

Hamiltonian Dynamics

[PHYS11012]

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Monday 10:00-10:50 lecture (LTC)

Thursday 10:00-10:50 lecture (LTC)

Friday 16:10-18:00 workshop (5327, wk 2 onwards)

[wk 1 16:10-17:00 lecture (LTC)]

TA: Matthew Thomas

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Syllabus

- Review of Lagrangian dynamics
Lagrangian equations of motion, generalised coordinates, constants of motion, symmetries and Noether's theorem
- Hamiltonian dynamics
Legendre transformations, Hamilton's equations, conservative systems, phase space, Liouville's Theorem
- Qualitative dynamics
Classification of the behaviours of 1 and 2 dimensional continuous time autonomous systems via linearisation of the system. Fixed points and limit cycles, simple Predator-Prey systems, Rayleigh's equation, Hamiltonian systems (eg vertical pendulum)
- Canonical Transformations
Variational principles, point transformations and generating functions, symplectic structure, Poisson brackets, infinitesimal canonical transformations, relation to symmetries and constants of motion (eg momentum, angular momentum, energy)
- Hamilton-Jacobi theory
Hamilton-Jacobi equation, Hamilton's principal function, Hamilton's characteristic function, separation of variables, connection with quantum mechanics
- Action-Angle variables
Constants of motion and integrability, libration and rotation, phase diagrams, frequencies and periods of motion, the harmonic oscillator and Kepler problem
- Canonical Perturbation theory
Simple minded time dependent and time independent perturbation theory, adiabatic invariants, the harmonic oscillator and vertical pendulum, the KAM theorem (descriptive)

A highly formal course developing tools/language necessary for the study of mechanical systems with a finite number of degrees of freedom.

Books

- H. Goldstein, C. Poole and J. Safko,
Classical Mechanics, (Addison–Wesley)
- L. N. Hand and J. D. Finch,
Introduction to Dynamics, (CUP)
- T. W. B. Kibble and F. H. Berkshire,
Classical Mechanics, (Imperial College Press)
- I. Percival and D. Richards,
Introduction to Dynamics, (CUP)

Further material:

- Integrability and chaos: the classical uncertainty, J. Masoliver and A. Ros,
arXiv:1012.4384 [nlin.CD].

Generating Function	Derivatives	K	Trivial case
$F_1(\vec{q}, \vec{Q}, t)$	$p_i = \frac{\partial F_1}{\partial q_i} \quad P_i = -\frac{\partial F_1}{\partial Q_i}$	$H + \frac{\partial F_1}{\partial t}$	$F_1 = \vec{q} \cdot \vec{Q} \Rightarrow (\vec{Q}, \vec{P}) = (\vec{p}, -\vec{q})$
$F_2(\vec{q}, \vec{P}, t)$	$p_i = \frac{\partial F_2}{\partial q_i} \quad Q_i = \frac{\partial F_2}{\partial P_i}$	$H + \frac{\partial F_2}{\partial t}$	$F_2 = \vec{q} \cdot \vec{P} \Rightarrow (\vec{Q}, \vec{P}) = (\vec{q}, \vec{p})$
$F_3(\vec{p}, \vec{Q}, t)$	$q_i = -\frac{\partial F_3}{\partial p_i} \quad P_i = -\frac{\partial F_3}{\partial Q_i}$	$H + \frac{\partial F_3}{\partial t}$	$F_3 = \vec{p} \cdot \vec{Q} \Rightarrow (\vec{Q}, \vec{P}) = (-\vec{q}, -\vec{p})$
$F_4(\vec{p}, \vec{P}, t)$	$q_i = -\frac{\partial F_4}{\partial p_i} \quad Q_i = \frac{\partial F_4}{\partial P_i}$	$H + \frac{\partial F_4}{\partial t}$	$F_4 = \vec{p} \cdot \vec{P} \Rightarrow (\vec{Q}, \vec{P}) = (\vec{p}, -\vec{q})$

Table 1: Properties of the four basic Canonical Transformations $(\vec{q}, \vec{p}) \rightarrow (\vec{Q}, \vec{P})$.

Conversion factors from SI units to Heaviside-Lorentz units

$$\vec{E} = \sqrt{\epsilon_0} \vec{E}_{SI}, \quad \vec{B} = c\sqrt{\epsilon_0} \vec{B}_{SI}, \quad \rho = \frac{1}{\sqrt{\epsilon_0}} \rho_{SI}, \quad \vec{j} = \frac{1}{\sqrt{\epsilon_0}} \vec{j}_{SI}, \quad [e = \frac{1}{\sqrt{\epsilon_0}} e_{SI}],$$

($c = 1/\sqrt{\epsilon_0 \mu_0}$) giving Maxwell's equations as

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \rho \\ \vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} &= 0 \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{B} &= \frac{1}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \end{aligned}$$

and

$$\vec{F} = e \left[\vec{E} + \frac{\vec{r}}{c} \times \vec{B} \right].$$

Also with

$$\phi = \sqrt{\epsilon_0} \phi_{SI}, \quad \vec{A} = c\sqrt{\epsilon_0} \vec{A}_{SI}$$

then

$$\vec{E} = -\vec{\nabla} \phi - \frac{1}{c} \dot{\vec{A}}, \quad \vec{B} = \vec{\nabla} \times \vec{A},$$

and

$$V = e \left(\phi - \frac{\vec{r}}{c} \cdot \dot{\vec{A}} \right).$$

Preliminary