Lecture 5
Quantum Electrodynamics (QED)

The quantum field theory of electromagnetic interactions

- QED rules for Feynman diagrams
- Relativistic electron-muon scattering $e^- \mu^- \rightarrow e^- \mu^-$
- Muon pair production $e^+ e^- \rightarrow \mu^+ \mu^-$
- Moeller ($e^- e^- \rightarrow e^- e^-$) and Bhabha ($e^+ e^- \rightarrow e^+ e^-$) scattering
- Higher order corrections
- Gyromagnetic ratio $g - 2$
- Renormalization & Gauge Invariance

QED rules for Feynman diagrams

- Incoming (outgoing) fermions have spinors u ($\bar{u}$).
- Incoming (outgoing) antifermions have spinors $\bar{v}$ (v).
- Incoming (outgoing) photons have polarisation vectors ϵ^μ ($\epsilon^{\mu *}$).
- Each vertex has a factor $ie\gamma^\mu$ where $e = \sqrt{4\pi\alpha}$
- A virtual photon propagator is $-ig_{\mu\nu}/q^2$.
- A virtual fermion of mass m has a propagator $i(\gamma_\mu p^\mu + m)/(p^2 - m^2)$.
- A minus sign is needed to antisymmetrise diagrams that differ only by the interchange of two identical fermions.

Electron-Muon scattering

Start from scattering of spinless particles (Lecture 3)

Add Dirac spinors, electromagnetic currents and couplings

Matrix element has one first order photon exchange diagram:

$$\mathcal{M} = e^2 (\bar{u}_3 \gamma^\mu u_1) \frac{1}{q^2} (\bar{u}_4 \gamma^\mu u_2) \qquad q^2 = (p_3 - p_1)^2 = t$$

For relativistic fermions scattering does not change helicity (proof on P.126 of Halzen & Martin)

Four spin configurations at high energies

$$\mathcal{M}(\uparrow\downarrow\uparrow\downarrow) \quad \mathcal{M}(\downarrow\uparrow\downarrow\uparrow) \quad \mathcal{M}(\uparrow\uparrow\uparrow\uparrow) \quad \mathcal{M}(\downarrow\downarrow\downarrow\downarrow)$$

Matrix element squared is product of electromagnetic currents

$$|\mathcal{M}|^2 = \frac{e^4}{q^4} L_e L_\mu \qquad L_i = [\bar{u}_i \gamma^\mu u_i][\bar{u}_i \gamma^\mu u_i]^*$$

Electron-Muon cross-section

Take average over initial spins and sum over final spins:

$$\overline{|\mathcal{M}|^2} = \frac{1}{(2s_1+1)(2s_2+1)} \sum_{spins} |\mathcal{M}|^2 = 2e^4 \frac{(s^2+u^2)}{t^2}$$

where s, t, u are Mandelstam variables (see Lecture 3)

$$\overline{|\mathcal{M}|^2} = 2e^4 \frac{[1+(1+\cos\theta)^2]}{(1-\cos\theta)^2} = 2e^4 \frac{[1+4\cos^4\theta/2]}{\sin^4\theta/2}$$

Differential cross-section in CM frame:

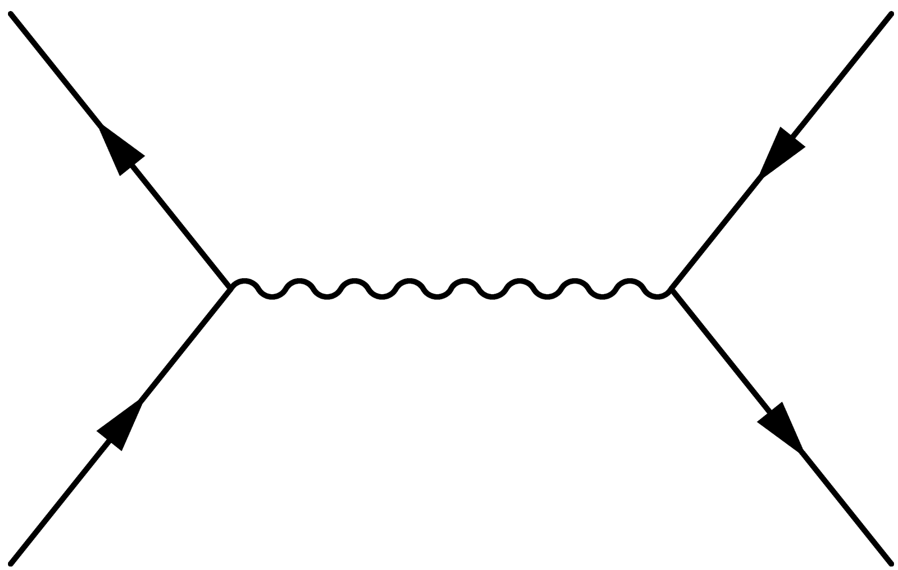
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{8s\sin^4\theta/2}\left(1+4\cos^4\theta/2\right)$$

where $1/s$ comes from flux and phase space factors

In Lab frame the no recoil limit $E_e \ll m_\mu$ gives the non-relativistic Rutherford scattering formula

Muon pair production

$e^+e^- \to \mu^+\mu^-$ occurs via annihilation of electron-positron into virtual photon, and then creation of a muon pair.



Related to $e^-\mu^- \to e^-\mu^-$ by *crossing symmetry* $s \leftrightarrow t$

$$\overline{|\mathcal{M}|^2}(e^+e^- \to \mu^+\mu^-) = 2e^4 \frac{(t^2+u^2)}{s^2}$$

Cross-section for $e^+e^- \to \mu^+\mu^-$

At high energies annihilating fermion/antifermion have opposite helicities. Again there are four possible spin combinations:

$$\mathcal{M}(\uparrow\downarrow\uparrow\downarrow) = \mathcal{M}(\downarrow\uparrow\downarrow\uparrow) = e^2(1+\cos\theta)$$

$$\mathcal{M}(\uparrow\downarrow\downarrow\uparrow) = \mathcal{M}(\downarrow\uparrow\uparrow\downarrow) = e^2(1-\cos\theta)$$

Spin-averaged matrix element squared: $\overline{|\mathcal{M}|^2} = e^4\left(1+\cos^2\theta\right)$

Differential cross-section in centre-of-mass:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right)$$

where $1/s$ comes from flux and phase space factors.

Total cross-section is simply:

$$\sigma = \frac{4\pi\alpha^2}{3s}$$

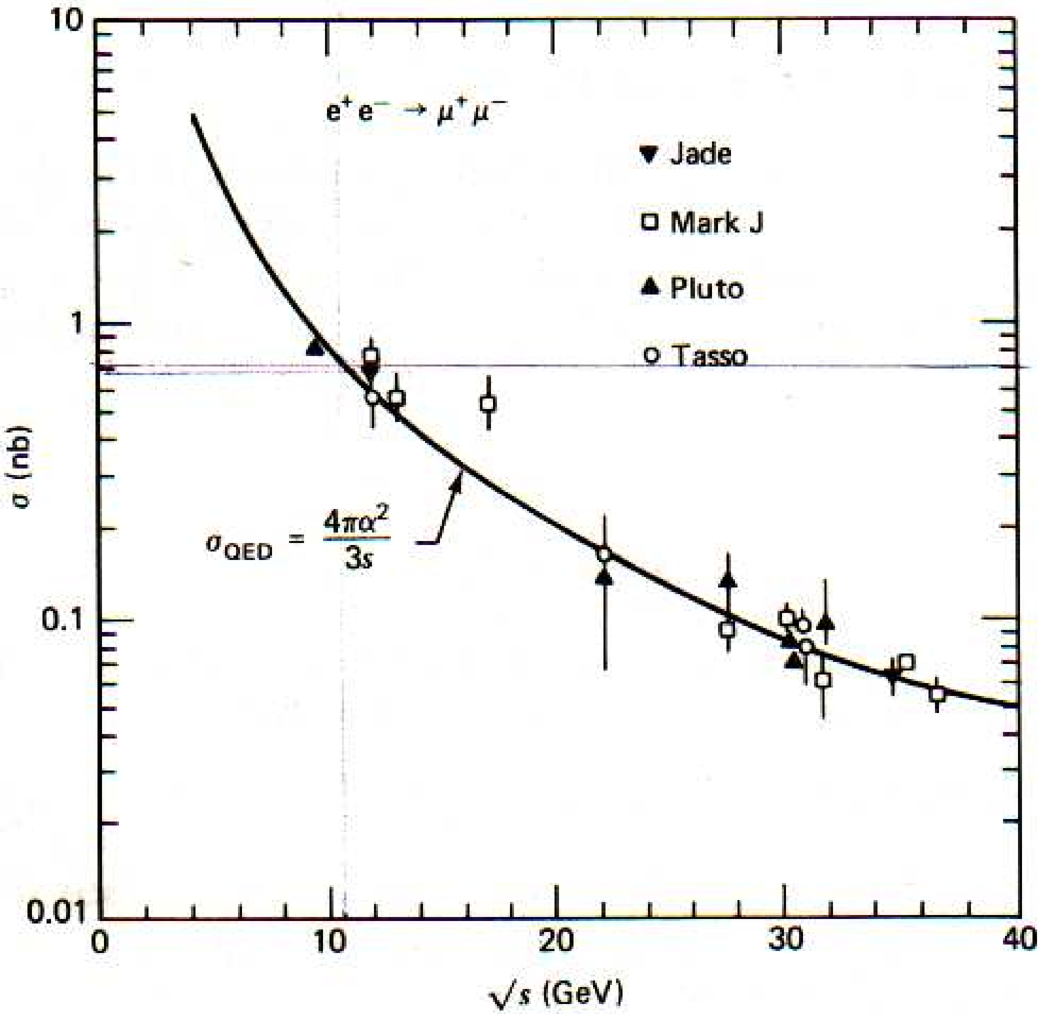
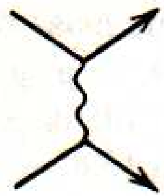
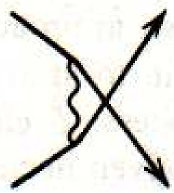
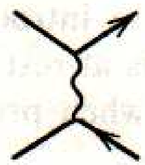
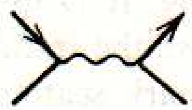
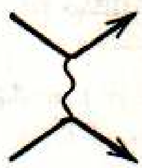
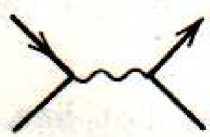


Fig. 6.6 The total cross section for $e^-e^+ \to \mu^-\mu^+$ measured at PETRA versus the center-of-mass energy.

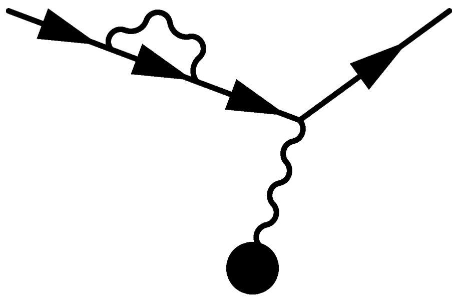
TABLE 6.1
Leading Order Contributions to Representative QED Processes

	Feynman Diagrams		$\overline{|\mathcal{M}|^2}/2e^4$		
	Forward peak	Backward peak	Forward	Interference	Backward
Møller scattering $e^-e^- \to e^-e^-$			$\frac{s^2+u^2}{t^2}$ +	$\frac{2s^2}{tu}$ +	$\frac{s^2+t^2}{u^2}$
				($u \leftrightarrow t$ symmetric)	
(Crossing $s \leftrightarrow u$) ↓	Forward	"Time-like"	Forward	Interference	Time-like
Bhabha scattering $e^-e^+ \to e^-e^+$			$\frac{s^2+u^2}{t^2}$ +	$\frac{2u^2}{ts}$ +	$\frac{u^2+t^2}{s^2}$
$e^-\mu^- \to e^-\mu^-$			$\frac{s^2+u^2}{t^2}$		
(Crossing ↓ $s \leftrightarrow t$)					
$e^-e^+ \to \mu^-\mu^+$					$\frac{u^2+t^2}{s^2}$

Halzen & Martin P.129

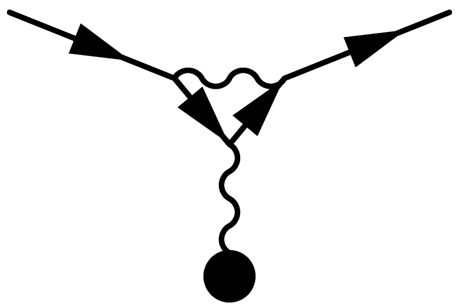
Higher order QED diagrams

“Dressed” fermions



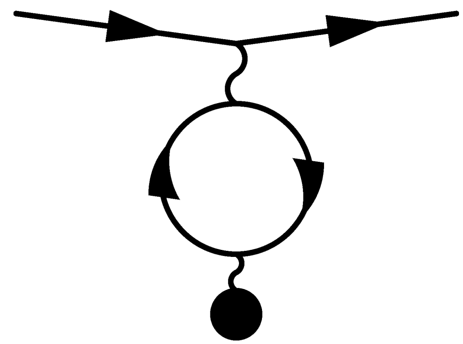
A real (or virtual) fermion emits and reabsorbs a virtual photon.

Vertex corrections



A virtual photon connects fermions across a previous vertex.

“Bubble” propagators



A real (or virtual) photon creates fermion/antifermion pairs.

Each pair of vertices + virtual particle adds a factor $\alpha = 1/137$

Sum of higher order QED corrections converges!

Anomalous Gyromagnetic Ratio $g - 2$

An example where higher order corrections are important

Measures the relationship between spin and magnetic moment

The magnetic moment of an electron is:

$$\vec{\mu} = g\mu_B \vec{S} \qquad \mu_B = \frac{e\hbar}{2m_e c}$$

where $\mu_B = 5.8 \times 10^{-11}\text{MeV/T}$ is the Bohr magneton.

From the Dirac equation the gyromagnetic ratio for pointlike fermions is exactly $g = 2$

Higher order QED diagrams give an “anomalous” value for g slightly different from 2.

Accuracy of QED

Anomalous moment of electron:

$$Experiment: \quad \left[\frac{g-2}{2}\right]_e = 0.0011596521869(41)$$

$$Theory: \quad \left[\frac{g-2}{2}\right]_e = 0.00115965213(3)$$

Anomalous moment of muon:

$$Experiment: \quad \left[\frac{g-2}{2}\right]_\mu = 0.0011659160(6)$$

$$Theory: \quad \left[\frac{g-2}{2}\right]_\mu = 0.0011659203(20)$$

The most precise test of any theory! Error is not from QED!

Comes from bubble diagrams with quark-antiquark pairs (QCD)

Renormalisation

Higher order contributions with large virtual four-momentum transfers give divergent integrals.

Absorbed by redefinitions of spinors, propagators & couplings:

$$e_R = e\left[1 - \frac{\alpha}{3\pi} ln\left(\frac{\Lambda^2}{m^2}\right) + \mathcal{O}(\alpha^2)\right]^{1/2}$$

where Λ is the **cutoff value** of the four-momentum.

This procedure is known as **renormalisation**

For QED it is usually sufficient to ignore terms $\mathcal{O}(\alpha^2)$

The electromagnetic current becomes:

$$J^\mu = e_R(\bar{u}\gamma^\mu u)$$

The running of α

A consequence of renormalisation is that the value of the coupling constant α becomes a function of q^2:

$$\alpha(q^2) = \frac{\alpha(\mu^2)}{1 - \frac{\alpha(\mu^2)}{3\pi} ln\left(\frac{q^2}{\mu^2}\right)}$$

where μ is a reference four-momentum transfer which is used to remove the dependence on the cutoff parameter Λ.

At low energies $\quad \alpha = 1/137$

At $M_Z = 90$ GeV $\quad \alpha = 1/128$

Can be though of as a correction to the "bare" electric charge to account for "screening" by higher order diagrams with virtual photons and fermion/antifermion pairs.

Gauge Invariance of QED

Electromagnetic interactions are invariant under:

- A **phase transformation** of the fermion wavefunctions:

$$\psi \rightarrow e^{ie\theta}\psi \qquad \bar{\psi} \rightarrow e^{-ie\theta}\bar{\psi}$$

- A **gauge transformation** of the vector potential:

$$A_\mu \rightarrow A_\mu + \partial_\mu \theta$$

The gauge invariance of QED is related to the conservation of charge and the masslessness of the photon.

In particle physics this is known as a U(1) group symmetry