)e^{-ip\cdot x} \qquad \downarrow$$

Positron with energy E and momentum $\vec{p}$

$$\psi = v^1(p)e^{ip\cdot x} = u^4(-p)e^{-i(-p)\cdot x} \qquad \uparrow$$

$$\psi = v^2(p)e^{ip\cdot x} = u^3(-p)e^{-i(-p)\cdot x} \qquad \downarrow$$

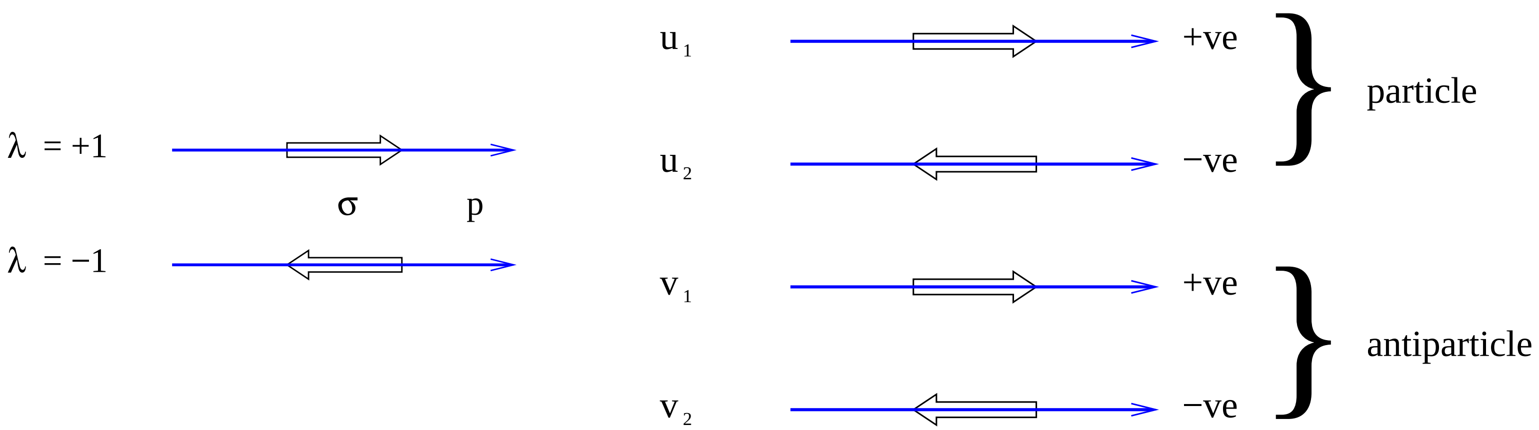
Note the reversal of the sign of p in both parts of the antifermion wavefunction and the change from u to v spinors

Helicity States

Choose axis of projection of spin along direction of motion z

Spinors $u^{1,2}$ describe electron states with spin parallel or antiparallel to momentum p_z.

Spinors $v^{1,2}$ describe positron states with spin parallel or antiparallel to momentum p_z.



Left and Right-handedness

The operator $(1-\gamma_5)$ projects out left-handed helicity $\mathcal{H}=-1$

$$\mathcal{H}=\frac{\vec{\sigma}.\vec{p}}{|\vec{\sigma}||\vec{p}|}$$

The operator $(1+\gamma_5)$ projects out right-handed helicity $\mathcal{H}=+1$

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} \qquad (\gamma^5)^2 = 1 \qquad \{\gamma^5, \gamma^\mu\} = 0$$

Massless fermions with $p=E$ are purely left-handed

Massless antifermions with $p=E$ are purely right-handed