Symmetries of Quantum Mechanics

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Contents

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Chapter 1

Preliminaries

1.1 Introduction

The theory of finite groups has historically arisen as an effort (by Galois) to classify different solutions of algebraic equations. The theory of Lie Groups was founded by Lie in an attempt to classify solutions of differential equations. Lie groups turned out to be continuous groups with a natural manifold structure. Both types will be discussed in this course.

Generally group theory is a standard algebraic structure which applies in many fields of mathematics and applied sciences. In physics, symmetries are naturally described by groups. The power of symmetries relies on the fact that they, partially, solve the dynamics; the symmetry restricts the solutions. This happens at the more sophisticated level of the celebrated Wigner-Eckart theorem (to be discussed in these lectures) as well as in simple integrals where the symmetries of the integrand restrict the form of the solutions.

An example is the following integral

$$I_{ij} = \int d^3k f(k \cdot p, k^2, p^2) k_i k_j = A(p^2)\delta_{ij} + B(p^2) p_i p_j \; , \tag{1.1}$$

where k_i, p_i are vectors in three dimensional space and δ_{ij} is the Kronecker symbol. The integral of a type I_{ij} is an integral that describes quantum correction, be it in quantum field theory or statistical mechanics. No matter how complicated the function f is, the solution has the form on the right hand side since the integrand is covariant under the three dimensional euclidian rotation group. If the symmetry is not assumed then one ought to write $3 \cdot 3 = 9$ coefficients instead of the two coefficients A and B above.

At the level of equations of motion, or more precisely the Lagrangian, symmetries are related to conservation laws by virtue of the famous Noether theorem. Most famously: time symmetry $\Leftrightarrow$ energy conservation, space translation symmetry $\Leftrightarrow$ momentum conservation, rotational symmetry $\Leftrightarrow$ spatial angular momentum conservation. The result can often be turned around in the sense that an unexpected symmetry in the result suggests that a symmetry in the Lagrangian has been overlooked.

Groups are abstract objects in principle. When realised on vector spaces they are referred to as representations. As quantum theory is described by Hilbert spaces which are specific, at times infinite dimensional, vector spaces representation theory plays a special rôle.[1] An example, that you should have seen in a course on quantum mechanics, is the Hydrogen atom for which the potential $V \sim 1/r$ has spherical symmetry (symmetry under the rotation group). The energy solutions are given by $E_n \sim 1/n^2$ which are in particular independent of the so-called third quantum numbers m (z-projection of the angular momentum) which singles out a direction in the physical space. Once an external magnetic field (an example of hyperfine splitting) is applied to the hydrogen atom, a direction is singled out. In this case, as expected, the energy depends on the magnetic quantum number m.

The course consists essentially of four parts.[2] An introduction into abstract group theory with focus on finite groups in chapter 2, generic remarks on representation theory 3, representation theory of finite groups and of continuous groups in chapters 4 and 5 respectively followed by applications to quantum mechanics in chapter 6. On the subject of continuous groups special focus is given on $U(1)$ (the symmetry group of quantum electrodynamics which is associated with charge conservation), $SO(3)$ (the rotation group) and $SU(2)$ (the rotation group of half integer spin objects e.g. the electron if spin 1/2).

Proofs are usually given or at least sketched. The proofs aim to give you insight about the theorems. More elaborate proofs which give only limited insight are not given but can looked up in textbooks on group theory. Throughout the text there are plenty of footnotes. Almost all of them could be omitted but they, hopefully, provide the reader with more insight on alternative views or a broader scope on the topic. In my personal experience this leads to higher appreciation of the topic in general. Comments on possible applications in physics, of the underlying mathematical topic, are given in italics. Please do not hesitate to give me comments on these notes either in person or via e-mail. Your effort is much appreciated, also by future students attending this course.

1.2 Basic mathematical notions

The aim of this section is to introduce the basic notions of mathematics used throughout the text.

1.2.1 Notation

Basic notation used throughout the course:

- The integer numbers $\mathbb{Z} \equiv \{.., -2, -1, 0, 1, 2, ..\}$
- The rational numbers $\mathbb{Q} \equiv \{p/q \mid p, q \in \mathbb{Z}\}$
- The unit circle in $n+1$ dimensions is denoted by $S^n = \{\vec{x} \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + .. + x_{n+1}^2 = 1\}$.
- $M_n(F)$ corresponds to the $n \times n$-matrices over the set F which is most often $\mathbb{R}$ or $\mathbb{C}$ in this course. $\mathbb{I}_n$ denotes the unit $n \times n$ matrix.
- $GL(n, \mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \det(A) \neq 0\}$ sometimes also denoted by $GL(V)$ where $\dim(V) = n$ is an n-dimensional vector space.

[1] Also for mathematics according to this quote: "all of mathematics is a form a representation theory " (Israel Gelfand (1913-2009)).

[2] The section on basic mathematical notions 1.2 will only be briefly discussed and mainly serves the purpose of having a common language. The basic revision of linear algebra will be discussed before starting with the chapter on representation theory 3 where it is really needed. Abstract group theory as presented in chapter 2 is independent of linear algebra.

- *Einstein summation convention:* when two identical indices (on the same side of the equation) are left "open", then they are meant to be summed over unless otherwise stated. For example $x_i^* y_i \to \sum_i x_i^* y_i$.

1.2.2 Definitions

- ***injective***-, ***surjective***- and ***bijective***-maps. Illustrated in Fig. 1.1.
  - A map $\varphi : A \to B$ is surjective if $\forall b \in B, \exists a \in A$ s.t. $\varphi(a) = b$.
  - A map $\varphi : A \to B$ is injective if $\forall a_1, a_2 \in A : \varphi(a_1) = \varphi(a_2) \Rightarrow a_1 = a_2$.
  - A map $\varphi : A \to B$ is bijective if and only if it is injective and surjective. Bijective is also known as one-to-one or invertible.

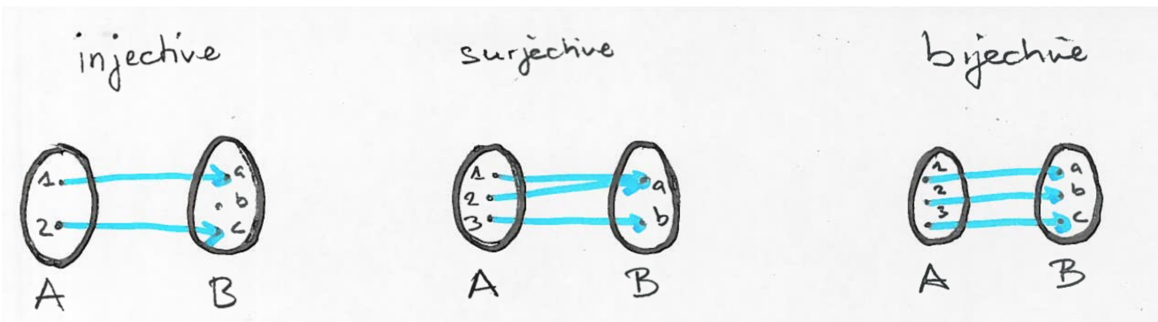

Figure 1.1: Injective (left), surjective (middle) and bijective (right) maps as explained in the text.

- The ***kernel*** of a map $\varphi : A \to B$ is the subset of A which maps to the trivial element of B. (which could be 0 for a vector space or the identity for a group). The kernel measures the degree to which φ is injective.
- The ***image*** of a map $\varphi : A \to B$ is $\varphi(A) \subseteq B$.
- The ***empty set*** is denoted by $\emptyset$.
- The ***Kronecker symbol*** δ_{ij} is widely used throughout the mathematical literature:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{1.2}$$

  In some sense it is a unit matrix.
- The ***Levi-Civita tensor*** in n-indices is the completely antisymmetric tensor which is fully defined by

$$\epsilon_{123..n} = 1 \ . \tag{1.3}$$

  Antisymmetry implies for instance a minus sign, $\epsilon_{213..n} = -1$, when the first two indices are permuted.

- An ***equivalence relation***: is a relation between two elements, say a and b denoted by $a \sim b$ which satisfies the following three properties

  1. $a \sim a$ (reflexivity)
  2. $a \sim b$ implies $b \sim a$ (symmetry)
  3. $a \sim b$ and $b \sim c$ implies $a \sim c$ (transitivity)

  The ***equivalence class*** of an element, say a, is the set of all elements which are equivalent to a. It is denoted by $[a] = \{x \mid a \sim x\}$. Any element of $[a]$ is said to be a representative of the equivalence class. An example for an equivalence relation is: two vectors being parallel.

1.3 Mini-revision of linear algebra

In this section we revise some notions from linear algebra which are particularly useful for the course.

- ***Vector space*** is a collection of elements (vectors) which add and are multiplied with objects of a field. A field is an algebraic body with certain properties. For the purpose of the course it is sufficient to think of the field either as the real numbers or the complex numbers. Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be such vectors (think of them as vectors in $\mathbb{C}^n$ for example) and k and k' complex numbers then the following properties define a vector space:

$$\begin{array}{ll} (1)\ \ (\vec{u}+v)+\vec{w} = \vec{u}+(\vec{v}+\vec{w})\ , & (5)\ \ k(\vec{u}+\vec{v}) = k\vec{u}+k\vec{v}\ , \\ (2)\ \ (\vec{u}+0) = \vec{u}\ , & (6)\ \ (k+k')\vec{u} = k\vec{u}+k'\vec{u}\ , \\ (3)\ \ \vec{u}+(-\vec{u}) = 0\ , & (7)\ \ (kk')\vec{u} = k(k'\vec{u})\ , \\ (4)\ \ \vec{u}+\vec{v} = \vec{v}+\vec{u}\ , & (8)\ \ 1\vec{u} = \vec{u}\ . \end{array} \tag{1.4}$$

  In particular the collection of vectors has a neutral element 0 (axiom (2)) and the field has a unit operators 1 (axiom (8)).

- ***Hilbert space*** is a vector space (see above) with a scalar or inner product, denoted here by $(\, ,\,)$, from $V \times V \to \mathbb{C}$ which satisfies the following axioms:

$$\begin{array}{ll} (1)\ \ (\vec{u}, k\vec{v}+k'\vec{w}) = k(\vec{u},\vec{v})+k'(\vec{u},\vec{w})\ , & \text{linearity}\ , \\ (2)\ \ (\vec{u},\vec{v}) = (\vec{v},\vec{u})^*\ , & \text{conjugation}\ , \\ (3)\ \ (\vec{u},\vec{u}) \geq 0\ , & \text{positivity}\ , \end{array} \tag{1.5}$$

  where the equality (3) holds $\Leftrightarrow \vec{u} = 0$. As an example we mention the vector space $\mathbb{C}^n$ over $\mathbb{C}$ with scalar product: $(\vec{u},\vec{v}) = \sum_{i=1}^n u_i^* v_i$.

  We ought to mention the ***bra and ket notation*** (introduced by P.A.M. Dirac and widely used in physics) which amounts to the notational identification,

$$u \leftrightarrow |u\rangle\ , \quad (u,v) \leftrightarrow \langle u|v\rangle\ , \tag{1.6}$$

  which we shall use at times when illustrating examples.

- A ***linear operator*** is an operator which is compatible with linearity:

$$A(k|u\rangle + k'|v\rangle) = kA|u\rangle + k'A|v\rangle \ . \tag{1.7}$$

In a finite dimensional vector space all linear operators can be represented by matrices. *This will be of importance when discussing finite representation theory.*

In order to illustrate the remaining notations we will consider a finite dimensional *complex vector space*, denoted by $V = \mathbb{C}^N$ where $N = \dim V$. Then there are N linearly independent vectors $\{|\epsilon_1\rangle, |\epsilon_2\rangle, ..., |\epsilon_N\rangle\}$[3] which form a basis in the sense that each $|v\rangle \in V$ can be written as

$$|v\rangle = \sum_{i=1}^{N} v_i |\epsilon_i\rangle \ , \quad v_i \in \mathbb{C} \ . \tag{1.8}$$

The *inner product* $(,) : V \times V \to \mathbb{C}$ is written in the Dirac notation as follows: $\langle v|w\rangle \equiv (|v\rangle, |w\rangle)$. The combination of a vector space and an inner product are also known as a Hilbert space.[4] A basis $\{|e_1\rangle, |e_2\rangle, ..., |e_N\rangle\}$ orthonormal to the inner product can be chosen:

$$\langle e_i|e_j\rangle = \delta_{ij} \ . \tag{1.9}$$

The scalar product of two vectors $|v\rangle = \sum_{i=1}^{N} v_i|e_i\rangle$ and $|w\rangle = \sum_{i=1}^{N} w_i|e_i\rangle$ is given by:

$$\langle v|w\rangle = \sum_{i=1}^{N} v_i^* w_i \ . \tag{1.10}$$

An orthonormal set provides a decomposition of the identity

$$\mathbf{1}_N = \sum_{i=1}^{N} |e_i\rangle\langle e_i| \ . \tag{1.11}$$

A *partition of the identity* is a collection of projectors $P_j \equiv \sum_{i=1}^{N_j} |e_i\rangle\langle e_i|$ satisfying:

$$P_i P_j = \delta_{ij} P_i \ , \quad P_j^\dagger = P_j \ , \quad \mathbf{1}_N = \sum_{j=1}^{N_P} P_j \ . \tag{1.12}$$

Note: $N = \sum_{j=1}^{N_P} N_j$. A *linear operator* $A : V \to V$ $(A(a|v\rangle + b|w\rangle) = a(A|v\rangle) + b(B|w\rangle)$ for $a, b \in \mathbb{C})$[5] can be represented as a *matrix*[6] and its *matrix elements* are given by

$$A_{ij} = \langle e_i|Ae_j\rangle \equiv \langle e_i|A|e_j\rangle \ , \tag{1.13}$$

[3]We shall use Dirac's bra and ket notation throughout this section.

[4]Subtleties arise in the case of infinite dimensional vector spaces which are beyond the scope of this course.

[5]An operator is anti-linear if $A(a|v\rangle + b|w\rangle) = a^*(A|v\rangle) + b^*(B|w\rangle)$. The operator that changes $t \to -t$, denoted by T (with t being the time), is an example of an anti-linear operator.

[6]In this section matrix stands for a square matrix.

where the notation on the right hand side is the one used in quantum mechanics (it is understood that A acts on the right i.e. $|e_j\rangle$). Hence a matrix A may be written as:

$$A = \sum_{i,j=1}^{N} A_{ij} |e_i\rangle\langle e_j| \; . \tag{1.14}$$

A fundamental fact is that the *eigenvalue equation*:

$$A|v_i\rangle = \lambda_i |v_i\rangle \; , \tag{1.15}$$

admits N solutions since the characteristic polynomial $P(t) = \det(A - t\mathbf{1}) = 0$ admits N solutions by virtue of the fundamental theorem of algebra. The λ_i are said to be *eigenvalues* and the $|v_i\rangle$ are the corresponding *eigenvectors*.

1.3.1 Some properties of matrices

- The ***transpose*** of a matrix is $(A^T)_{ij} = A_{ji}$. A matrix is *(anti)symmetric* $\Leftrightarrow A^T = \pm A$ (with symmetric for plus sign).
- The ***hermitian conjugate*** of a matrix is $(A^\dagger)_{ij} = A^*_{ji}$. A matrix is *hermitian* $\Leftrightarrow A^\dagger = A$.
- The ***inverse*** of a matrix A^{-1} is a matrix that satisfies: $A^{-1}A = AA^{-1} = \mathbf{1}_N$.
- A ***unitary*** matrix is a matrix whose hermitian conjugate is its inverse $A^\dagger = A^{-1} \Leftrightarrow AA^\dagger = A^\dagger A = \mathbf{1}_N$.
- The ***trace*** of matrix is $\mathrm{Tr}[A] = \sum \langle e_i|A|e_i\rangle$ where $|e_i\rangle$ is an orthonormal basis.
- The ***determinant*** of a matrix is $\det A = \epsilon_{i_1 \dots . i_N} A_{1 i_1} \dots . A_{N i_N}$ where summation over the indices i_x is implied and $\epsilon_{123..N} = 1$ is the completely antisymmetric Levi-Civita tensor. The definition, which is not as simple as other definitions in this list, is given for completeness only. You should be familiar with its basic properties given in section 1.3.1.
- A matrix is ***diagonal*** $\Leftrightarrow A_{ij} = 0$ for $i \neq j$.
- A matrix A is ***block diagonal*** if it can be written as:

  $$A = \begin{pmatrix} A_1 & 0_{m\times n} \\ 0_{n\times m} & A_2 \end{pmatrix} \tag{1.16}$$

  where A_1 and A_2 are $n \times n$ and $m \times m$ matrices. The symbol $0_{i\times j}$ stands for a $i \times j$ matrix with entries equal to zero.
- The ***direct sum*** of two vector spaces $V \oplus W$ has the following meaning: To each $|v_i\rangle \in V$ and $|w_a\rangle \in W$ we associate a state $|v_i\rangle \oplus |w_a\rangle \equiv |v_i \oplus w_a\rangle$ (the last equality is non-standard notation) and the inner product is extended to:

  $$\langle v_i \oplus w_a | v_j \oplus w_b\rangle_{V\oplus W} = \langle v_i|v_j\rangle_V + \langle w_a|w_b\rangle_W \tag{1.17}$$

The dimension is $\dim(V \oplus W) = v + w$ ($v \equiv \dim(V)$ and $w \equiv \dim(W)$) Given $A \in GL(V)$ and $B \in GL(W)$ then

$$(A \oplus B)|v_i\rangle \oplus |w_a\rangle = (A|v_i\rangle) \oplus (B|w_a\rangle) \; . \tag{1.18}$$

Hence the direct sum may be thought of as a $v + w$-dimensional vector space where operators are represented as $(v + w) \times (v + w)$ matrices in block diagonal form:

$$A \oplus B = \begin{pmatrix} A & 0_{v\times w} \\ 0_{w\times v} & B \end{pmatrix} \tag{1.19}$$

The inverse problem, can a vector space be written as a direct sum, plays an important rôle in discussing the irreducibility of a representation.

- The ***direct product*** of two vector spaces $V \otimes W$ has the following meaning: To each $|v_i\rangle \in V$ and $|w_a\rangle \in W$ we associate a state a state $|v_i\rangle \otimes |w_a\rangle \equiv |v_i \otimes w_a\rangle$ (the last equality is non-standard notation) and the inner product is extended to:

$$\langle v_i \otimes w_a | v_j \otimes w_b\rangle_{V\otimes W} = \langle v_i|v_j\rangle_V \cdot \langle w_a|w_b\rangle_W \tag{1.20}$$

The dimension is $\dim(V \otimes W) = v \cdot w$. The direct product of the two operators from the proceeding item acts on the space as follows:

$$(A \otimes B)|v_i\rangle \otimes |w_a\rangle = A|v_i\rangle \otimes B|w_a\rangle \; . \tag{1.21}$$

An example in quantum mechanics is the spatial wave function of the electron $\Psi(x)$ times ($\otimes$) its spin 1/2 part. Direct products play a rôle in representation theory in terms of, what we shall call, Kronecker products. They are widely used in physics. For example by taking Kronecker products (tensor products of representations to be defined throughout the course) the so-called fundamental representations all other representations of the group can be obtained.

Properties associated with definitions above:

1. For a hermitian matrix the eigenvalues are real ($\lambda_i \in R$) and the eigenvectors can be chosen to be orthonormal. In this basis the matrix A is diagonal. For a two 2×2 hermitian matrix in the orthonormal eigenbasis reads:

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \equiv \mathrm{diag}(\lambda_1, \lambda_2) \; , \tag{1.22}$$

with obvious generalisation to $n \times n$-matrices.

It is straightforward to construct the explicit transformation matrix. Suppose A is a hermitian and a non-diagonal matrix and λ_i and $|v_i\rangle$ are its orthonormal eigenvectors, then

$$A = \sum_{l=1}^{N} \lambda_l |v_l\rangle\langle v_l| \tag{1.23}$$

is diagonalised by the following unitary basis transformation[7]

$$\mathrm{diag}(\lambda_1, ..., \lambda_N) = V^\dagger A V \ , \quad V = \sum_{l=1}^{N} |v_l\rangle\langle e_l| \ . \tag{1.24}$$

2. The inverse of a matrix A exists $\Leftrightarrow$ A has no zero eigenvalues. For a hermitian matrix, the inverse in the orthonormal eigenbasis reads:

$$A^{-1} = \begin{pmatrix} \lambda_1^{-1} & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix} \ . \tag{1.25}$$

3. The trace of a matrix is the sum of its eigenvalues $\mathrm{Tr}[A] = \sum_{i=1}^{N} \lambda_i$.

4. The trace is cyclic $\mathrm{Tr}[AB] = \mathrm{Tr}[BA]$.[8]

5. The determinant of a matrix is the product of its eigenvalues $\det(A) = \prod_{i=1}^{N} \lambda_i$.

6. The determinant of a product of matrices is the product of its determinants.

$$\det(AB) = \det(A) \cdot \det(B) (= \det(BA)) \ . \tag{1.26}$$

7. The following relation holds:

$$\langle v|Aw\rangle = \langle A^\dagger v|w\rangle \tag{1.27}$$

This implies that for a unitary matrix U:

$$\langle Uv|Uw\rangle = \langle v|w\rangle \ . \tag{1.28}$$

Hence unitary transformations are the natural objects to implement symmetry transformations on Hilbert spaces since they preserve the inner product (probability measure).

Exercise 1.3.1 ***Basic linear algebra:***

a) Make sure you understand all the definitions in the list given in this section and that the properties 1) to 7) are familiar to you.

b) Show that $SL(V) = \{A \in GL(V) \mid \det(A) = 1\}$ is a group. The letter "S" stands for special since $SL(V)$ is a special linear group.

c) Verify that the determinant is a group homomorphism: $\det : GL(V) \to \mathbb{C}^*$.

[7] In the case where A is not hermitian it can happen that the eigenvectors, because of linear dependence, cannot be brought into a orthonormal form. In such cases a so-called Jordan normal form can be obtained with eigenvalues on the diagonal and entries equal to one for $A_{(i+1)(i-1)}$-elements on subspaces of degenerate eigenvalues. For more details consider textbooks as this topic is beyond the scope of this course.

[8] The trace is also linear and $\langle A|B\rangle = \mathrm{Tr}[A^\dagger B]$ is an inner product on the vector space of matrices.

d) Show that

$$\langle v|Aw\rangle^* = \langle w|A^\dagger v\rangle \ . \tag{1.29}$$

e) As additional exercises you might want to show that:

- $\det(A^{-1}) = \det(A)^{-1}$, $\det(aA) = a^N \det(A)$ where $a \in \mathbb{C}$.
- $\mathrm{Tr}[A^T] = \mathrm{Tr}[A]$, $\mathrm{Tr}[A^n] = \sum_{i=1}^N \lambda_i^n$.
- The trace and the determinant are invariants under similarity transformations.
- Properties of hermitian conjugation $(aA)^\dagger = a^* A^\dagger$ for $a \in \mathbb{C}$, $(A+B)^\dagger = A^\dagger + B^\dagger$, $(AB)^\dagger = B^\dagger A^\dagger$, $(A^\dagger)^\dagger = A$, $(a|\alpha\rangle\langle\beta|)^\dagger = a^*|\beta\rangle\langle\alpha|$.

Chapter 2

Group theory

This chapter aims to familiarise the student with the basic notions of group theory. This part resembles an undergraduate course on group theory in algebra except that the proofs are not as detailed and that comments on applications in physics are inserted here and there. Most proofs are straightforward from the viewpoint of the mathematician. The exceptions are Cauchy's theorem and the theorem on the alternating groups. In those cases previous results (isomorphism and orbit-stabiliser theorem) give rise to more elegant proofs.

2.1 Basics of group theory

Let us start from the formal definition of a *group* .

Definition 2.1.1 ***Group:***
Let G be a set on which a binary operation $G \times G \to G$ is defined. G is called a group if and only if

$(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$	$\forall g_1, g_2, g_3 \in G$,	product rule is *associative*
$e \cdot g = g \cdot e = g$,	$\exists e \in G : \forall g \in G$,	there is an *identity* element in G.
$g \cdot g^{-1} = g^{-1} \cdot g = e$,	$\forall g \in G, \exists g^{-1} \in G$,	every element of G has an *inverse* in G.

Exercise 2.1.1 To be discussed in the workshop

a) Show that $(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$.

Examples 2.1.1 ***(basic ones)***

- The integers $\mathbb{Z}$ form a group with respect to the binary operation of addition, denoted by $(\mathbb{Z}, +)$. The unit element is given by 0. The inverse of n is $-n$.

- The integers $\mathbb{Z}$ are *not* a group with respect to the binary operation of multiplication. First the 0 element $0 \cdot \mathbb{Z} = 0$ does not comply with the group axioms. Taking $\mathbb{Z}^* \equiv \mathbb{Z} \backslash \{0\}$ is not a group either since the inverse of say 2, which is $1/2$, is not contained in the set.
- The rational numbers $\mathbb{Q}^*$ (with notation understood as above), the real numbers $\mathbb{R}^*$ or the complex numbers $\mathbb{C}^*$ form a group with respect to the binary operation of multiplication, which we denote by $(\mathbb{Q}^*, \cdot)$ and so on.
- The group of permutation of n elements, denoted by S_n, forms a group.[1] The permutation groups will be discussed in a separate section 2.3.
- The group $\mathbb{Z}_n$ (cyclic group of order n) with n elements $e^{2\pi i(m/n)}$ ($0 \leq m < n$ integer) forms a group under multiplication.
- The rotation group of three dimensional space $O(3) = \{O \in M_3(\mathbb{R}) \mid O^T O = \mathbb{I}_3\}$. More precisely this corresponds to the so-called fundamental representation of $O(3)$. This viewpoint will be discussed at length in chapter 3 on representation theory.

The examples are concrete examples of structures that you might already have known. Below we give a few basic definitions that help to characterise a group. This can be helpful for example as specific statements, e.g. theorems, might only be valid for certain classes of groups.

Definition 2.1.2 ***Basic defintions:***

- A group is called ***finite***, ***discrete*** or ***continuous*** depending on whether the set G is finite, discrete (isomorphic to $\mathbb{Z}$) or uncountable.
- The ***order of a group*** G (denoted by $|G|$) is the number of elements of the group. Thus the order of a group is a sensible concept for a finite group.
- The ***order of a group element*** $g \in G$ is the smallest integer n for which $g^n = e$.
- A group G is called ***abelian*** $\Leftrightarrow$ $\forall g_1, g_2 \in G: \; g_1 \cdot g_2 = g_2 \cdot g_1$.[2] The elements g_1 and g_2 are said to *commute* with each other.
- A group H is called ***subgroup*** of G (denoted by $H \subset G$) $\Leftrightarrow H$ is a group and its elements are a subset of the elements of G. A proper subgroup is a subgroup which is not the group itself nor the group with one element (often called the trivial subgroup).
- Two elements $g_1, g_2 \in G$ are ***conjugate*** to each other $\Leftrightarrow$ $\exists g \in G$ s.t. $g_1 = g \cdot g_2 \cdot g^{-1}$.
- The ***generators*** of a group are a set of elements from which all other elements can be generated through the group multiplication law. A set of generators is not unique in general. In particular the number of elements of a generating set might differ. The minimal number of elements that generate a certain group is called the ***rank*** of a group.[3] It is a fact that all groups which are of rank 1 are isomorphic to the cyclic group $\mathbb{Z}_n$ mentioned above.

[1] You may think of rearranging n objects with different colours which are placed on one line. The group concept seems very natural for investigating this object.

[2] In honour of the Norwegian mathematician Niels Henrik Abel (1802-1829).

[3] For Lie groups this notion is extended to the Lie Algebra which is often finite dimensional unlike the group itself.

- The ***center of a group*** is the subset of elements which commute with all elements of the group: $Z(G) = \{z \in G \mid \forall g \in G : z \cdot g = g \cdot z\}$

Exercise 2.1.2 To be discussed in the workshop

a) Which groups in examples 2.1 are finite, discrete or continuous?

b) Give an element in $O(3)$ (c.f. example 2.1 last one) which is of order 2. That is to say write down a 3×3 matrix in $O(3)$, other than the identity, which squares to one.

c) Give one example of a proper and continuous subgroup of $O(3)$. Give an example of a proper subgroup of $\mathbb{Z}_4$.

d) Give all minimal set of generators of $\mathbb{Z}_4$ and $\mathbb{Z}_7$ using the representation $e^{2\pi i(n/4)}$ and $e^{2\pi i(n/7)}$

e) Show that conjugation within a group forms an equivalence relation. This fact will become particularly important when discussing the representation theory of finite groups.

f) Show that for a group where *each* element, except the identity, is of order 2 is abelian.

Theorem 2.1.1 ***subgroup criteria***
A subset H of a group G is a subgroup $\Leftrightarrow$ $g_1 \cdot g_2^{-1} \in H$ for $g_1, g_2 \in H$.

The proof of theorem 2.1.1 does not provide any insight and so we shall skip it.

Finite groups can be represented in terms of so-called ***multiplication tables*** which simply list all possible multiplications of the groups in terms of a rows and columns system.

$$\begin{array}{c|ccc} G & e & a & b\ .. \\ \hline e & e & a & b\ .. \\ a & a & a^2 & a \cdot b\ .. \\ .. & .. & .. & .. \end{array} \tag{2.1}$$

Examples of the cyclic group $\mathbb{Z}_2$ and the permutation group S_3 are given in tables 2.1 and 2.2 respectively. The multiplications are such that column multiplies the row which is relevant in the case where the two elements do not commute. One may therefore envisage the program of writing down all multiplication tables at any finite order and verify that the result satisfies the group axioms. Whereas in principle possible you might guess that this is not the most efficient nor elegant way to characterise groups. In essence this amounts to finding the subset of permutations which is compatible with the group axioms. In addition in each row and column one find each elements exactly once as otherwise the inverse would not be unique. For example in the table if $a^2 = a \cdot b$ then by the uniqueness of the inverse of a we would conclude $a = b$ which is clearly wrong. This simple fact is sometimes referred to as the *rearrangement theorem* throughout the literature. This insight makes the following theorem obvious:

Theorem 2.1.2 ***Cayley's thm:***
Any finite group is isomorphic (to be defined below) to a subgroup of the symmetric group S_n.

$\mathbb{Z}_2$	1	-1
1	1	-1
-1	-1	1

Table 2.1: Cyclic group $\mathbb{Z}_2$ as described in Examples 2.1.

S_3	()	(1,2)	(2,3)	(1,3)	(1,2,3)	(1,3,2)
()	()	(1,2)	(2,3)	(1,3)	(1,2,3)	(1,3,2)
(1,2)	(1,2)	()	(1,2,3)	(1,3,2)	(2,3)	(1,3)
(2,3)	(2,3)	(1,3,2)	()	(1,2,3)	(1,3)	(1,2)
(1,3)	(1,3)	(1,2,3)	(1,3,2)	()	(1,2)	(2,3)
(1,2,3)	(1,2,3)	(1,3)	(1,2)	(2,3)	(1,3,2)	()
(1,3,2)	(1,3,2)	(2,3)	(1,3)	(1,2)	()	(1,2,3)

Table 2.2: The group of three permutation S_3. The element () denotes the identity, $(1, 2)$ permutes or cycles the first two elements and $(1, 2, 3)$ is understood as a cyclic permutation in the three elements. The operation (a, b) is called a transposition

Remark: this theorem is reminiscent of the Whitney embedding theorem that states that any manifold can be embedded into a euclidian space $\mathbf{R}^n$ for suitable n.

Exercise 2.1.3 Exercises on permutation groups and multiplication tables (more: section 2.3)

a) Show that S_3 is a non-abelian group (by using the multiplication table 2.2 for example). It is sufficient to find one example for which $a \cdot b \neq b \cdot a$.

b) What is the order of the group S_n?

c) Show the group of four elements $\{e, g_1, g_2, g_3\}$ (with $|G| = 4$), where all elements, except the element e, are of order two is uniquely determined. *Hint:* Build up the multiplication table. This group is known as the Vierergruppe: $\mathbb{Z}_2 \times \mathbb{Z}_2$ (after Felix Klein).

Definition 2.1.3 ***group morphisms***

- A ***group homomorphism*** is a map between two groups that respects the group structure. More precisely let G_1 and G_2 be two groups then $\phi : G_1 \to G_2$ is a group homomorphism $\Leftrightarrow$

$$\forall a_1, b_1 \in G_1 : \quad \phi(a_1 \cdot b_1) = \phi(a_1) \cdot \phi(b_1) \tag{2.2}$$

  Note that the "·" on the left and right hand side operate on G_1 and G_2 respectively.

- A ***group isomorphism*** is a bijective group homomorphism (one-to-one correspondence or invertible). Notation: $G_1 \simeq G_2$ means that the two groups G_1 and G_2 are isomorphic to each other.

An isomorphism really means that two groups are the same and the classification of groups are up to isomorphisms. We shall see that the symmetry groups of the platonic solids are isomorphic to permutation groups. A major application of these notions is the first isomorphism theorem 2.2.8 which we are going to discuss once the notation of a normal subgroup has been introduced.

Exercise 2.1.4 To be discussed in the workshop

a) Two finite groups which admit an isomorphism, obviously, have the same number of elements. For continuous groups the issue is more subtle as is often the case. Can you give an isomorphism between the integer number $\mathbb{Z}$ and the even integers $2\mathbb{Z}$?

b) Show that for $g \in G$, $g \to g^{-1}$ (inversion) defines an isomorphism $\Leftrightarrow$ G is abelian.

2.1.1 Group presentation

One may define groups from generators and their relation in a completely algebraic way. More precisely consider a set of generators $\{a, b, c, ..\}$, then the group is defined as all words (arrangement of letters) that can be formed modulo certain constraints e.g. $a^2 = e, b^3 = e, abc = e, ...$ This way of defining a group is called the ***group presentation***. The presentation, for a given group, is not unique. For example S_3 can be defined as:

$$S_3 = < a, b, c \mid a^2 = b^2 = c^3 = abc = e > \tag{2.3}$$

or

$$S_3 = < A, B \mid A^2 = B^2 = (AB)^3 = e > \tag{2.4}$$

Presentations, at least for finite groups, will not play a major part in this course. Yet it seems good practice to do the following exercise:

Exercise 2.1.5 To be discussed in the workshop

a) Identify candidates for generators a, b, c and A, B in Eqs. (2.3) and (2.4) (by using the multiplication table 2.2) respectively.

2.2 Notions of group theory

2.2.1 Cosets

Definition 2.2.1 ***(left)-coset:***
Given a subgroup $H \subset G$ one can associate to each $g \in G$ a left-coset $gH = \{g \cdot h \mid h \in H\}$

Remarks and properties:

- one can define in completely analogous manner a ***right-coset***. Below we shall continue to reference left cosets only.

- Only for a certain subclass of groups, called normal (to be defined), one can define a group multiplication on cosets. Hence cosets do in general not admit a group structure and are therefore called cosets rather than "cogroups".

Definition 2.2.2 ***partition***
A partition of a set is a decomposition into (non-empty) subsets which do not intersect. In terms of symbols: $P_i \subset X$ form a partition of X if $X = P_1 \cup P_2 \cup ..$ and $P_i \cap P_j = \emptyset$ for $i \neq j$.

Theorem 2.2.1 (a) The left- gH and also the right-cosets Hg partition G. (b) the partition sets are of equal size: $|g_iH| = |g_jH| = |H|$.

Proof: (a) We need to show that for $g_1 \notin g_2H$, $g_1H \cap g_2H = \emptyset$. Ad absurdum: suppose $g_1H \cap g_2H \neq \emptyset$ then there $\exists h_1, h_2 \in H$ s.t. $g_1 \cdot h_1 = g_2 \cdot h_2$, but then $g_1 = g_2 \cdot h_2 \cdot h_1^{-1} \in g_2H$ which contradicts our original assumption. Therefore g_iH partition G. (b) The map $g_1H \to g_2H$: $g_1 \cdot h \to g_2 \cdot h$ for $h \in H$ is clearly invertible and thus $|g_1H| = |g_2H|$. The assertion for the right cosets is obviously true as all steps in the proof follow in complete analogy. q.e.d.

Thus gH partition G in sets of equal size as illustrated in Fig. 2.1. From this observation follows the

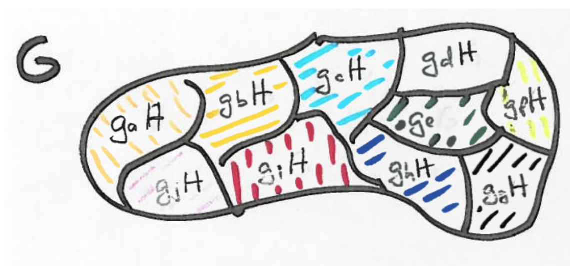

Figure 2.1: The partition of G into its left-cosets g_iH as written symbolically in Eq. (2.5).

famous Lagrange theorem:

Theorem 2.2.2 ***Lagrange's thm:***
If H is a subgroup of G ($H \subset G$) $\Rightarrow |G|/|H|$ is an integer.

According to theorem 2.2.1 the integer corresponds to the number of non-identical set gH which partition G. Summarising the most important content of this section in terms of a formula:

$$G = \cup_{i=1}^{|G|/|H|} g_{n_i} H \ , \tag{2.5}$$

with n_i an appropriate map from integers to integers.[4]

The collection of all g_iH is called the coset space and often denoted by G/H. We will see in section 2.2.3 that this space has a group structure $\Leftrightarrow$ the group H has a property which is called normal. *In*

[4]Note the converse problem of theorem 2.2.2: Is there a subgroup of order p if $|G|/p$ is integer is related to the so-called Sylow theorems and are one of the tools used to classify finite groups. The study thereof is beyond the scope of this course.

particle physics and statistical mechanics cosets naturally arise when a continuous global symmetry G is broken by the vacuum state to a subgroup H. Then one can formulate an effective theory of the low lying spectrum on the coset space.

Exercise 2.2.1 ***Exercises on Cosets and Lagrange's thm.***

a) Why can $\mathbb{Z}_7$ *not* be a subgroup of S_6? (you need a result from exercises 2.1.3).

b) Consider $H = \{e, (2,3)\}$ subgroup of S_3. Find the other partitions by choosing elements which are not in H and then form the left and right cosets.
(Observe in particular that the right and left cosets are not equal to each other. This will be relevant in a forthcoming exercise.)

2.2.2 Group action

Definition 2.2.3 ***Group action:***
A ***group action*** is a map of the group on a set compatible with the group structure.
More precisely, let G be a group and X a set then $\phi : G \times X \to X$, denoted by $\phi(g, x) \to g.x$ for short[5] (with $g \in G$ and $x \in X$), is a group action[6] $\Leftrightarrow$

$$\begin{array}{lll} (1) & e.x = x \;, & \text{identity} \\ (2) & (g_1 \cdot g_2).x = g_1.(g_2.x) & \text{,compatibility } (g_1, g_2 \in G) \;. \end{array}$$

Let us illustrate this concept with a few examples:

Examples 2.2.1 ***(basic ones)***

- Trivial action $g.x = x$.
- Group acts on itself ($X = G$) by:
  - left(right) multiplication $g.x = g \cdot x$ ($g.x = x \cdot g$).
  - conjugation $g.x = g \cdot x \cdot g^{-1}$.
- The *group action on a linear space* is called ***representation*** and plays a distinct rôle in physics. *For example the Hilbert space of quantum mechanics and quantum field theories are linear spaces. In mathematical language particles corresponds to representations of the Lorentz groups and internal symmetry groups.*
- The permutation group S_n acts on the set $\{1, ..., n\}$ by permutation.[7]

[5] You must pay attention to distinguish the group multiplication "$\cdot$" from the group action ".".
[6] An alternative definition (for finite groups): A group action is a group homomorphism of G into $S_{|X|}$.
[7] Or one may work in $\mathbb{R}^n$ and associate permutations of unit vectors $e_1 = (1, 0.., 0) \in \mathbb{R}^n$ with the permutation group. One would be tempted to think that this is a good start for the representation theory of S_n. We will see in chapter 3 that this representation is not the smallest building block (irreducible) of the representations of S_n.

- The group $(\mathbb{Z}, +)$ acts on the set $\mathbb{R}$ as follows: $m \in \mathbb{Z}$ and $r \in \mathbb{R} : m.r \to (-1)^m r$.

In gauge theories such as electrodynamics the gauge symmetry, which is $U(1)$ (its elements are $e^{i\phi(x)}$ and the binary operation is just the ordinary multiplication) for electrodynamics, acts on the gauge potential $A_\mu(x) \to A_\mu(x) + \frac{\partial}{\partial x^\mu}\phi(x)$. All the terminology in this section can be found in the physics literature. Some further definitions are useful for group actions:

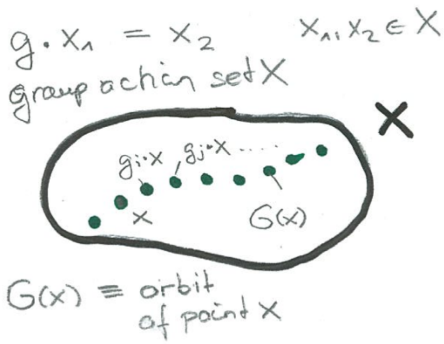

Figure 2.2: Group acting on a point $x \in X$ defines the corresponding orbit denoted by $G(x) \subseteq X$.

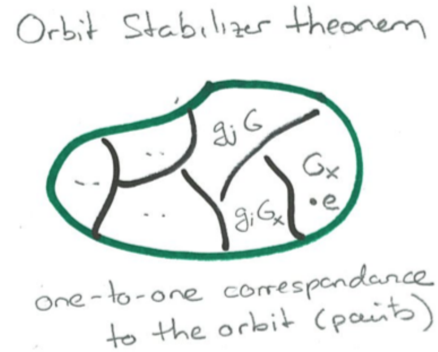

Figure 2.3: The orbit-stabiliser theorem 2.2.3 states that there is a one-to-one correspondence between each point on the orbit (c.f. Fig. 2.2) and the left (right) cosets gG_x $(G_x g)$ of the stabiliser subgroup.

Definition 2.2.4 ***further terminology***
The ***orbit*** of $x \in X$, illustrated in Fig. 2.2, is the set of elements that are reached by the group acting on x: $G(x) = \{g.x \mid g \in G\} \subset X$.[8][9] The ***stabiliser (subgroup)***[10] of $x \in X$ is the subset in G which leaves x invariant under the group action $G_x = \{g \in G \mid g.x = x\}$.

Examples 2.2.2 ***orbit of*** $O(2)$

- Let us consider the action of the two dimensional group $O(2) = \{O \in M_2(\mathbb{R}) \mid O^T O = \mathbb{I}_2\}$ (analogous to $O(3)$ in examples 2.1) on a two dimensional plane around a point a. Then the orbit of a point b in the plane is the circle of radius $|a - b|$ centred around a. The point a is a fixed point of the orbit.

Theorem 2.2.3 ***Orbit-Stabilizer thm:***
The map $g.x \to gG_x$ defines a bijection (one-to-one correspondence).

The theorem states that each point in the orbit corresponds to a coset gG_x of the stabiliser subgroup.

Proof: The map is surjective by definition since gG_x partitions G by virtue of theorem 2.2.1. To show that the map is injective, and therefore bijective, we need to show that $g_1 G_x = g_2 G_x$ implies $g_1.x = g_2.x$. If $g_1 G_x = g_2 G_x$ then $\exists g \in G_x$ s.t. $g_1 = g_2 g$. But then $g_1.x = (g_2 \cdot g).x = g_2.(g.x) = g_2.x$ and therefore the map is also injective. q.e.d.

[8] A group action is ***transitive*** if there is only a single orbit.
[9] Note that the orbit corresponds to an equivalence class under the group action.
[10] The stabiliser subgroup is also known as the isotropy group.

Corollary 2.2.1 If G is a finite group then the size of each orbit is a divisor of the order of the group. I.e. $|G_x| = |G|/|G(x)|$ is an integer.

Proof: By theorem 2.2.3 there is a one-to-one correspondence between the size of the orbit and the number of left (right) cosets i.e. $|G(x)| = |G|/|gG_x|$. Hence $|G|/|G(x)| = |G_x|$ is an integer and $|G(x)|$ a divisor of the order of the group. q.e.d.

As one application we shall consider the proof of the famous Cauchy theorem which reads:

Theorem 2.2.4 ***Cauchy's thm:***
If $|G|/p$ is an integer and p is prime then there exists an element of order p. ($\exists g \in G$ s.t. $g^p = e$.)

Proof: Let $(x_1, x_2, ...x_p) \in X$ (with $x_i \in G$) be the set of ordered strings for which $x_1 \cdot x_2 .. \cdot x_p = e$. Note that the number of such strings is $|G|^{p-1}$ since we may freely choose $p-1$ elements out of G and then fix the last one to be the inverse. In particular $|X|/p$ is an integer since $|G|/p$ is. Next, we define a group action of $\mathbb{Z}_p$ on X; $m \in \mathbb{Z}_p$ acts on X by cyclic translation: $m.(x_1, x_2, .., x_p) = (x_{m+1}, .., x_p, x_1, x_2, .., x_m)$. Since the size of the orbit divides the order of the group which is $|\mathbb{Z}_p| = p$ in the case at hand, the size of each orbit is either 1 (all elements are equal) or p (not all elements are equal). An example of the former is $(e, .., e)$. But it cannot be the only example as otherwise $|X|$ is not divisible by p (that is to say $np + 1$ for n integer is not divisible by p). Hence there exists a configuration for which $x_1 = .. = x_p \neq e$. This implies that $x_1 (\neq e) \in G$ with $x_1^p = e$ and ends the proof. q.e.d.

This is proof is elegant and short and provides an example of the power of the notions introduced previously in this section. Trying to understand is rewarding!

Corollary 2.2.2 A group G of prime order $|G| = p$ (p is prime) is isomorphic to the cyclic group $\mathbb{Z}_p$.

Before presenting the proof let us refine the notion of the cyclic group $\mathbb{Z}_p$ by stating its presentation and representations. The latter notion will be discussed throughout this lecture in more detail for generic groups for which it is more complicated. The cyclic group has the *group presentation*,

$$\mathbb{Z}_p = \langle a \mid a^p = e \rangle \ . \tag{2.6}$$

The element a can be *represented* by any of the $p-1$: $a = e^{2\pi i m/p}$ for $0 \leq m < p$.

Proof: By Cauchy's thm a group of order p has an element of order p i.e. $\exists a \in G$ s.t. $a^p = e$. (And since p is prime $a^m = e$ for $m < p$ cannot exist since m does not divide p.) Hence the group, i.e. all multiplication rules, are completely determined by $a^p = e$ c.f. (2.6) q.e.d.

Let us summarise to this end some of the notions introduced. Let there be a group G a set X and a group action "." then the orbit of a an element $x \in X$ is $G(x) = \{g.x \mid g \in G\} \subset X$ a subset of X, $G_x = \{g \in G \mid g.x = x\}$ is as subgroup called the stabiliser subgroup. c.f. definition 2.2.4. The orbit stabiliser thm states that to each element $g.x$ there corresponds one and only one left coset gG_x (or right coset $G_x g$).

Exercise 2.2.2 Cauchy's thm and orbits

a) The cyclic group $\mathbb{Z}_6$ is a subgroup of $O(2)$ in example 2.2.2. Consider $\mathbb{Z}_6$ acting on points on the two dimensional plane by rotation centred at a point a. Draw the *two* characteristic orbits and give the corresponding stabiliser subgroups. Give the size of the orbits and check that they satisfy corollary 2.2.1.

b) Find the order of a proper subgroup H of a group G ($|G| = 100$) given the following hints: 1) H is not cyclic 2) H has got no element which is its own inverse.

c) Following up on a previous exercise 2.1.3, show that the order of a group of four elements $\{e, g_1, g_2, g_3\}$ admits only four different multiplication tables. Argue (or show) that three of those are isomorphic to each other. Hence there are two inequivalent groups of order four. They are isomorphic to $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ where the latter is the Vierergruppe from exercise 2.1.3 as introduced by Felix Klein.

d) Given a group G of order 52, what is the order of the proper non-abelian subgroup? The claim is that the order is unique. *Hint:* you will need to use exercise c).

2.2.3 Normal subgroups

Definition 2.2.5 ***Normal subgroup*** = invariant subgroup:
A normal subgroup N is a subgroup of a group G, denoted by $N \lhd G$, which is invariant under the group action of conjugation. I.e.

$$N = \{n_i \in G \mid g \cdot n_i \cdot g^{-1} = n_j, \forall g \in G\} \ . \tag{2.7}$$

Theorem 2.2.5 Left- and right-cosets of subgroup H are identical $\Leftrightarrow$ H is normal in G.

We shall omit the proof of this theorem. Comment: To some extent a normal subgroup is a subgroup which is invariant under relabelling (conjugation is a kind of relabelling of the group when it comes to representations at least). For the permutation groups this is most obvious. Having this in mind it is not surprising that a normal subgroup can be factored retaining a group structure:

Theorem 2.2.6 ***left (right) cosets of normal subgroup allow a group structure.***
Given $N \lhd G$ one can define a so-called ***quotient group*** (factor group) G/N of order $|G|/|N|$ by imposing the following binary operation (group multiplication):

$$g_1 N \cdot g_2 N = g_1 \cdot g_2 N \ . \tag{2.8}$$

Another way of writing (2.8) is $g_1 N \cdot_{G/N} g_2 N := (g_1 \cdot_G g_2) N$ where $\cdot_G$ emphases in which group the product is to be understood. In other words it is the group product in G that is used to define the group product in G/N.

Proof: G/N is a group: Since $g_1 N, g_2 N \in G/N \;\; \Rightarrow \;\; (g_1 N) \cdot (g_2^{-1} N) \stackrel{(2.8)}{=} (g_1 g_2^{-1}) N \in G/N$ and thus G/N is a subgroup by virtue of theorem 2.1.1. It remains to be shown that (2.8) is well defined multiplication law. Step 1: show that $g_1 N \cdot g_2 N \supseteq g_1 \cdot g_2 N$: $g_1 \cdot g_2 N \ni g_1 \cdot g_2 \cdot n$ for $n \in N$. The latter may be written as $(g_1 \cdot e) \cdot (g_2 \cdot n)$ which is clearly contained in $(g_1 N) \cdot (g_2 N)$. Step 2: show that $g_1 N \cdot g_2 N \subseteq g_1 \cdot g_2 N$: Let $g_1 \cdot n_1 \cdot g_2 \cdot n_2$ be an element of $g_1 N \cdot g_2 N$. This element can be written as $(g_1 \cdot g_2) \cdot (g_2^{-1} \cdot n_1 \cdot g_2) \cdot n_2 \stackrel{(2.7)}{=} g_1 \cdot g_2 \cdot (n_j \cdot n_2)$ for some $n_j \in N$. This proves the assertion that (2.8) is a well-defined multipilciation law and ends the proof. q.e.d.

In essence in the quotient group the group elements are the cosets, the sets shown Fig. 2.1, and a group multiplication is defined by (2.8). The identity corresponds to N (which is H is Fig. 2.1).

Definition 2.2.6 ***Simple group*** = no normal subgroups
A group which has no other normal subgroups other than the identity and the group itself is called a ***simple group***.[11] The term simple is appropriate, in view of theorem 2.2.6. The simple groups are to the classification of groups what the prime numbers are to the integer numbers. Namely a structure that cannot be divided any further.

Theorem 2.2.7 If H is a subgroup of G and $|G|/|H| = 2$ then H is a normal subgroup.

Proof: Left and right cosets partition the group (theorem 2.2.1) ,

$$G = H \cup gH = H \cup Hg \;, \tag{2.9}$$

see also (2.5). Hence left and right cosets are the same and theorem 2.2.5 implies that H is normal in G. Note that this implies that the quotient group G/H is isomorphic to $\mathbb{Z}_2$ since the latter is the only group of order two.

Theorem 2.2.8 ***First isomorphism thm:***[12] Let there be homomorphism $\varphi : G \to \tilde{G}$ then the following statements are true:

a) (the image) $\mathrm{Im}(\varphi)$ is a subgroup of $\tilde{G}$.

b) (the kernel) $K = \mathrm{Ker}(\varphi)$[13] is a *normal* subgroup of G

c) G/K is isomorphic to $\mathrm{Im}(\varphi)$

Proof: We shall content ourselves in proving b) from which a) at least is obvious. b) Show that K is a subgroup. Assume $k_1, k_2 \in K$ then $k_1 \cdot k_2^{-1} \in K$ since $\varphi(k_1 \cdot k_2^{-1}) = \varphi(k_1) \cdot \varphi(k_2^{-1}) = e \cdot e = e$.[14] By theorem 2.1.1 K is a subgroup. Show that K is normal: For $k_1 \in K$ and $g \in G$, then $\varphi(g \cdot k_1 \cdot g^{-1}) = \varphi(g) \cdot \varphi(k_1) \cdot \varphi(g^{-1}) = \varphi(g) \cdot e \cdot \varphi(g^{-1}) = \varphi(g \cdot g^{-1}) = e$ and hence K is normal. q.e.d.

The isomorphism theorem is illustrated in Fig. 2.4. The essence of the first isomorphism theorem is that to show that a group, say N_i is a normal subgroup, it is often simplest to find the homomorphism where the kernel corresponds to N_i. The theorem will be applied to show that even and off permutations are well defined (c.f. section 2.3.2).

[11] An interesting fact that emerged from the classification of finite groups is that all simple finite groups admit a set of generators which consists of at most two elements. From the simple groups all other finite groups can be built.
[12] There are two further isomorphism theorems frequently stated in textbook which we do not discuss in this course.
[13] For a group homomorphism the Ker $= \{g \in G | \phi(g) = e_{\tilde{G}}\}$.
[14] In the last step we, silently, used $\varphi(k_2^{-1}) = \varphi(k_2)^{-1}$. You are going to fill in this step in exercise 2.2.3.

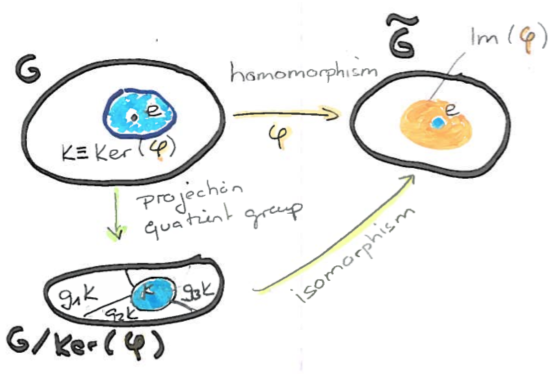

Figure 2.4: The isomorphism theorem 2.2.8 illustrated.

Exercise 2.2.3 normal subgroups

a) Is $\{e, (1,2)\}$ a normal subgroup of S_3? To assess this question you could compare left and right cosets, which you have assessed in a previous problem for an equivalent situation, and use a theorem of current section.

b) Show that $(n\mathbb{Z}, +)$ is a normal subgroup of $(\mathbb{Z}, +)$

c) Show that the factor group $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to $\mathbb{Z}_n$. It is sufficient to state the isomorphism (that is to say the map).

d) Show that for a homomporphism, $\varphi : G \to \tilde{G}$, the following relation holds: $\varphi(g^{-1}) = \varphi(g)^{-1}$. This fills a gap in the proof of thm 2.2.8.

2.2.4 Direct and semidirect products of groups

Definition 2.2.7 ***Direct products:***
The direct product of two groups H and J, denoted by $H \times J$, is the set

$$H \times J = \{(h, j) \mid h \in H \ , j \in J\}$$

with the binary operation

$$(h, j) \cdot (h', j') = (h \cdot h', j \cdot j') \ , \quad h, h' \in H \ , j, j' \in J \ .$$

Note that $H \times J$ is a group of order $|H \times J| = |H||J|$. For finite groups you may think of the size of the group $H \times J$ as the size of the rectangle with lines $|H|$ and $|J|$.

Definition 2.2.8 ***Semidirect products:***
Let ϕ be a group action of a group H on J then one can define a semi-direct product denoted by $H \ltimes J$. $H \ltimes J$ is the set $H \times J$ with the following binary operation:

$$(h, j) \cdot (h', j') = (h \cdot h', j \cdot \phi(h, j')) \equiv (h \cdot h', j \cdot (h.j')) \ , \quad h, h' \in H \ , j, j' \in J \ .$$

The order of $H \ltimes J$ is $|H| \cdot |J|$ in close analogy to order of the direct product groups.

Examples 2.2.3 ***of semidirect products***

- An example of the semidirect product is the group of euclidian isometries which consist of a rotation R and a translation $\vec{a}$ which act on some vector space. Two consecutive applications give rise to

$$((R', \vec{a}') \cdot (R, \vec{a})) \circ \vec{x} = (R', \vec{a}') \circ ((R, \vec{a}) \circ \vec{x}) = (R', \vec{a}') \circ (R\vec{x} + \vec{a}) = R'R\vec{x} + R'\vec{a} + \vec{a}' \tag{2.10}$$

  and hence $(R', \vec{a}') \cdot (R, \vec{a}) = (R'R, R'\vec{a} + \vec{a}') \neq (R'R, \vec{a} + \vec{a}')$ is not equal to a direct product structure. One writes $\mathrm{ISO}(n) = O(n) \ltimes \mathbb{R}^n$. *In physics rotation and boosts plus space-time translations correspond to the so-called Poincaré group whose invariants, the mass and the spin, characterise a particle. When the translations are omitted the group is referred to as the Lorentz group which is obviously a subgroup of the Poincaré group. Hence the Poincaré group is the semidirect product of the Lorentz and the time-space translation group.*

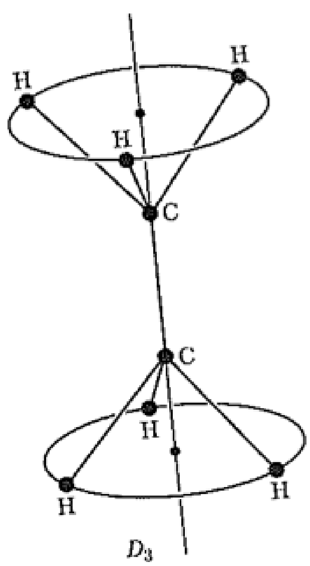

Figure 2.5: C_2H_6-molecule with D_3-dihedral symmetry. Taken from **[Ham62]**.

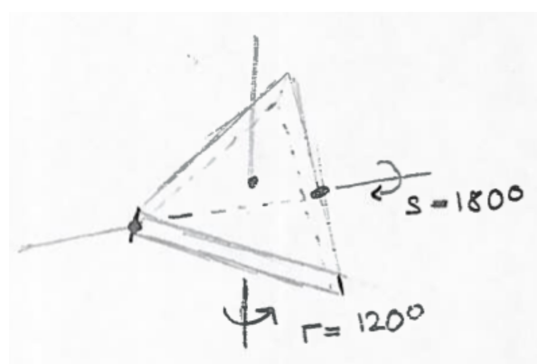

Figure 2.6: Triangle with D_3-dihedral symmetry. The dihedral symmetry D_n can be understood as the symmetry of the regular n-polygon.

- Another important example comes from chemistry where many molecules (c.f. Fig. 2.5) are characterised by what is called the ***dihedral group***, denoted by D_n, of order $2n$ which is a semidirect product $\mathbb{Z}_n \rtimes \mathbb{Z}_2$.[15] The group homomorphism of $\mathbb{Z}_2$ on $\mathbb{Z}_n$ is given by inversion.[16] In Fig. 2.5 the inversion is literally the inversion through the centre point of the molecule. Alternatively the dihedral group D_n is the symmetry group of a regular n-polygon as illustrated for the triangle in Fig.2.6. The inversion (operation s in Fig. 2.6) corresponds to rotating the polygon by 180° around a straight line from a vertex through its centre. Examples for $(r_i, s_i) \in \mathbb{Z}_n \rtimes \mathbb{Z}_2$ are

$$\begin{aligned} &(e^{2\pi i3/n}, 1) \cdot (e^{2\pi i1/n}, 1) &&= (e^{2\pi i4/n}, 1) \\ &(e^{2\pi i3/n}, -1) \cdot (e^{2\pi i1/n}, 1) &&= (e^{2\pi i3/n} e^{2\pi i(-1/n)}, -1) = (e^{2\pi i2/n}, -1) \ , \end{aligned} \tag{2.11}$$

and hopefully clarify the text above. A presentation is given by,

$$D_n = < r, s \mid r^n = s^2 = s^{-1} r s r = e > \ . \tag{2.12}$$

The group element r corresponds to a rotation of $360°/n$ around the axis of the molecule and s corresponds to the rotation around 180° of the centre of the molecule. For D_3 the operations are illustrated in Fig. 2.6. *As already mentioned the dihedral group is the symmetry group of molecules. As another example we shall mention the symmetry of a snowflake which corresponds to D_6 (hexagon).*

Exercise 2.2.4 Products of groups.

a) Show that S_3 is isomorphic to D_3 by finding the elements of S_3 (from the multiplication tables) that satisfy the presentation 2.12. You could also attempt a geometric approach and identify the symmetries in either the C_2H_6-molecule Fig. 2.5 or Fig. 2.6.

b) As a matter of fact (up to isomorphisms) there are two groups of order six. One of them is D_3. Can you find or give the other one?

c) The Vierergruppe $\mathbb{Z}_2 \times \mathbb{Z}_2$ has got the following group presentation:

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b \mid a^2 = b^2 = (ab)^2 = e \rangle \ .$$

Write down explicit generators for a, b and ab in terms of the direct product notation: $(e^{2\pi i m/2}, e^{2\pi i n/2})$.

2.3 Permutation groups

2.3.1 The symmetric permutation group S_n

We have already mentioned the permutation group S_n in examples 2.1 and shown the multiplication table of S_3 in table 2.2. In this chapter we shall discuss some more details.

[15] This is why some authors also denote the dihedral group of $2n$ elements by D_{2n}.

[16] Recall from exercises 2.1.4 that inversion, itself, is a group homomorphism for abelian groups which is clearly the case for $\mathbb{Z}_n$.

Consider a set of n elements $X = \{x_1, ..., x_n\}$, then all possible permutations of this set, obviously, define a group which is denoted by S_n and called the ***symmetric permutation group***.[17][18] $..S_n$ leaves symmetric polynomials $P(x_1, x_2, ..x_n)$ The number of elements of this group corresponds to the number of possible permutations which is $n!$. The elements of the group can be denoted by brackets, as in table 2.2, $(a_1, a_2, a_3..)$ with $1 \leq a{-}i \leq n$ which correspond to cyclic permutations; best explained by example:

$$\underbrace{(1,4,2)\cdot(1,2,3)}_{=(2,3,4)} \circ \{x_a, x_b, x_c, x_d\} = (1,4,2) \circ \{x_c, x_a, x_b, x_d\} = \{x_a, x_d, x_b, x_c\} \ . \tag{2.13}$$

Once understood we can define multiplication directly within the group without reference to the set X, as indicated in Eq.(2.13). Other example which you may want to verify are:

$$(1,2)\cdot(2,3) = (1,2,3) \ , \quad (2,3)\cdot(1,2) = (1,3,2) \ . \tag{2.14}$$

A cycle $(a_1, .., a_k)$ is referred to as a ***k-cycle*** and a 2-cycle is called a ***transposition***. Without proof: All elements of S_n can be written as the product of disjoint cycles. By two disjoint cycles we mean cycles $(a_1, .., a_n)$ and $(b_1, .., b_m)$ for which $a_i \neq b_j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. any which do not have any permutation element in common. Note that disjoint cycles commute with each other e.g. $(1,2)\cdot(3,4) = (3,4)\cdot(1,2)$. Hence whenever two cycles are not disjoint one can commute them through other elements and then multiply them into a single object as done above (2.14). You will provide some more evidence in exercise 2.4.1. Problems such as the question of how many m-cycles there are in S_n are very typical combinatorial problems that are best broken into parts. For example we may first look at how many ways there are to get m distinct numbers out of n which is $n!/(n-m)!/(m!)$. Second we take into account the ordering. We may freely choose one element to be in the first entry by convention but then the order of all the others matters and there are $(m-1)!$ possibilities. Hence the answer is $n!/(n-m)!/m$. For example there $4!/(4-3)!/3 = 8$ 3-cycles in S_4:

$$\{(1,2,3),(1,3,2),(1,2,4),(1,4,2),(1,3,4),(1,4,3),(2,3,4),(2,4,3)\} \ . \tag{2.15}$$

Theorem 2.3.1 The transpositions (2-cycles) generate S_n.

Proof: The elements of S_n consist of products of k-cycles with $1 \leq k \leq n$. An arbitrary k-cycle can be obtained from the following product of transpositions.

$$(a_1, a_2, ..., a_k) = (a_1, a_k)\cdot(a_1, a_{k-1})\cdot ... \cdot(a_1, a_2) \ , \quad \text{q.e.d.}$$

More specifically

Theorem 2.3.2 (a) The transpositions $\{(1,2),(1,3),..,(1,n)\}$ generate S_n.
(b) The transpositions $\{(1,2),(2,3),..,(n-1,n)\}$ generate S_n.
(c) The transposition $(1,2)$ and the n-cycle $(1,2,..,n)$ generate S_n.

Proof: (a) Note that $(a_1, a_2) = g\cdot(1,a_2)\cdot g^{-1}$, $g = (1, a_1)$ and then use theorem 2.3.1. (b) Note that $(1,k) = g\cdot(1,2)\cdot g^{-1}$, $g = (k-1,k)\cdot..\cdot(3,4)\cdot(2,3)$ and then use (a). (c) Note that $(k,k+1) = g\cdot(1,2)\cdot g^{-1}$ where $g = (1,2,..,n)^k$ and then use (b) q.e.d.

The proof of (b) and (c) are in essence implementing a change of basis (relabelling the permutation elements).

[17] A more abstract definition for S_n are the maps that are bijections (one-to-one) on a set of n-elements.
[18] S_n is the group which leaves so-called symmetric polynomials in n variables invariant.

2.3.2 The alternating group A_n

We have learned that any element of the symmetric group can be written as a product of k-transpositions. The number of transposition that occur are unambiguously even or odd and are called ***even-*** and ***odd-permutations*** respectively. This can be seen by using the first isomorporhism theorem 2.2.8. Consider the Vandermonde polynomial $P(x_1, .., x_n) = (x_1-x_2)\cdot..\cdot(x_1-x_n)\cdot(x_2-x_n)\cdot..\cdot(x_{n-1}-x_n)$[19] of all possible products of $(x_i - x_j)$ with $i \neq j$. Then the map ϕ : acting on $P/|P|$ by permutation is a group homomorphism from S_n to $\mathbb{Z}_2$ since the image of S_n acting on P is $\pm P$ and therefore isomorphic to $\mathbb{Z}_2$. The kernel of this map are the even permutations i.e. A_n. The latter are therefore, according to theorem 2.2.8, a normal subgroup (i.e. $A_n \lhd S_n$) of order $|A_n| = |S_n|/|\mathbb{Z}_2| = n!/2$.[20] Stated as a theorem:

Theorem 2.3.3 ***alternating group A_n:*** The even-permutations generate a normal subgroup of S_n of order $n!/2$

Proof: Given above.

Theorem 2.3.4 The 3-cycles generate A_n

Proof: According to theorem 2.3.2 S_n is generated $\{(1,2),(1,3)..\}$. In the case of A_n each element, consists of an even number of transpositions. Since $(1,a_1)\cdot(1,a_2) = (1,a_2,a_1)$ the statement follows. q.e.d.

2.4 Platonic solids

A platonic solid is a regular, convex geometric body in three dimensions which consists of faces of a *single* polygon. There are five solids which meet this criteria, each named after the number of polygons: tetrahedron, hexahedron (cube), octahedron dodecahedron, icosahedron, c.f. Fig. 2.7. Further information can be found in table 2.3. The symmetry of the platonic solids seems predestined to be explained by group theory. The symmetry groups are given in table 2.3 on the right and correspond all to groups of the A_n- and S_n-type. An obvious start to characterise the symmetry group is to note that any symmetry permutes the vertices of the body. Hence the symmetry group of a platonic solid with n vertices is a subgroup of S_n (in accordance with Cayley's theorem 2.1.2). Intuitively the reason why it is not necessarily S_n is that the platonic solids have a certain rigidity so that one can not arbitrarily permute the edges.

Moreover, platonic solids which are equivalent under interchange of vertices and surfaces are dual to each other; they have the same symmetry group. As can be seen from table 2.3 the tetrahedron is dual to itself and the hexahedron and octahedron (c.f. Fig. 2.8 in addition) as well as the dodecahedron and icosahedron are dual to each other respectively.

We would like to illustrate the symmetry group by looking at one specific example that is the tetrahedron. The symmetry operations are shown in Fig. 2.9. They consist of a rotation symmetry of $120°$ (denoted by r) around the axis of a vertex through the centre of the opposite face and a $180°$ rotation through around an axis that goes through the middle of two opposite edges. From the figure it is easily deduced that r and s correspond to $(2,3,4)$- and $(1,4)(2,3)$-permutation respectively.

[19] This is a so-called alternating polynomial. The name derives from the fact that the permutation of any two variables alternates the sign and this is what gave the alternating group its name c.f. definition above.

[20] $A_n \lhd S_n$ also follows from theorem 2.2.7.

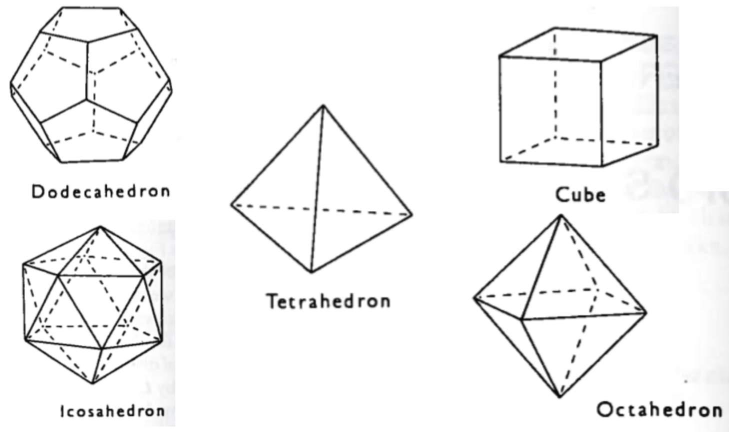

Figure 2.7: Platonic solids – photocopied from [**Arm88**].

solid	faces	vertices	edges	polygon	symmetry group
tetrahedron	4	4	6	triangle	A_4
hexahedron	6	8	12	square	S_4
octahedron	8	6	12	triangle	S_4
dodecahedron	12	20	30	pentagon	A_5
icosahedron	20	12	30	triangle	A_5

Table 2.3: Platonic solids (c.f. Fig.2.7) with number of faces (F), vertices (V) and edges E), the polygon and their symmetry group. The former are related by the famous Euler formula: $V - E + F = 2$ which can be proven in many ways. The hexahedron and octahedron as well as the dodecahedron and icosahedron are dual under interchange of the number of faces and vertices. This is the reason they admit the same symmetry group.

Those permutations are even and provide some evidence that the subgroup could be A_4. You will complete a more thorough proof in the exercises below.

Exercise 2.4.1 Permutation groups.

a) In the notes it was stated that any element of S_n can be written as a product of disjoint cycles for which we have not given a proof. This exercise aims to give some evidence by example.
Take $(1, 2, 3)$ and $(2, 4, 5)$ (common element 2) and write down the the 5-cycle corresponding to $(1, 2, 3) \cdot (2, 4, 5)$. *Hint:* In order to establish the multiplication rule let the cycles act on a set of five ordered elements.

b) Exercise a) will help you to answer the following question: What is the largest order of an element in S_{10}?

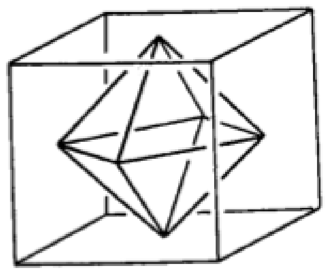

Figure 2.8: Octahedron and cube being dual under interchange of faces and vertices. Figure from [**Arm88**].

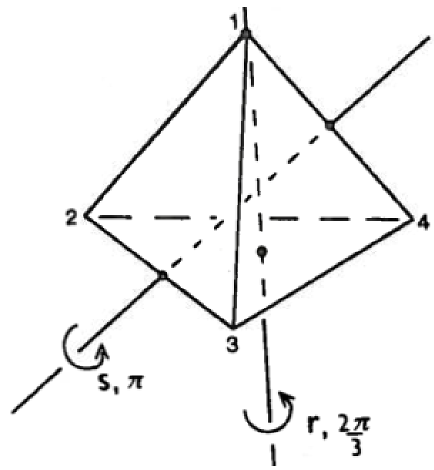

Figure 2.9: Tetrahedron (photocopied out [**Arm88**])

c) *The symmetry group of the tetrahedron is isomorphic to* A_4. A presentation of A_4 is given by:

$$A_4 = < a, b \mid a^2 = b^3 = (ab)^3 = e > \quad . \tag{2.16}$$

Show that $r = (2,3,4)$ and $s = (1,4)(2,3)$, as defined above these exercises, satisfy the presentation relation in (2.16).

d) *The symmetry group of the cube is isomorphic to* S_4. For the cube it is appropriate to consider the space diagonals as the group elements (They retain the rigidity of the cube. Think of the space diagonals as a scaffolding of the cube.) According to theorem 2.3.2 S_4 is generated by $(1,2)$ and $(1,2,3,4)$. Identify those symmetry operations geometrically. *Hint:* Whereas the operation $(1,2,3,4)$ should be immediate, $(1,2)$ could be a bit more difficult. You could try to look for a symmetry operation which is of order two since $(1,2)^2 = e$.

e) List all 24 elements of S_4 in the form $()$, (x,y), (x,y,z), (x,y,z,w) and $(x,y)\cdot(z,w)$ with $x,y,z,w \in \{1,2,3,4\}$.

f) Let S_4 act on the sets 1) $x_1 = \{a,a,a,a\}$ 2) $x_2 = \{a,a,a,b\}$, 3) $x_3 = \{a,a,b,c\}$ and 4) $x_4 = \{a,b,c,d\}$ by permutation. Give the stabiliser subgroup and the size of the orbit (write down all elements of the orbits) for each set $x_{1,2,3,4}$ and verify corollary 2.2.1.

2.5 Applications

Most applications of finite group theory involve representation theory which are going to treat in subsequent chapters. We shall content ourselves alluding at one (two) example(s) from chemistry which can be found in [**Atk08**] chapter 11.

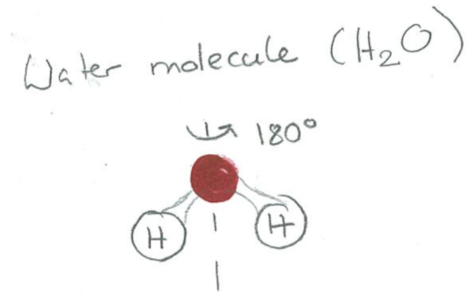

Figure 2.10: Water molecule H_20 with $\mathbb{Z}_2$ symmetry. The water molecule has a permanent electric dipole moment along the symmetry axis as further discussed in examples 2.5.1.

Examples 2.5.1 ***Polarity and chirality of molecules.***.

- A polar molecule is a molecule with a permanent electric dipole. The basic idea is that the molecule has to be asymmetric to a certain degree for it to admit an asymmetric charge distribution. The latter is the source of a permanent electric dipole. For instance any molecule that has a $\mathbb{Z}_{n\geq 2}$-symmetry cannot have an electric dipole that is perpendicular to the symmetry axis. So it appears at first that C_2H_6 Fig. 2.5 and H_2O (water) in Fig.2.10 could have an electric dipole along the rotation axis of $\mathbb{Z}_3$ and $\mathbb{Z}_2$ respectively. For C_2H_6 the s-symmetry operation Fig.2.6 takes one end of the molecule to the other and this obstructs the formation of a electric dipole. Hence only H_20, and not C_2H_6, has got a permanent electric dipole.

  As a matter of fact only molecules with $\mathbb{Z}_n$-symmetry types (c.f. [**Atk08**] for more details) posses a permanent electric dipole along the rotation axis.

- Another example is the *chirality* of a molecule (mirror image not the same as the original); c.f. [**Atk08**]. A molecule may be chiral if does not admite an improper rotation axis. An improper rotation, is a rotation around an axis followed by an inversion.[21] Since we did not discuss those kind of symmetry groups we shall not go into any further details here.

[21] These groups are denoted, somewhat unfortunately, by S_n in the Schoenfliess notation used in chemistry literature. They are of order n and not to be confused with the permutation group of order $n!$.

Chapter 3

Representation theory

Representations are realisation of groups on linear vector spaces. Applications within mathematics, physics, computer science and so on are enormous. Especially from the viewpoints of sciences, other than pure mathematics, representation are natural since computations are performed in concrete settings such as linear spaces. *From your course on quantum theory you are familiar with the concept of discussing physics (equivalently) in different linear spaces. E.g. the harmonic oscillator can be discussed in the coordinate space $|x\rangle$, momentum space $|p\rangle$ or the number of particle representation $|n\rangle$. Each one of them has its own merits. When in addition the system displays a symmetry, e.g. rotational symmetry of the potential, then the symmetry transformation acts as a representation on the physical eigenstates of the Hilbert space. This will become clearer when we proceed to examples.*

Discussion of finite representation theory, which consists to a large extent of character theory, is presented in chapter 4; followed by chapter 5 on Lie groups. In the current chapter we give the definition of a representation and discuss it in the, by now, familiar setting of S_3. Basic definitions are given in section section 3.2. Section 3.3 discusses Schur's lemma and decomposability, which are relevant both to finite representation theory of chapter 4 and the Lie groups of chapter 5. Applications are presented at the end of the chapters.

Definition 3.0.1 ***representation:***

A representation ρ is a homomorphism (c.f. definition 2.1.3) from a group G into the set of linear (invertible) operators on a complex vector space V (the latter are denoted by $GL(V)$ and form a group structure themselves under composition).

$$\begin{array}{rccl} \rho : & G & \longrightarrow & GL(V) \\ & g & \longrightarrow & \rho(g) \, . \end{array}$$

Since ρ is a homomorphism the map is compatible with the group structure, (c.f. Eq. (2.2)),[1]

$$\rho(g_1 \cdot g_2) = \rho(g_1)\rho(g_2) \, . \tag{3.1}$$

The relevant[2] representations of finite groups are finite dimensional and therefore $\rho(g)$'s are nothing but matrices acting on a finite dimensional vector space; i.e. $V = \mathbb{C}^n$ with $n = \dim V$.

[1] Alternatively we may think of a representation as a group action on the linear space V.

[2] i.e. irreducible, a terms to defined shortly hereafter

3.1 Representations the pedestrian way

Going back to the permutation group of three elements S_3. According to theorem 2.3.2 a), S_3 is generated by the elements $(1,2),(1,3)$. Associating the permutation with permutation of unit vectors of a three dimensional space we may represent those elements by:

$$\rho((1,2)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \quad \rho((1,3)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \ . \tag{3.2}$$

All other elements may be generated from (3.2). Whereas this is certainly *a* representation, known as the ***permutation representation***, it does nor answer the following two questions: i) are there any other representations? ii) can the representation be further reduced? We shall see that representation theory of finite groups gives concrete and definite criteria for answering questions i) and ii). Before presenting the theory let us point out that one can see that the representation (3.2) can be further reduced. Both matrices, and therefore all group elements, have the eigenvector $\vec{v} = (1,1,1)^T$, with eigenvalue $\lambda_{\vec{v}} = 1$, in common:

$$\rho((1,2))\vec{v} = \vec{v} \ , \quad \rho((1,3))\vec{v} = \vec{v} \ . \tag{3.3}$$

Hence one can construct a change of basis, $\rho \to \rho'$, such that the matrices are block diagonal

$$\rho'((1,2)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_1 & a_2 \\ 0 & a_3 & a_4 \end{pmatrix} \ , \quad \rho'((1,3)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & b_1 & b_2 \\ 0 & b_3 & b_4 \end{pmatrix} \ . \tag{3.4}$$

It is noted that the original three dimensional representation has split into a one and a two dimensional representation. Representations which can be split as the one in Eq. (3.2) are called *reducible*. A representation which cannot be split any further is called an irreducible representation. An aim of representation theory is to find, and if possible construct, all irreducible representation of the group. It is now time to proceed to the more formal representation theory.

Exercise 3.1.1 ***Representations the pedestrian way***

a) Verify through explicit matrix multiplication that the product $\rho((1,2))\rho((1,3))$ in (3.2) generates an element which can be interpreted as $\rho((1,3,2))$ in terms of permutation of the unit vectors.

3.2 Basic definitions of representation theory

Representations have been defined in definition 3.0.1 and we have seen a first example in section 3.1. Below $\rho(g)$ denotes a representation of $g \in G$ on the vector space V.

Definition 3.2.1 ***of representation theory***

- A representation is ***faithful*** if the homomorphism ρ is injective (inverse map from image is defined). If the latter is not the case the representation is called ***unfaithful*** accordingly. If $\rho(g) = \rho(e)$ for $\forall g \in G$ then the representation is called the ***trivial representation***.
- A ***unitary representation*** is a representation for which $\rho(g)$ is a unitary operator on the vector space.
- Two representations ρ and ρ' are called ***equivalent*** if they can be mapped into each other by a ***similarity transformation***. I.e. if there exists $S \in GL(V)$ s.t. $\forall g \in G$, $\rho(g) = S\rho'(g)S^{-1}$.[3]
- An ***invariant subspace*** is a subspace $W \subset V$ which is left invariant by the representation acting on it i.e.
$$\rho(g)\vec{w} \subset W \ , \quad \forall \vec{w} \in W \ , \forall g \in G \ . \tag{3.5}$$
- A representation is ***irreducible*** if there are no invariant subspaces.

  Fact: If there are invariant subspaces one can find a basis where $\rho(g)$ can be written as
$$\rho(g) = \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix} \ , \quad \forall g \in G \ , \tag{3.6}$$
  in a block diagonal form.[4] If a representation can be written in the form (3.6) the representation is said to be ***reducible***.

  In essence a representation is either irreducible or reducible. For finite groups (c.f. theorem 3.3.3) and continuous groups which are compact the reducible representations are a direct sum of the irreducible representations.[5]
- A ***real representation*** is a representation which can be brought (similarity transformation) into a form where the matrix entries are all real numbers. If this is not the case we speak of a ***complex representation***.[6] The complex conjugate representation of ρ is denoted by $\bar{\rho} = \rho^*$ and is given by complex conjugation (of the matrix entries).

Unitary representations are of importance for quantum physics. Unitary transformations are the natural language to describe symmetries on the physical Hilbert space since they preserve the inner product. Irreducible representation are not only the atomic objects of mathematical representation theory but they are also the "atoms" in physics. According to Wigner an elementary particle corresponds to an irreducible representation of the Poincaré group (i.e. space-time symmetry group including translations.) In physics, or particle physics, complex representation are associated with charges; real representations carry no charge.

Exemplifying some of these terms for the representation discussed in (3.2). The representation (3.2) is *reducible* since it can be written in block diagonal form (3.4). The two representations (3.2) and (3.4) are equivalent. The representation ρ' (and therefore also ρ since it is equivalent) contain the trivial representation which is in particular *unfaithful*. The representation ρ is real since the corresponding matrices have real entries. The representation ρ' is real since it is equivalent to the representation ρ which is real.

[3] As the wording suggest the equivalence of representations is an equivalence relation.
[4] We omit the proof of this statement.
[5] We shall give an example of a non-reducible representation for the translation group (which is non-compact) in chapter 5.
[6] For $SU(2)$ we will encounter a refinement of this notion to so-called pseudo-real representations.

3.3 Basics of representation theory Maschke's thm, Schur's Lemma and decomposability

In this section we state a few facts about representation theory of finite groups which are also relevant for representation theory of the Lie groups (compact Lie groups $U(1)$, $SO(2)$, $SO(3)$ and $SU(2)$) which are going to discuss in the chapter 5. The theorems are stated for finite groups.

Theorem 3.3.1 ***Maschke's thm:***
Any representation of a *finite* group is equivalent to a unitary representation.

Proof: Suppose (x, y) is the inner product of the vector space. One can define a new inner product

$$(x, y)_G \equiv \frac{1}{|G|} \sum_{g \in G} (\rho(g)x, \rho(g)y) \tag{3.7}$$

by averaging over the group (Weyl's trick). It is easily verified (exercises) that $\rho(g)$ is unitary i.e. $(\rho(g)x, \rho(g)y)_G = (x, y)_G$ with respect to the new inner product. q.e.d.

Theorem 3.3.2 ***Schur's Lemma***
Let ρ and ρ' be two irreducible representations of vector spaces V and V' and $T : V \to V'$ be a linear map satisfying $\rho' T = T\rho$ then

a) T is either an isomorphism, or $T = 0$ is trivial.

b) If $V = V'$ then $T = \lambda \cdot \mathbf{1}$ (for $\lambda \in \mathbb{C}$ and $\mathbf{1}$ the identity map on V).

Schur's lemma states in essence that there is no room for any non-trivial homomorphisms between irreducible representations of a group. Schur's Lemma is the fundament of representation theory. It is the essential ingredient to many powerful theorems. Often you will find Schur's lemma stated as follows: let ρ be a matrix irreducible representation and T a matrix in the same space with $[T, \rho(g)] = 0$ for $\forall g \in G$ then $T = \lambda\mathbf{1}$. This is clearly equivalent to theorem 3.3.2b).

Proof: a) The first statement follows from the fact that $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$ are invariant subspaces (to be discussed below) and therefore $\mathrm{Ker}(T) = 0, V$ and $\mathrm{Im}(T) = 0, V'$ since irreducible representation do not admit non-trivial invariant spaces. If $\mathrm{Ker}(T) = 0$ then $\mathrm{Im}(T) = V'$ and T is an isomorphism by virtue of the first isomorphism theorem 2.2.8. If $\mathrm{Ker}(T) = V$ then $\mathrm{Im}(T) = 0$ and $T = 0$ (maps all elements into zero vector of V'.)
Remains to be shown that Ker(T) $= \{v_0 \in V \mid Tv_0 = 0\}$ is an invariant subspace i.e. $T\rho(g)v_0 = 0$. This follows from $T\rho(g)v_0 = \rho'(g)Tv_0 = 0$ by using the premise $T\rho(g) = \rho'(g)T$.
b) If $T = 0$ then $\lambda = 0$. If $T \neq 0$, T must have at least one eigenvalue λ. Hence for $U \equiv T - \lambda\mathbf{1}$, $\dim[\ker(U)] \geq 1$ and U is not an isomorphism; by virtue of a) $U = 0$; i.e. $T = \lambda\mathbf{1}$ q.e.d.

Theorem 3.3.3 ***Decomposability thm***

a) Any reducible representation of a finite group decomposes into irreducible representations,

$$\rho(g) = m_1\rho_1(g) \oplus .. \oplus m_k\rho_k(g) \; , \tag{3.8}$$

with ***multiplicities*** $m_i \in \mathbb{Z}^+$.[7]

[7] A representation is simply reducible if $m_i = 0, 1$.

b) The decomposition into k factors is unique whereas the decomposition into a direct sum of m_i copies is not.

Proof: a) We content ourselves with the remark that the proof uses the same inner product as in theorem 3.3.1. b) Consider another representation of $\rho' = m'_1\rho'_1 \oplus .. \oplus m'_K\rho'_K$ of G in V and T a map between those representations. Then it follows from Schur's lemma that T must map ρ_i to ρ'_i which are isomorphic to each other. (One cannot map a representations of different dimensions to each other as otherwise T would not correspond to an isomorphism.) q.e.d.

Exercise 3.3.1 a) Argue that a representation $\rho(g)$ is unitary with respect to the inner product (3.7). I.e. $(\rho(g)x, \rho(g)y)_G = (x, y)_G$.

Chapter 4

Representation theory of finite groups

4.1 Character theory for finite groups

The character theory is the heart of the representation theory of finite groups.

Definition 4.1.1 The trace of a representation ρ is called the ***character*** χ

$$\chi(g) \equiv \mathrm{Tr}[\rho(g)] \ . \tag{4.1}$$

Hence to each group element a character is associated. The following properties are immediate:

Corollary 4.1.1 (of definition 4.1.1):

a) The characters of two equivalent representation (c.f. 3.2.1) are equal:

$$\chi(S\rho(g)S^{-1}) = \mathrm{Tr}[S\rho(g)S^{-1}] = \mathrm{Tr}[\rho(g)] = \chi(g) \ . \tag{4.2}$$

b) The characters of conjugate elements are identical for the same reason as in a) (replace $S \to \rho(g')$ above). It is therefore sufficient to consider conjugacy classes only. The character is said to be a ***class function*** (its value is constant on any conjugation class of a representation).

c)

$$\chi(\rho_a \oplus \rho_b) = \chi(\rho_a) + \chi(\rho_b) \ , \tag{4.3}$$

$$\chi(\rho_a \otimes \rho_b) = \chi(\rho_a) \cdot \chi(\rho_b) \ , \tag{4.4}$$

$$\chi(\rho(g^{-1})) \ = \chi(\rho(g))^* \ . \tag{4.5}$$

Above the index a and b are labels referring to a type of representation (e.g. trivial, permutation representation in the case of S_n). The first two properties follow from (1.18) and (1.20) and the definition of the trace respectively. The last property (4.5) is a consequence of Maschke's theorem 3.3.1:

$$\chi(\rho(g^{-1}) \stackrel{exercise (2.2.3)}{=} \chi(\rho(g)^{-1}) = \chi(\rho(g)^{\dagger}) \stackrel{\mathrm{Tr}[A^T]=\mathrm{Tr}[A]}{=} \chi(\rho(g)^*) = \chi(\rho(g))^* \ . \tag{4.6}$$

Maschke's thm was used in the second equality.

- d) For one dimensional representations the character is equal to the representation matrix since the latter is a 1×1-matrix. Note that a one dimensional representation is not necessarily the trivial representation.

The characters characterise groups to a large extent. In fact the mathematician Gerhard Brauer asked the question whether the characters together with the order of the elements would completely determine a finite group. It took decades to find a counterexample and this shows how important character tables are.

Definition 4.1.2 One can define an ***inner product on the space of characters*** as follows:

$$\begin{aligned} \langle \chi(\rho_a) | \chi(\rho_b) \rangle &\equiv \frac{1}{|G|} \sum_{g \in G} \chi(\rho_a(g))^* \chi(\rho_b(g)) \\ &= \frac{1}{|G|} \sum_{k=1}^{N_c} c_k \chi_{ak}^* \chi_{bk} \ . \end{aligned} \tag{4.7}$$

Above $\chi_{ak} = \chi(\rho_a(g_k))$ with g_k belonging to the k^{th}-class with c_k elements. The number N_c denotes the total number of classes of the group. The second line exploits the fact that the character is a class function. The character $\chi(g)$ can be thought of as a vector in $\mathbb{C}^{N_c}$ where each class represent a linearly independent direction:

$$\chi(\rho) = (\chi(\rho(e)), ..., \chi(\rho(g_{N_c})) \in \mathbb{C}^{N_c} \ , \tag{4.8}$$

where g_{N_c}, as above, is a representative of the ${N_c}^{\text{th}}$-class. Note, the first class is usually assigned to the identity element.

Theorem 4.1.1 ***Orthogonality theorem***
In terms of the inner product (4.7) the irreducible representations of G are orthonormal, i.e.

$$\langle \chi(\rho_a) | \chi(\rho_b) \rangle = \delta_{ab} \ . \tag{4.9}$$

The proof is given for the interested reader only as it is too abstract for a first course on representation theory. Let us just mention that the main ingredient is Schur's lemma.

*for the interested reader (**not** part of the course)*

Proof: Let us assume that V_a and V_b are irreducible representations of G. Then the space of all homomorphism from $V_a \to V_b$ is equal to $\mathrm{Hom}(V_a, V_b) = V_a^* \otimes V_b$ where V_a^* is the dual vector space of V_a (going from ket to bra in Dirac notation). Let V_0 denote the vector space of trivial representations i.e. $\rho(g)|_{V_0} = \mathbf{1}_{V_0}, \ , \forall g \in G$.
The rest of the proof builds on three observations: i) Schur's lemma states that the only non-trivial homomorphism, compatible with the group structure (i.e. invariant), there can be is an isomorphism. Hence $\dim(\mathrm{Hom}(V_a, V_b)_0) = \delta_{ab}$ where the subscript zero denotes the trivial representation subspace as before. ii) The character of the $V_a^* \otimes V_b$ representation is given by

$$\chi(\rho_{V_a^* \otimes V_b}) \stackrel{(4.4)}{=} \chi(\rho_{V_a^*}) \cdot \chi(\rho_{V_b}) = \chi(\rho_{V_a})^* \cdot \chi(\rho_{V_b}) \ , \tag{4.10}$$

where $\rho_{V_a} \equiv \rho_a$ in the statement of the theorem. iii) The operator $\phi = \frac{1}{|G|} \sum_{g \in G} \rho_V(g)$ is a projector ($\phi^2 = \phi$) on V_0. I.e. $\phi v_0 = v_0 \ , \forall v_0 \in V_0$ and $\phi V_0^\perp = 0$ where $V_0^\perp \oplus V_0 = V$. To see this note that applying ϕ to a vector results in an invariant vector and the only invariant vectors corresponds to the subspace of trivial representation. Moreover, a second application of ϕ just results in a rearrangement which has though no effect since it is averaged over the group and therefore $\phi^2 = \phi$. Hence the number of trivial representation is given by

$$\dim(V_0) = \mathrm{Tr}[\phi] = \frac{1}{|G|} \sum_{g \in G} \chi(\rho_V(g)) \ . \tag{4.11}$$

Assembling

$$\begin{aligned} \delta_{ab} &\stackrel{i)}{=} \dim(\mathrm{Hom}(V_a, V_b)_0) = \dim((V_a^* \otimes V_b)_0) \stackrel{iii)}{=} \frac{1}{|G|} \sum_{g \in G} \chi(\rho_{V_a^* \otimes V_b}(g)) \\ &\stackrel{ii)}{=} \frac{1}{|G|} \sum_{g \in G} \chi(\rho_{V_a}(g))^* \cdot \chi(\rho_{V_b}(g)) \stackrel{def}{=} \langle \chi(\rho_{V_a}) | \chi(\rho_{V_b}) \rangle \ , \quad \text{q.e.d.} \end{aligned} \tag{4.12}$$

Corollary 4.1.2 (of the orthogonality theorem 4.1.1)
Let $\rho = m_1 \rho_1 \oplus .. m_k \rho_k$ stand for a generic representation.

a) A representation is fully characterised by its character since the distinct ρ_i's correspond to linearly independent vectors in $\chi(\rho(g)) \stackrel{(4.3)}{=} m_1 \chi(\rho_1(g)) + .. + m_k \chi(\rho_k(g))$. You should think of $\chi(\rho(g))$ as a vector in $\mathbb{C}^{N_c}$.

b) The multiplicity m_a of ρ_a in ρ is given by

$$m_a = \langle \chi(\rho_a) | \chi(\rho) \rangle \ . \tag{4.13}$$

c) The norm of a representation is

$$||\chi(\rho)||^2 = \langle \chi(\rho) | \chi(\rho) \rangle = \sum_i m_i^2 \ . \tag{4.14}$$

Hence ρ is irreducible $\Leftrightarrow ||\chi(\rho)||^2 = 1$.

d) The character of the identity element corresponds to the dimension of the representation:

$$\chi(\rho_i(e)) = \dim V_i \tag{4.15}$$

e) The characters of the trivial representation are all one. I.e.

$$\chi(\rho_{\mathbf{1}}) = (1, .., 1) \in \mathbb{C}^{N_c} \ . \tag{4.16}$$

Exercise 4.1.1 ***Characters:***

a) Verify statements b),c), d) and e) in corollary 4.1.2

b) Obtain all six matrices of the representation (3.2). Two are given, one is trivial and one of them you have already computed in exercise 3.1.1. Compute the remaining two and compute the norm of the character using the inner product (4.7).

c) According to the decomposability theorem 3.3.3 the permutation representation is either a direct sum of i) three one dimensional or ii) one one dimensional and a two dimensional irreducible representation. I.e. $\rho_{\text{perm}} = \rho_{\mathbf{1}} + \rho_{\mathbf{1}'} + \rho_{\mathbf{1}''}$ or $\rho_{\text{perm}} = \rho_{\mathbf{1}} + \rho_{\mathbf{2}}$ where the subscript denotes the dimension of the representation. Using the result in b) and a statement in corollary 4.1.2, show that ii) is the case.

d) Identify the equivalence classes of S_3. You may for instance guess them and verify them either through 1) explicit computation of the matrices 2) the multiplication laws from the multiplication table 2.2 or 3) you may proceed along the lines of proof of theorem 2.3.2 and find the equivalence transformations. You might even want to try more than one method to gain some confidence in the formalism. Note for S_3 members of an equivalence class corresponds to a renumbering of the elements so that guessing the equivalence classes is possible.

In order to proof the powerful dimensionality theorem we are going to introduce the regular representation

Definition 4.1.3 The ***regular representation*** corresponds to the left-action of the group on itself. More precisely to each element $g \in G$ we associate a vector $\vec{e}_g$ (declaring $\vec{e}_g$ to be linearly independent of $\vec{e}_{g'}$ for $g \neq g'$) and the representation acts as follows:

$$\rho_R(g)\vec{e}_{g'} = \vec{e}_{g \cdot g'} \ . \tag{4.17}$$

The regular representation has got the following property

$$\chi(\rho_R(g)) = \begin{cases} 0 & g \neq e \\ |G| & g = e \end{cases} , \tag{4.18}$$

which follows from the fact the $\rho_R(e) = \mathbf{1}_{|G|}$ and that no other element than $\rho_R(e)$ has any entry on the diagonal (to be verified in the exercises).

This definition together with the observation (4.18) is sufficient to proof the dimensionality theorem:

Theorem 4.1.2 ***Dimensionality theorem***
The sum of the dimension squared of all irreducible representations of a finite group is equal to the order of the group

$$|G| = \sum_{i=1}^{N_c} \dim(V_i)^2 \; . \tag{4.19}$$

Proof: First we observe that

$$m_i \overset{(4.13)}{=} \langle \chi(\rho_i) | \chi(\rho_R) \rangle \overset{\text{def}}{=} \frac{1}{|G|} \sum_{g \in G} \chi(\rho_i(g))^* \chi(\rho_R(g)) \overset{(4.18)}{=} \chi(\rho_i(e)) \overset{(4.15)}{=} \dim(V_i) \; . \tag{4.20}$$

Second by virtue of the decomposability theorem 3.3.3 we may write

$$\chi(\rho_R(g)) = \sum_i m_i \chi(\rho_i(g)) \overset{(4.20)}{=} \sum_i \dim(V_i) \chi(\rho_i(g)) \tag{4.21}$$

and choosing $g = e$ with Eqs. (4.18) and (4.15) we get

$$|G| = \sum_i \dim V_i^2$$

which completes the proof q.e.d..

Note, had we known the dimensionality theorem earlier, the reducibility of the representation (3.2) would have been clear. Moreover exercise 4.1.1 could have been solved more efficiently. You can complete the argument in exercise 4.1.2 a).

Exercise 4.1.2 ***dimensionality thm***

a) Use the dimensionality theorem to argue that the largest irreducible representation of S_3 is of dimension two.

b) What is the dimension of the permutation and the regular representation of S_n?

c) Is the regular representation always reducible?

4.2 Character tables

The information on characters is commonly presented in terms of so called ***character tables***; e.g. table 4.1 (a few pages ahead). First all group elements are divided into equivalence classes under conjugation. Note the identity is always a conjugation class on its own. At the top row a representative of each equivalence class is listed. Sometimes a second row is present stating the order of the representative. To each successive row an irreducible representation is associated where the corresponding characters are listed. The fact that the table, without counting the top rows, can be interpreted as a unitary square matrix (to be made precise) is going to be one of the main results of this chapter.

The rows in the character tables can be thought of as linearly independent vectors by virtue of the orthogonality theorem 4.1.1. Each conjugation class defines a linearly independent direction. Hence it is clear that the number of irreducible representations of G is less or equal to the number of equivalence classes as otherwise the rows would not be linearly independent. In fact they are equal:

Theorem 4.2.1 ***Character table is a square:*** The number of irreducible representations of G is equal to the number of conjugacy classes of G.

An equivalent statement is that the characters form an orthonormal basis for the class function space. Much alike $\exp(2\pi ix/Ln)$, for $n \in \mathbb{Z}$, form a basis for periodic function $f(x+L) = f(x)$ in Fourier theory. The proof of this theorem will be presented below but is **not** really part of the course and is given for the interested reader only. It is powerful but a bit abstract for an introductory course on the subject.

*for the interested reader (**not** part of the course)*

Proof: Assume that there was a class function $f : G \to \mathbb{C}$ (c.f. definition 4.1.1) s.t.

$$\langle f|\chi(\rho_i)\rangle = 0 \tag{4.22}$$

on all irreducible representations. If we can show that $f = (0, ..., 0) \in \mathbb{C}^{N_c}$, then the theorem is proven as this implies that the characters form a complete set of vectors on the class function space. Consider the following map,

$$\phi = \sum_{g\in G} f(g)^* \cdot \rho_i(g) : V_i \to V_i \ , \tag{4.23}$$

on the vector space V_i of the irreducible representation ρ_i. From $\rho_i(g')\phi = \phi\rho_i(g')$ and Schur's lemma (theorem 3.3.2) it follows that $\phi = \lambda\mathbf{1}$ and we may write:

$$\lambda \dim(V_i) = \mathrm{Tr}[\phi] = \sum_{g\in G} f(g)^* \chi(\rho_i(g)) = |G|\langle f|\chi(\rho_i)\rangle \overset{(4.22)}{=} 0 \ . \tag{4.24}$$

Hence either $\lambda = 0$ (i.e. $f(g) = 0$ for $\forall g \in G$) or $\sum_{g\in G} f^*(g)\rho_i(g) = 0$ has to hold on *any* irreducible representation and by virtue of decomposability on any representation; in particular the regular representation $\rho_R(g)$ (which is a direct sum of the irreducible representations). Since all $\rho_R(g)$ are linearly independent by definition this enforces $f(g) = 0$ for all $g \in G$ and this completes the proof q.e.d.Note, we have therefore shown that no f other than $f = (0, .., 0)$ obeys (4.22).

Another powerful theorem, somewhat reminiscent, of Lagrange's theorem is:

Theorem 4.2.2 The dimension of any irreducible representation divides the order of the group:

$$\frac{|G|}{\dim V_i} \in \mathbb{Z}^+ \ , \quad \rho_i \text{ is an irreducible representation on } V_i \tag{4.25}$$

We shall omit the proof[1] and state simple corollary:

Corollary 4.2.1 All irreducible representations of an abelian group are one dimensional.

Proof: First we notice that for an abelian group each element forms a conjugacy class on its own since $a = g \cdot a \cdot g^{-1} \Leftrightarrow a \cdot g = g \cdot a$. Hence for an abelian group there are as many irreducible representations as there are elements by virtue of theorem 4.2.2. In order to obey the constraint (4.19) all those irreducible representations are necessarily one dimensional $\sum_{i=1}^{|G|} \dim(V_i)^2 = \sum_{i=1}^{|G|} 1^2 = |G|$ q.e.d.

Exercise 4.2.1 ***Representations of abelian groups*** By virtue of corollary 4.2.1 representations of abelian groups are rather straightforward. You might find the following exercises useful as they clarify previous matters.

a) Write down all irreducible representations of $\mathbb{Z}_7$. By virtue of corollary 4.2.1 they are one dimensional and in fact you have been using them in a previous exercise.

b) How many of the irreducible representations of the previous exercise are faithful?

c) How many irreducible representations of $\mathbb{Z}_4$ are faithful?

The amount of a priori information on the dimension of the irreducible representations is rather impressive and is largely grounded in the Schur's important Lemma (theorem 3.3.2). In summary, given a finite group its irreducible representation ρ_i on V_i are constrained as follows:

	Diophantine equations for $\dim V_i$ *of* G	
(1)	$\|G\| = 1 + \dim {V_2}^2 + .. + \dim {V_k}^2$,	(theorem 4.1.2)
(2)	$k = N_c \equiv$ number of conjugacy classes	(theorem 4.2.1)
(3)	$\|G\|/\dim(V_i) \in \mathbb{Z}^+$,	(theorem 4.2.2)

(4.26)

Note the one in $|G| = 1 + ..$ stands for the trivial representation which is always present. The equations above, diophantine nature, are in many cases sufficient to obtain the dimensions of the irreducible representations (without working them out explicitly). From the orthogonality 4.1.1 and the fact that the character table is essentially a unitary matrix follows a second orthogonality theorem:

[1] In fact an even stronger statement applies: $|G/Z(G)|/\dim V_i$ is an integer where $Z(G)$ is the centre of the group which is always normal in the group itself.

Theorem 4.2.3 ***Second orthogonality thm:*** We can define an inner product on the class space, with symbols as in (4.7), as follows:

$$\langle \chi(\rho([g])|\chi(\rho([g'])\rangle_C = \frac{c_{[g]}}{|G|}\sum_{i=1}^{N_c} \chi(\rho_i([g])^* \chi(\rho_i([g']) \; . \tag{4.27}$$

which is orthonormal,

$$\langle \chi(\rho([g])|\chi(\rho([g'])\rangle_C = \delta_{[g][g']} \; , \tag{4.28}$$

in the class directions. The subscript C stands for class and helps to avoid confusion with the previous inner product (4.7). Note, the slightly asymmetric definition in (4.27) anticipates the result (4.28).

Proof: The proof is constructive and uses the fact that for a unitary $n \times n$ matrix U:

$$UU^\dagger = U^\dagger U = \mathbf{1}_n \; . \tag{4.29}$$

The second equation follows from the first by hermitian conjugation. The matrix U in our case is

$$U_{a[g]} = \left(\frac{c_k}{|G|}\right)^{1/2} \chi(\rho_a([g]) \; , \tag{4.30}$$

where we remind the reader that a and $[g]$ are indices referring to an irreducible representation and a class representative respectively. The first and second orthogonality relations (4.9) and (4.28) can be written as

$$\begin{aligned} \langle \chi(\rho_a)|\chi(\rho_b)\rangle \quad &= (UU^\dagger)_{ab} \quad = \sum_{[g]} U_{a[g]}U^*_{b[g]} \;= \delta_{ab} \; , \\ \langle \chi(\rho([g]))|\chi(\rho([g']))\rangle &= (U^\dagger U)_{[g][g']} = \sum_{a=1}^{N_c} U^*_{a[g]}U_{a[g']} = \delta_{[g][g']} \; , \end{aligned} \tag{4.31}$$

where we implied the notation $(U^\dagger)_{ab} = U^*_{ba}$. Hence (4.30) guaranties that the two orthogonality relations imply each other. Stated differently: (4.9) $\Leftrightarrow$ (4.28) q.e.d.

4.2.1 Character table of S_3 as an illustration

We shall exemplify the character table of S_3. The conjugacy classes correspond, for any permutation group S_n, to the number of ways in which n can be written in terms of integers. The latter is known as a ***partition of an integer number*** $n = \lambda_1 + ...\lambda_k$, $\lambda_1 \geq .. \geq \lambda_k \geq 1$. For S_3 we have got:

$$3 = 1+1+1 \quad \leftrightarrow \quad [()] \; , \qquad 1 \text{ element} \; , \tag{4.32}$$

$$3 = 2+1 \quad \leftrightarrow \quad [(1,2)] \; , \qquad 3 \text{ elements} \tag{4.33}$$

$$3 = 3 \quad \leftrightarrow \quad [(1,2,3)] \; , \qquad 2 \text{ elements} \; . \tag{4.34}$$

Hence there are three irreducible representations. We can start building the character table:

S_3 class	$()$	$(1,2)$	$(1,2,3)$
dim class	1	3	2
trivial	1	1	1
alternating	1	-1	1
..	..	..	..

(4.35)

S_3 class	$()$	$(1,2)$	$(1,2,3)$
dim class	1	3	2
trivial	1	1	1
alternating	1	-1	1
standard	2	0	-1

Table 4.1: Character table of S_3 (which is isomorphic to D_3).

With the class in the first row and the dimension of the class in the second row. The latter is often omitted. The trivial representation is always present and all its entries are equal to 1. For the permutation groups S_n there is always a second one dimensional representation which acts as follows:

$$\rho_{\text{alt}}(g)\vec{v} = \text{sgn}(g)\vec{v} \;, \tag{4.36}$$

where $\text{sgn}(g) = \pm 1$ depending on whether g is an even or odd permutation. There are many ways to know that the remaining irreducible representation is two dimensional c.f. exercise 4.2.2. In fact the permutation representation (3.2) decomposes into the trivial and the two dimensional irreducible representation : $\rho_{\text{perm}} = \rho_{\text{triv}} \oplus \rho_{\text{stand}}$ (permutation, trivial and standard have been abbreviated.) The ***standard representation*** for the permutation groups is the non-trivial irreducible representation contained in the permutation representation.

In the case of S_3 the standard representation is two dimensional (c.f. exercise 4.1.2a)). By virtue of exercise 4.2.2 the character of the permutation representation is equal to the number of elements that the operation leaves in invariant it follows that $\chi(\rho_{\text{perm.}}) = (3,1,0)$ in the order of the classes as in the character table. Therefore

$$S_3: \quad (3,1,0) = \chi(\rho_{\text{perm.}}) \overset{(4.3)}{=} \chi(\rho_{\text{triv.}}) + \chi(\rho_{\text{stand}}) = (1,1,1) + \chi(\rho_{\text{stand}}) \;, \tag{4.37}$$

which yields $\chi(\rho_{\text{stand}}) = (2,0,-1)$ and completes the S_3 character table 4.1.

Exercise 4.2.2 ***characters and orthogonality***

a) Let ρ_{perm} be the permutation representation, as given for S_3 in (3.2). Argue that $\chi(\rho_{\text{perm}}(g))$ corresponds to the number of elements that are left invariant by $\rho_{\text{perm}}(g)$ acting on its vector space. The answer can be as short as one sentence.

b) Verify the second orthogonality theorem 4.2.3 on the example of the character table of S_3 given in table 4.1.

Exercise 4.2.3 ***Step by step construction of the S_4 character table.***

a) Find all the character classes of S_4 following the same recipe as for S_3. *Hint:* You should find that there are five of them.

b) Argue that the irreducible representations satisfy the following equation:

$$24 = 1^2 + 1^2 + d_3^2 + d_4^2 + d_5^2 \tag{4.38}$$

Hint: the two 1^2 correspond to two one dimensional irreducible representations. Which ones?

c) Find the dimension $d_{3,4,5}$ of the remaining three irreducible representations. The equation are "diophantine enough" to be solved.

d) Begin to build up the character table by writing down the first two irreducible representations.

e) Find the character of the standard representation using the same method as for S_3 (recall: $\rho_{\text{perm}} = \rho_{\text{triv}} \oplus \rho_{\text{stand}}$).

f) You can find another representation by multiplying the standard representation by the non-trivial one dimensional representation.

g) You can find the character entries of the last irreducible representation by making sure that it is orthogonal to the other entries. *Hint:* the class-characters assume the following values: $\{0, 0, -1, 2, 2\}$.

4.3 Restriction to a subgroup – branching rules

Before addressing the problem of restriction to a subgroup it will prove useful (in this context as well as for the rest of the course) to expand on the definitions of real and complex representations given in definition 3.2.1.

4.3.1 Complex and real representations

For a complex representation the complex conjugate representation $\bar{\rho}_a = \rho_a^*$ is inequivalent. For a real representation one can find a basis where ρ_a is real and therefore two complex conjugate representation are equivalent to each other. This knowledge gives a simple criterion, which we call the ***complex-real***

criterion, for determining the type of representation[2]

$$\langle \chi(\rho_a) | \chi(\bar{\rho}_a) \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(\rho_a(g)^2)^* = \left\{ \begin{array}{cc} 0 & \rho_a \text{ complex} \\ 1 & \rho_a \text{ real} \end{array} \right. . \tag{4.39}$$

The first equality is by no means trivial but follows from the decomposition of $V \otimes V$ into a symmetric and antisymmetric part as discussed in the footnote. In practice representations are often denoted by their dimension and if there is ambiguity then one uses prime or other symbols to distinguish them. In this spirit we write the complex conjugate representation as

$$\text{complex conjugate representation of } [\mathbf{3}] \equiv \bar{\mathbf{3}} \,, \tag{4.40}$$

where we have taken a three dimensional representation as an example for the sake of explicitness.

4.3.2 Branching rules

When a representation $\rho(g)$ for $g \in G$ is restricted to a subgroup $H \subset G$, $\rho(h)$ certainly forms a representation since ρ is still a group homomorphism on a linear space. The crucial point is though that it is not necessarily an irreducible representation. By virtue of the decomposability theorem the representation of G decomposes into irreducible representations of H

$$\rho(g)|_H = m_1 \rho_1(h) \oplus ... \oplus m_k \rho_k(h) \,. \tag{4.41}$$

where k is the number of equivalence classes of H.[3]

Branching rules can be constructed systematically from the character tables of the group G and H. In some cases, rather often, one is able to guess the branching rules with less information. We illustrate this after pursuing the systematic construction.

We consider the example of $\mathbb{Z}_3 \subset S_3$ and identify the branching rules. In doing so many concepts of the previous sections get illustrated. First of all we note that since $\mathbb{Z}_3$ is abelian, c.f. corollary 4.2.1, the irreducible representations of $\mathbb{Z}_3$ are all one dimensional. The group $\mathbb{Z}_3$ is generated by either $(1,2,3)$ or $(1,3,2)$ and has therefore no transpositions (2-cycles). We denote the irreducible

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[2] For completeness (non-examinable) let us mention that there is in addition there is a pseudo-real (or quaternionic) type representation where $\bar{\rho}_a = S^{-1}\rho_a S$ exists but it is not a real representation. Fortunately there is a criterion for pseudo-reality: $\rho(g)$ is pseudo-real if and only if $\frac{1}{|G|}\sum_{g\in G}\chi(\rho(g)^2) = -1$. Note $\langle \chi(\rho)|\chi(\bar{\rho})\rangle = 1$ for a pseudo-real representation. Pseudo-real representations will be discussed in the context of the Lie group $SU(2)$ in the following chapter. In case you want to understand of why a pseudo-real representations are a logical possibility you may want to consider exercise 3.38 in **[Ful91]**. The essential idea is that if the V is the vector space of a non-complex representation then $V \otimes V$ contains a trivial representation. The tensor product $V \otimes V = \text{Sym}(V\otimes V) \oplus \text{Asym}(V \otimes V)$ decomposes into a symmetric and antisymmetric subspace similar to the case of a matrix that can be decomposed uniquely into a symmetric plus a antisymmetric matrix. In the case where the trivial representation falls into the $\text{Sym}(V \otimes V)$ the representation is real and if it falls into $\text{Asym}(V \otimes V)$ it is pseudo-real. The criterion for pseudo-reality is $\frac{1}{|G|}\sum_{g\in G}(\rho(g^2)) = -1$. An example is given by the two dimensional representation of the Quaternions. The Quaternions are defined through the presentation $Q_8 =< I, J, K, -e \mid I^2 = J^2 = K^2 = IJK = -e, (-e)^2 = e >$. The two dimensional representation is given by $\rho_2(I) = i\sigma_3$, $\rho_2(J) = i\sigma_2$ and $\rho_2(K) = i\sigma_1$ where σ_i are the well-known Pauli-matrices to be defined later on. One readily obtains that $\frac{1}{|G|}\sum_{g\in G}\chi(\rho_2(g)^2) = \frac{1}{8}(2\cdot 2 + 6\cdot(-2)) = -1$.

[3] In the physics literature such decompositions are frequently called ***branching rules***. Branching rules are important in the case where a large symmetry breaks into a smaller one such as in grand unification e.g. $SU(5) \to SU(3) \otimes SU(2) \otimes U(1)$ with the latter being the gauge group of the so-called Standard Model.

representations by $\mathbf{1}_{\mathbb{Z}_3}$, $\mathbf{1}'_{\mathbb{Z}_3}$, $\mathbf{1}''_{\mathbb{Z}_3}$. The irreducible representations of $\mathbf{1}'_{\mathbb{Z}_3}$, $\mathbf{1}''_{\mathbb{Z}_3}$ are easily guessed to be

$$\begin{aligned} \rho(1,2,3)|_{\mathbf{1}'_{\mathbb{Z}_3}} &= e^{2\pi i/3} \quad \Rightarrow \quad \rho(1,3,2)|_{\mathbf{1}'_{\mathbb{Z}_3}} = e^{-2\pi i/3} \ , \\ \rho(1,2,3)|_{\mathbf{1}''_{\mathbb{Z}_3}} &= e^{-2\pi i/3} \quad \Rightarrow \quad \rho(1,3,2)|_{\mathbf{1}''_{\mathbb{Z}_3}} = e^{2\pi i/3} \ . \end{aligned} \tag{4.42}$$

Notice that the $\mathbf{1}'_{\mathbb{Z}_3}$, $\mathbf{1}''_{\mathbb{Z}_3}$ are *complex* representations (c.f. definition 3.2.1). In the case under discussion they are necessarily complex conjugate representations of each other since they originate from S_3 whose representations are real. In terms of equations, using the notation introduced in Eq. (4.40),

$$\bar{\mathbf{1}}'_{\mathbb{Z}_3} = \mathbf{1}''_{\mathbb{Z}_3} \ , \qquad \bar{\mathbf{1}}''_{\mathbb{Z}_3} = \mathbf{1}'_{\mathbb{Z}_3} \ , \tag{4.43}$$

where the second equation is just the complex conjugate of the first one. This knowledge can help in guessing the branching rules as we shall see below.

The characters are equal to the expressions in (4.42) since the representation is one dimensional. Below we give the character table for $\mathbb{Z}_3$ and reprint the character table of S_3 in tables 4.2 and 2.2 respectively. Note the elements $(1,2,3)$ and $(1,3,2)$ which are in the same equivalence class in S_3 constitute different equivalence classes in $\mathbb{Z}_3$. This happens frequently and is due, in this case, to the fact that the transpositions (x,y), which guarantee the equivalence under S_3 are not in $\mathbb{Z}_3$! Given both character tables one can compute the branching rules by taking scalar products of the

$\mathbb{Z}_3$ class	$()$	$(1,2,3)$	$(1,3,2)$
dim class	1	1	1
$\mathbf{1}$	1	1	1
$\mathbf{1}'$	1	$e^{2\pi i/3}$	$e^{-2\pi i/3}$
$\mathbf{1}''$	1	$e^{-2\pi i/3}$	$e^{2\pi i/3}$

Table 4.2: Character table of $\mathbb{Z}_3$.

S_3 class	$()$	$(1,2)$	$(1,2,3)$
dim class	1	3	2
trivial ($\mathbf{1}$)	1	1	1
alternating ($\mathbf{1}'$)	1	-1	1
standard ($\mathbf{2}$)	2	0	-1

Table 4.3: Character table of S_3 reprinted from table 2.2 for the readers convenience

corresponding characters. The question of how many times a representation j of $\mathbb{Z}_3$ is contained in the representation i of S_3,

$$\rho_{(S_3)_i}|_{\mathbb{Z}_3} = \bigoplus_j m_{ij}\rho_{(\mathbb{Z}_3)_j} \tag{4.44}$$

can be obtained by the scalar product of the characters restricted to the subgroup $\mathbb{Z}_3$:

$$m_{ij} = \langle \chi(\rho_{(S_3)_i} | \chi(\rho_{(\mathbb{Z}_3)_j} \rangle_{\mathbb{Z}_3} = \frac{1}{|\mathbb{Z}_3|} \sum_{h \in \mathbb{Z}_3} \chi(\rho_{(S_3)_i}(h))^* \chi(\rho_{(\mathbb{Z}_3)_j}(h)) \ , \tag{4.45}$$

where the right hand side resolves the potential ambiguity in notation between $\rho_{(S_3)}$ and $\rho_{(\mathbb{Z}_3)}$ in the same scalar product. This leads to the following branching rules:

$$\textbf{branching rules:} \quad \begin{aligned} \rho_{\mathbf{1}_{S_3}}|_{Z_3} &= 1 \cdot \rho_{\mathbf{1}_{\mathbb{Z}_3}} \ , & \mathbf{1}_{S_3} &\to \mathbf{1}_{\mathbb{Z}_3} \\ \rho_{\mathbf{1}'_{S_3}}|_{Z_3} &= 1 \cdot \rho_{\mathbf{1}_{\mathbb{Z}_3}} \ , & \mathbf{1}'_{S_3} &\to \mathbf{1}_{\mathbb{Z}_3} \\ \rho_{\mathbf{2}_{S_3}}|_{Z_3} &= 1 \cdot \rho_{\mathbf{1}'_{\mathbb{Z}_3}} \oplus 1 \cdot \rho_{\mathbf{1}''_{\mathbb{Z}_3}} \ , & \mathbf{2}_{S_3} &\to \mathbf{1}'_{\mathbb{Z}_3} \oplus \mathbf{1}''_{\mathbb{Z}_3} \ , \end{aligned} \tag{4.46}$$

since the only non-vanishing scalar products are (to be verified in exercise 4.3.1):

$$\begin{aligned} &\langle \chi(\rho_{\mathbf{1}_{S_3}}) | \chi(\rho_{\mathbf{1}_{\mathbb{Z}_3}}) \rangle = 1 \;, \quad \langle \chi(\rho_{\mathbf{1}'_{S_3}}) | \chi(\rho_{\mathbf{1}_{\mathbb{Z}_3}}) \rangle = 1 \;, \\ &\langle \chi(\rho_{\mathbf{2}_{S_3}}) | \chi(\rho_{\mathbf{1}'_{\mathbb{Z}_3}}) \rangle = 1 \;, \quad \langle \chi(\rho_{\mathbf{2}_{S_3}}) | \chi(\rho_{\mathbf{1}''_{\mathbb{Z}_3}}) \rangle = 1 \;. \end{aligned} \tag{4.47}$$

As promised, when character tables are known the computation of branching rules is straightforward. The notation on the right hand side of Eq.(4.46) is the one used in the physics literature. The notation on the left hand side is a bit extensive but presumably good for the sake of clarity and connected to the initial one in Eq. (4.41).

Alternative reasoning: Often it is possible to guess the branching rules in indirect ways. We shall illustrate this below.

- General rule: the trivial representation is always mapped to the trivial representation. Hence: $\mathbf{1}_{S_3} \to \mathbf{1}_{\mathbb{Z}_3}$.
- The alternating representation $\mathbf{1}'_{S_3}$ is distinct from the trivial representation by a minus sign on odd permutations. $\mathbb{Z}_3$ is generated by $(1,2,3)$ or $(1,3,2)$ (as stated above), which are even and hence the trivial and the alternating representation are the same when restricted to $\mathbb{Z}_3$. Hence: $\mathbf{1}'_{S_3} \to \mathbf{1}_{\mathbb{Z}_3}$.
- The standard representation $\mathbf{2}_{S_3}$ has to branch into two one dimensional representations of $\mathbb{Z}_3$. Given the fact that $\mathbf{2}_{S_3}$ is a real representation it can either split into $2 \cdot \mathbf{1}$ or $\mathbf{1}' + \mathbf{1}''$ of $\mathbb{Z}_3$. The split $\mathbf{1} + \mathbf{1}'$ is not possible because it is not a real combination: $(\mathbf{1}_{\mathbb{Z}_3} + \mathbf{1}'_{\mathbb{Z}_3})^* = \bar{\mathbf{1}}_{\mathbb{Z}_3} + \bar{\mathbf{1}}'_{\mathbb{Z}_3} = \mathbf{1}_{\mathbb{Z}_3} + \mathbf{1}''_{\mathbb{Z}_3}$. A split into two identities would indicate that $\mathbf{2}_{S_3}$ is not a faithful representation. This cannot be the case as one representation has got to be faithful ($\mathbf{1}$ and $\mathbf{1}'$ of S_3 aren't) and therefore $\mathbf{2}_{S_3} \to (\mathbf{1}' + \mathbf{1}'')_{\mathbb{Z}_3}$ is unavoidable.

Exercise 4.3.1 ***branching rules***

a) Compute or confirm the scalar products in (4.47).

b) Make sure you understand that (4.42) are complex representations. To do so you should consult the definition.

c) Why does a complex representation have at least one character which is complex valued (by which we mean that it is not real)? *Hint:* A representation and its complex conjugate representation are inequivalent to each other i.e. $\langle \rho_a | \bar{\rho}_a \rangle = 0$.

4.4 Constructing representations

Constructing all irreducible representations is not the main aim of this course. An appreciation of some of the tools is nevertheless very helpful and this is what we aim for in this section. For example direct product representations. *The latter naturally appear in problems of physics e.g. when two particle in given representations form bound states. The latter behaves like the irreducible representations of the product representations.* The diophantine rules (4.26) are of great help in starting to construct the character table as illustrated in section 4.2.1 for S_3. As we shall see for S_n there is a beautiful one to one correspondence between so-called Young diagrams and all irreducible representations. The magic of Young tableau's also migrates, for example, into Lie group representations of $SU(n)$.

4.4.1 A short notice on the irreducible representations of S_n

We are going to state a few basic facts about the irreducible representation of the symmetric group in this section without giving further proofs. The simplicity of the results and their appeal hint at a much deeper underlying structure which is beyond the scope of this course.

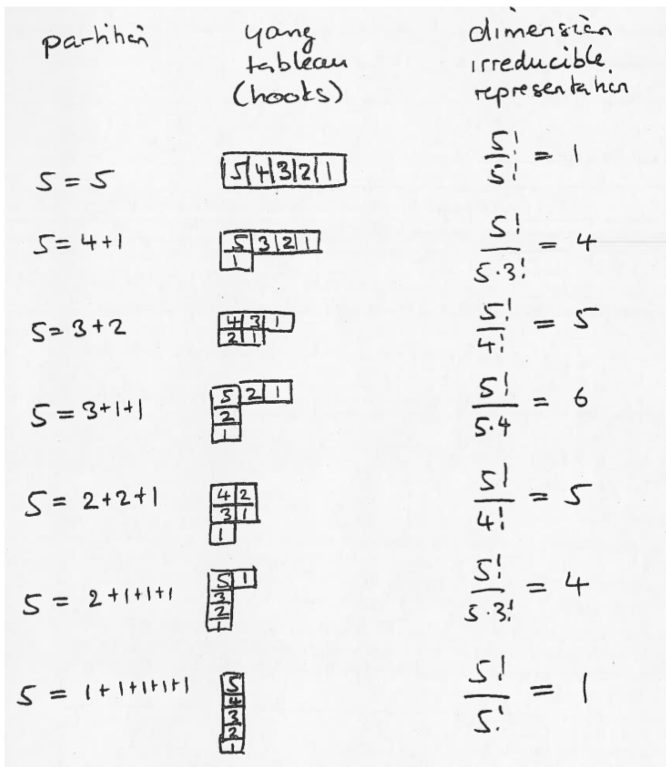

Figure 4.1: Partition of 5 on the left and in the middle the associated Young tableaus including the hook length of each square. (The number of vertical boxes in a row corresponds to a summand of the partition. From top to bottom the size of the corresponding decreases. The hook length is explained in the text). On the right the dimension of the irreducible representation is computed according to formula (4.48). A non-trivial check is the verification of the dimensionality theorem in (4.49).

In section 4.2.1 we have stated that the number of the conjugacy classes corresponds to the number of ways one can partition an integer. By virtue of theorem 4.2.1 this number corresponds to the number of irreducible representations. In fact to each partition one can associate a so-called Young tableau, illustrated for S_5 in Fig. 4.1. In addition a number, called the ***hook length***, is associated to each square. The hook length counts the number of squares to the right and below the square including the square. The amazing property is that the dimension of the irreducible representation corresponds to

$$\dim V_i[S_n] = \frac{n!}{\prod \text{hook lengths}} \ , \tag{4.48}$$

where Π stands for the product (of all hook lengths). All of which, the partition, the Young tableau, hooks and formula (4.48), are illustrated in Fig. 4.1 for S_5. A non-trivial check is whether it satisfies the dimensionality theorem 4.1.2:

$$|S_5| = 5! = 120 = 2 \cdot 1^2 + 2 \cdot 4^2 + 2 \cdot 5^2 + 6^2 \ . \tag{4.49}$$

Indeed it does! Certain symmetries are apparent in Fig. 4.1 and demand a few explanations. Representations whose Young tableau can be mapped into each other by a mirror reflection at an axis going from top left to bottom right corner correspond to the parity partner representation. For example $n = n$, which always corresponds to the trivial representation, is mirror dual to $n = 1 + .. + 1$ which corresponds to the alternating representation. The complete set of relations is given by:

$$\begin{aligned}
&5 = 5 && \overset{\text{mirror}}{\leftrightarrow} && 5 = 1+1+1+1+1 \ , \\
&5 = 4+1 && \overset{\text{mirror}}{\leftrightarrow} && 5 = 2+1+1+1 \ , \\
&5 = 3+2 && \overset{\text{mirror}}{\leftrightarrow} && 5 = 2+2+1 \ .
\end{aligned} \tag{4.50}$$

Hence the characters of mirror dual representation are

$$\chi(\rho_i(g)) = \text{sgn}(g)\chi(\rho_{\text{mirror}(i)}(g)) \tag{4.51}$$

with $\text{sgn}(g)$ denoting the parity $g \in S_n$. Further we note that the $5 = 3 + 1 + 1$ is mirrored to itself (self-dual in a certain sense). With all this information we could in principle (re)construct the character table of S_5 without too much effort. In view of limited time we shall not do so and move on, but the interested reader is referred to **[Ful91]** chapter 3. The program of this section is to be exercised for S_4, by you, in exercise 4.4.1.

Exercise 4.4.1 ***Young tableau for S_4 etc***

a) Repeat the program of determining the number and dimensions of the irreducible representations for S_4 (or any other $S_{n\neq 5}$) as outlined in this section. i) write down all partitions ii) draw the corresponding Young tableaus with hook lengths iii) compute the dimension according to formula (4.48) and then, finally, check the dimensionality theorem.

b) Argue that $\chi(\rho_a(g)) = 0$ for $g \notin A_n$ (i.e. g odd permutation), if ρ_a is a self-dual (identical mirror image of Young tableau) irreducible representation of S_n. *Hint:* If this was not the case then one could construct a further irreducible representation.

4.4.2 Direct product representations – Kronecker product

So far we have been concerned with finding the smallest building block of representation theory: the irreducible representations. Tensor product representations $\rho_a \otimes \rho_b$ acting on $V_a \otimes V_b$ is a straightforward way to generate new representations which are generically reducible but decompose, by virtue of theorem 3.3.3, into irreducible representations.

First we notice that the direct product of representations is a representation since the (direct) product of two group homomorphism is still a group homomorphism. By theorem 3.3.3 we have:

$$\rho_a \otimes \rho_b = m^1_{ab}\rho_1 \oplus .. \oplus m^k_{ab}\rho_k = \bigoplus_c m^c_{ab}\,\rho_c \;, \tag{4.52}$$

where $m^c_{ab} \in \mathbb{Z}^+$ are the multiplicities. Direct product of representations are known as ***Kronecker products*** or ***Clebsch-Gordan series***. The former refers more to the left hand side and the latter to the right hand side respectively. By virtue of properties of direct products and sums (c.f. section 1.3) we have the following sum rule:

$$\mathrm{dim}V_a \cdot \mathrm{dim}V_b = \sum_c m^c_{ab}\,\mathrm{dim}V_c \;, \tag{4.53}$$

which is yet another diophantine equation.

Motivation for decomposition of direct product representations

We shall motivate the fact that direct products decompose into smaller representations in two different ways: one formal argument (for finite groups) and one physical argument (addition of angular momenta).

- *the largest irreducible representation:* The direct product of the largest irreducible representation of a finite group has to decompose as otherwise it is not the largest irreducible representation. In concrete cases conflicts with the dimensionality theorem could arise.

- *addition of angular momenta:* From your course on quantum mechanics you should know that when two particles of a certain angular momentum form a bound state, the angular momenta add to a total angular momentum. We shall illustrate this below with two states of angular momentum $l = 1$, i.e. states which transform as vectors under $SO(3) = \{O \in M_3(\mathbb{R}) \mid O^T O = \mathbb{I}_3 \;, \det O = 1\}$. Strictly speaking we should be addressing this example in the next chapter, where we discuss continuous groups, but I believe there is some merit in doing it here.

  Consider three dimensional vectors $\vec{v} \in \mathbb{R}^3$ which transforms under a $l = 1$ irreducible representations of the rotation group $SO(3)$.[4] The direct product of two such irreducible representations, say $\vec{v}$ and $\vec{w}$ can be written,[5]

  $$v_i w_j = A_{ij} + B_{ij} \;, \quad A_{ij} = (v_i w_j - \frac{\delta_{ij}}{3}\vec{v}\cdot\vec{w}) \;, B_{ij} = \frac{\delta_{ij}}{3}\vec{v}\cdot\vec{w} \;, \quad i,j = 1..3 \;, \tag{4.54}$$

  unambiguously as a sum of a trace free (A_{ij}) and a trace part (B_{ij}). Intuitively it is clear that the trace part, the term on the righthand side, does not transform under rotation since it is

[4] Be warned: physicists will often state that $\vec{v}$ in the case above is in a $l = 1$ irreducible representation of $SO(3)$.

[5] We omit the $\oplus$ sign in the equation below. Strictly speaking one should write $v_i \otimes w_j$ etc.

a scalar product. Formally this follows from δ_{ij} being just a unit matrix which multiplied by any matrix (representation) yields back the matrix. Hence the term is a *scalar* (= invariant) corresponding to $l = 0$ irreducible representation of $SO(3)$.[6]

Multiplicities from the scalar product on characters

The multiplicity $m_{ab}^c \in \mathbb{Z}^+$, according to corollary 4.1.2 b), is is given by

$$m_{ab}^c = \langle \chi(\rho_a \otimes \rho_b) | \chi(\rho_c) \rangle \quad = \quad \frac{1}{|G|} \sum_{g \in G} \chi(\rho_a(g) \otimes \rho_b(g))^* \chi(\rho_c(g)) \tag{4.55}$$

$$\stackrel{(4.4)}{=} \quad \frac{1}{|G|} \sum_{g \in G} \chi(\rho_a(g))^* \chi(\rho_b(g))^* \chi(\rho_c(g)) \; . \tag{4.56}$$

Now follows a corollary of immense practical use:

Corollary 4.4.1 The Kronecker product of two irreducible representations $\rho_{a,b}$ contains the identity (with multiplicity one),

$$\rho_a \otimes \rho_b = 1 \cdot \rho_{\mathbf{1}} \oplus ... \tag{4.57}$$

if and only if $\rho_b = \bar{\rho}_a$ (ρ_b is the complex conjugate representation of ρ_a). Note: if the $\rho_{a,b}$ are not irreducible then the multiplicity of $\rho_{\mathbf{1}}$ can be larger than one.

Proof:

$$m_{ab}^{\mathbf{1}} \quad \stackrel{(4.55)}{=} \quad \frac{1}{|G|} \sum_{g \in G} \chi(\rho_a(g))^* \chi(\rho_b(g))^* \chi(\rho_{\mathbf{1}}(g)) \stackrel{Cor.4.1.2e)}{=} \frac{1}{|G|} \sum_{g \in G} \chi(\rho_a(g))^* \chi(\rho_b(g))^* = 1$$

$$\stackrel{Cor.4.1.2c)}{\Leftrightarrow} \quad \chi(\rho_b(g))^* = \chi(\rho_a(g)) \; , \quad \forall g \in G \; , \tag{4.58}$$

where the last line is the very definition of the complex conjugate representation. q.e.d.

Generalisations of corollary 4.4.1 to reducible representations are immediate and left to the reader.

Remark: To this end we note that Kronecker products allow us to understand what we called mirror representation in section 4.4.1 in the following manner:

$$\rho_{\text{mirror}(a)} = \rho_{1'} \otimes \rho_a \; , \quad G = S_n \tag{4.59}$$

where $1'$ is the alternating representation $\rho(g)_{1'} = \text{sgn}(g)$.

[6] Risking an outlook towards the next chapter: the full story is that in the trace free part there is a $l = 1$ (antisymmetric) and $l = 2$ (symmetric) irreducible representation . In fact we know that one can get a vector from two vectors via the cross product $\vec{x} = \vec{v} \times \vec{w}$ and this corresponds to the $l = 1$ part. The qualitative discussion above is also coherent with the formula of addition of angular momenta $|\vec{l}_1 + \vec{l}_2| = |l_1 - l_2|, ...l_1 + l_2 = 0, 1, 2$ for $l_1 = l_2 = 1$. The reason why angular momenta add when representations are multiplied will become clear when we discuss the corresponding Lie group and Lie Algebra of $SO(3)$.

Exercise 4.4.2 Exercises on Kronecker products

a) Check that

$$\mathbf{1}'_{\mathbb{Z}_3} \otimes \mathbf{1}''_{\mathbb{Z}_3} = \mathbf{1}_{\mathbb{Z}_3} \qquad (4.60)$$

and hence $\bar{\mathbf{1}}'_{\mathbb{Z}_3} = \mathbf{1}''_{\mathbb{Z}_3}$ and vice versa.

In an earlier version of the notes we did state the Wigner-Eckart theorem (valid for any group with finite dimensional representation) at this stage. In order to reduce the level of abstraction we shall solely state the Wigner-Eckart theorem for $SU(2)(SO(3))$ in section 6.3 with brief comments on the general case stated thereafter.

4.5 A few sample applications

4.5.1 Distortion of lattices

Consider a model of a solid with a cubic lattice where you can imagine the atoms to be located, in equilibrium, at the corners of a cube c.f. figure 4.2 According to section 2.4 and table 2.3 such a lattice exhibits a S_4 symmetry instead of the $SO(3)$-rotation symmetry of the free space.

Let us consider a cartesian coordinate system whose coordinates are aligned with the sides of the cube. We consider two type of distortions for which we are interested in the reduction of the symmetry group $S_4 \to H$.

- *A distortion in the* $(1,0,0)$*-direction:* $S_4 \to D_4$*.* The effect on the symmetry is like losing a direction and hence the symmetry reduces to the one of the planar square. The latter (c.f. figure 2.6) corresponds to the dihedral group D_4 of eight elements.
- *A distortion in the* $(1,1,1)$*-direction along the diagonal:* $S_4 \to D_3$*.* The effect is similar to the previous case though not quite as obvious. Imagine looking at a cube onto a vertex then you would see the three faces of that cube. You can convince yourself that the symmetry corresponds to the symmetry of a triangle. The latter corresponds to the dihedral group D_3 of six elements according to c.f. figure 2.6.

4.5.2 Fermi and bose statistics

Identical particles in quantum theory give rise to rather curious phenomenon. In classical theory we may follow the trajectory of each particle and one may therefore attach a definite number to each particle.

In quantum theory this is not possible anymore. Let us imagine an ensemble of non-interacting identical particles. States of that space may be written as tensor product of single particle states. E.g. $|\phi_1 \otimes .. \otimes \phi_N\rangle$ with ϕ_i being a single particle state. The generic state vector can then a priori be in any irreducible representation of the symmetric permutation group S_N. How do we know which representation describes which type of particle configuration? In particular if N is large

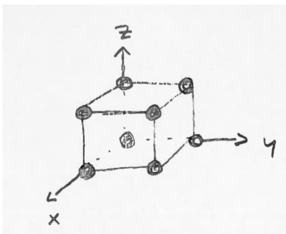

Figure 4.2: Lattice with cubic symmetry with only one shell shown.

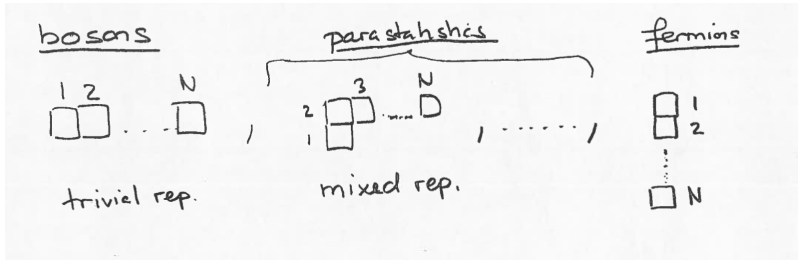

Figure 4.3: Possible arrangements (irreducible representations) of an identical particle state. (right) trivial representations as chosen by identical bosons. (middle) mixed representations do not appear in nature or are equivalent to boson and fermion statistics times a global symmetry, (left) alternating representation as chosen by identical fermions.

($N \simeq 10^{23}$ Avogadro number for systems of macroscopic size) then the number of choices of irreducible representations is rather large.

A very important fact of quantum theory: If the identical particles are *bosons* (i.e. particles of integer spin $s = 0, 1, 2...$) then the wave function is in the trivial representation and if the identical particles are *fermions* (i.e. particles of half integer spin $s = 1/2, 3/2...$) then the wave function is in the alternating representation. For $N = 2$ we have got:

$$\begin{aligned} \Phi_{12}|_{\text{bosons}} &= \frac{1}{\sqrt{2}} \left(|\phi_1 \otimes \phi_2\rangle + |\phi_2 \otimes \phi_1\rangle \right) \ , \\ \Phi_{12}|_{\text{fermions}} &= \frac{1}{\sqrt{2}} \left(|\phi_1 \otimes \phi_2\rangle - |\phi_2 \otimes \phi_1\rangle \right) \end{aligned} \tag{4.61}$$

This is the essence of it. A few comments seem in order:

- This implies that no two fermions can be in the same state as otherwise the wave function vanishes on grounds of antisymmetry. This entails Pauli's famous ***exclusion principle*** that was postulated to give the atomic a shell model a more solid foundation within non-relativistic quantum mechanics. The so-called occupation number n_{E_α} of a given fermion state of fixed energy E_α is either $n^{\text{fermion}}_{E_\alpha} = 0, 1$ whereas for bosons $n^{\text{boson}}_{E_\alpha} = 0, 1, 2, 3, ..$ it can be any integer.
- The fact that all the bosons can condense into (or occupy) the same state at low energies has spectacular consequences and is known as Bose-Einstein condensation. The crucial bit is that the bosons are all coherent in the ground state and act as one entity. This is the basis of such phenomenon as superfluidity, superconductivity and many others.

  For fermions the fact that the occupation number per state is either one or zero leads to the so-called zero point pressure. In neutronstars (neutrons are fermions) it is the zero point pressure of the fermions that balances the gravitational attraction and avoids the star imploding into a black hole.
- The general connection between spin and statistics is called, not surprisingly, the ***spin statistics theorem***. It's proof is deep down in the structure of quantum field theory. If one tries to go the other way then one can show (Pauli 1940) that this leads to violation of Einstein causality

(commutators of quantum field have support for space-like separation points). The most rigorous proof is found in axiomatic quantum field theory (Jost 1957) and uses a complexification (analytic continuation) of the Lorentz symmetry group $SO(3,1)$.

- In fact at first it might seem somewhat surprising that of the many representation of the symmetry group of S_N (for N large) there are only two types which are realised for identical particles. The other representation (mixed representations) were also considered, and referred to as *parastatistics*, but were found to be equivalent to the trivial and alternating representation times a global symmetry and therefore do not seem to bring in a new element. Note that adding fermions and bosons coherently is not possible because the boson and fermion number is a conserved quantum number. This is known as a *superselection rule* and will be briefly discussed towards the end of the course.

Exercise 4.5.1 Invariants and mixed representations.

a) The group S_4 is generated by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} , B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} , C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \tag{4.62}$$

Note, in order to check invariance of a potential invariant it is sufficient to check invariance for a set of group generators. Check that A, B, C both leave $\mathcal{I}_2 = x^2 + y^2 + z^2$ $\mathcal{I}_3 = xyz$ invariant.

b) *Explicit construction of mixed representation* We shall construct the standard representation of S_3 (which corresponds to a mixed representation) explicitly. The construction is similar to the permutation representation although it is algebraically more involved. This exercise also contains the seeds to understand the hook formula (4.48) for the S_n.

The representations (4.61) correspond to the trivial and alternating representation of S_2:

$$S_2 : \quad \mathbf{1} : \begin{array}{|c|c|}\hline & \\ \hline\end{array} , \mathbf{1}' : \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$$

Let us consider the situation for three particles and construct the *mixed representation* (parastatistics) of S_3:

$$S_3 : \quad \mathbf{2} : \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$$

The representation is two dimensional and the numbers are not the hook lengths but an enumeration to be used below.

The meaning of the arrangement of the boxes for S_n is as follow: First take a vector with n directions say $|\phi_i\rangle$ $i = 1..n$ and then take a tensor product of n of them $|\phi_{i_1} \otimes .. \otimes \phi_{i_n}\rangle$ and the boxes give the rules of anti(symmetrisation). Indices associated with rows are symmetrised and indices which are in one column are antisymmetrised over.

Let us implement this on the following state: $|\phi_1 \otimes \phi_2 \otimes \phi_3\rangle$ (or you can use the notation $|\phi_a \otimes \phi_b \otimes \phi_c\rangle$ if you find it less confusing) and call the state e_1:

$$e_1 = |\phi_1 \otimes \phi_2 \otimes \phi_3\rangle + |\phi_2 \otimes \phi_1 \otimes \phi_3\rangle - |\phi_1 \otimes \phi_3 \otimes \phi_2\rangle - |\phi_2 \otimes \phi_3 \otimes \phi_1\rangle . \tag{4.63}$$

A second basis vector might be obtained by starting with $|\phi_1 \otimes \phi_3 \otimes \phi_2\rangle$

$$e_2 = |\phi_1 \otimes \phi_3 \otimes \phi_2\rangle + |\phi_3 \otimes \phi_1 \otimes \phi_2\rangle - |\phi_1 \otimes \phi_2 \otimes \phi_3\rangle - |\phi_3 \otimes \phi_2 \otimes \phi_1\rangle . \tag{4.64}$$

The action of $\rho_{\text{stand.}}(1,2) = \rho(1,2)$ on e_1, e_2 is then just to interchange labels 1 and 2 (or a and b in the alternative notation).

S_4 class	$[()]$	$[(1,2)]$	$[(1,2)(3,4)]$	$[(1,2,3)]$	$[(1,2,3,4)]$
size class	1	6	3	8	6
$\chi(\rho_{\mathbf{1}})$	1	1	1	1	1
$\chi(\rho_{\mathbf{1}_1})$	1	-1	1	1	-1
$\chi(\rho_{\mathbf{3}_1})$	3	1	-1	0	-1
$\chi(\rho_{\mathbf{3}_2})$	3	-1	-1	0	1
$\chi(\rho_{\mathbf{2}})$	2	0	2	-1	0

Table 4.4: Character table of S_4 as obtained in a previous exercise. Note $\mathbf{1}' \equiv \mathbf{1}_1$ as with regard to previously used notation.

– Show that this leads to

$$\rho(1,2) = \begin{pmatrix} .. & -1 \\ .. & -1 \end{pmatrix} \tag{4.65}$$

with dots to be completed by you. Obtain $\rho(2,3)$ (which takes on a rather simple form by construction) and $\rho(1,2,3)$. Verify that

$$\rho(1,2,3) = \rho(1,2)\rho(2,3) = \begin{pmatrix} .. & 1 \\ .. & 0 \end{pmatrix} \tag{4.66}$$

is indeed satisfied by explicit matrix multiplication.

– (optional) For those who feel that they need further practice can obtain the other elements by matrix multiplication using the rules in the multiplication table given in the lecture and then check that the representation is indeed irreducible.

c) Practice Kronecker products. Verify a few of the following Kronecker products

$$\begin{aligned} \mathbf{3}_1 \otimes \mathbf{3}_1 &= (\mathbf{1} \oplus \mathbf{2} \oplus \mathbf{3}_1)_s \oplus (\mathbf{3}_2)_a \,, \\ \mathbf{3}_1 \otimes \mathbf{2} &= \mathbf{3}_1 \oplus \mathbf{3}_2 \,, \\ \mathbf{3}_1 \otimes \mathbf{3}_2 &= \mathbf{1}_1 \oplus \mathbf{2} \oplus \mathbf{3}_1 \oplus \mathbf{3}_2 \,, \end{aligned}$$

using the S_4 character table 4.4. Note the subscripts s and a stand for the symmetric and antisymmetric subspace and are not of importance for this exercise per se.

Chapter 5

Lie groups: $U(1) \simeq SO(2)$, $SO(3) \simeq SU(2)/\mathbb{Z}_2$

Lie groups were discovered by the Norwegian mathematician Sophus Lie (1842-1899) in attempting to find a Galois theory (solving algebraic equations) for differential equations. Technically they correspond to manifolds which are compatible with a group structure. By no means does every manifold admit a group structure.

In some sense Lie groups are more complicated than finite groups but the classification of the simple and compact Lie groups due to Cartan and Dynkin is remarkably simple and ingenious at the same time. The rigidity of the group axioms and the manifold structure are restrictive enough for a short and concise classification unlike the thousand of research papers that lead to the classification of simple finite groups. The key observation is that the linearisation of the Lie group, which corresponds to the so-called Lie Algebra, is sufficient to study many of the properties of the Lie groups. Hence the apparatus of linear algebra comes into play and allows to make strong statements.

We will only be concerned with the four Lie groups $U(1)$, $SO(2)$, $SU(2)$ and $SO(3)$ which are related, as groups as follows:

$$U(1) \simeq SO(2) \ , \quad SO(3) \simeq SU(2)/\mathbb{Z}_2 \ , \tag{5.1}$$

where the symbol $\simeq$ stands for isomorphic. In the first section 5.1 we shall, nevertheless, keep the definitions general so that you can apply them in your further studies to more complicated Lie groups. Lie groups are of major importance in particle physics in term of gauge symmetries (the Standard Model, that is to say the most fundamental theory of particles, has a gauge symmetry $U(1) \otimes SU(2) \otimes SU(3)$) or so-called global symmetries (e.g. isospin $SU(2)$ the approximate symmetry under interchange of up and down quarks).

5.1 Basic definitions

Definition 5.1.1 ***Continuous group:*** a ***continuous group*** is a group G with an infinite number of elements which depends continuously on some parameters $\{\alpha\} \equiv \{\alpha_1, .., \alpha_{\mathrm{dim}G}\}$. The number of continuous parameters $\mathrm{dim}G$ is known as the ***dimension*** of the group.

Note that it does not make sense to talk of the order of the group as it is simply infinite.

Definition 5.1.2 ***Lie group:*** a ***Lie group*** is a continuous group which admits an analytic structure in the parameters $\{\alpha\}$. (Therefore corresponds to a manifold).

Examples of Lie groups are:

Examples 5.1.1 ***Lie groups acting on*** $\vec{x} \in \mathbb{R}^3$ [1]

a) Spatial translations, $x_i \mapsto x_i + a_i$; $\mathrm{dim}G = 3$ (three parameters a_i correspond to the α_i above).

b) Spatial rotations ($O(3)$), $x_i \mapsto R_{ij}(\theta)x_j$; $R^T R = \mathbb{I}$; $\mathrm{dim}G = 3$ (three Euler angles θ_i correspond to the α_i above).

Definition 5.1.3 ***Basic characterisation of Lie groups:*** Lie groups are distinguished by several characteristic properties, which we list in the table below. In addition we indicate the properties of the groups we will be concerned with in this course.

properties	$U(1)$	$SO(2)$	$SU(2)$	$SO(3)$
<u>f</u>inite or <u>i</u>nfinite dimensional (as a manifold)	f	f	f	f
<u>r</u>eal or <u>c</u>omplex (as a manifold)	r	r	r	r
<u>c</u>ompact or <u>n</u>on-<u>c</u>ompact (as a manifold)	c	c	c	c
<u>s</u>imply-<u>c</u>onnected, <u>c</u>onnected or <u>d</u>isconnected (as a manifold)	c	c	sc	c
<u>s</u>imple, <u>s</u>emi-<u>s</u>imple or <u>c</u>omposite	s	s	s	s

Of these properties the first two should be clear and the last three demand some explanation. Further the statement that $SU(2)$ is complex is refined to $SU(2)$ being pseudo-real later on.

Compactness On a metric space (a space with a distance) ***compact*** is equivalent to closed[2] and bounded (finite distance). For example $[a,b]$ is compact whereas the intervals $[a,b[$, $]a,b]$ or $]a,b[$ are not compact. More concretely and relevant for our discussion: the circle S^1 is compact where the real line $\mathbb{R}$ is not. Hence example 5.1.1a) is not a compact Lie group whereas example 5.1.1b) is a compact Lie group. We will only be concerned with compact Lie groups in this course. The important point about compact Lie groups are that all the theorems in chapter 3, such as Maschke's theorem (unitary representations), decomposability etc, apply. This is the case since the replacement of the discrete sum $\sum_{g\in G}$ with the so called *Haar measure* $\int d\{\alpha\}$ is unproblematic, in the sense, that all proofs are essentially the same. For pedagogic reason let us give a counterexample for a non-compact group.

Examples 5.1.2 ***A non-compact non-decomposable Lie group*** Consider the translation group $\mathbb{R}$ then the following representation,

$$\rho(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \ , \quad \rho(a)\rho(b) = \rho(a+b) \tag{5.2}$$

but does *not* decompose subspaces. The direction $\vec{e}_x = (1,0)$ is an invariant subspace but the orthogonal complement $\vec{e}_y = (0,1)$ is not. This is due to the fact that the translation group is non-compact!

[1] Be warned the fact that $\mathrm{dim}G = 3$ and $\mathrm{dim}R^3 = 3$ is not a rule (though certainly related). The rotation group in two ($O(2)$) and four dimension ($O(4)$) have got $\mathrm{dim}O(2) = 1$ and $\mathrm{dim}O(4) = 6$ respectively.

[2] Every converging sequence converges to a value in the set: C closed, $a_n \in C$, $a = \lim_{n\to\infty} a_n \in C$.

Connectedness A set is ***connected*** if there is a smooth path from each point of the set to any other point of the set. It is in particular ***simply connected*** if all loops can be contracted in a smooth fashion to a point. A set which is not connected is disconnected. In Fig. 5.1 three illustrative and hopefully clarifying examples are given. We shall only be concerned with connected Lie groups. We

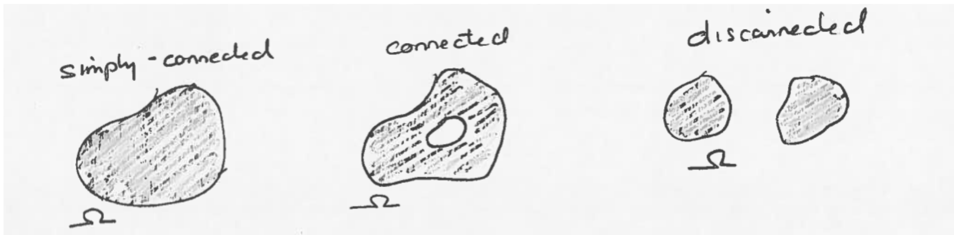

Figure 5.1: (left) simply connected set (middle) connected set which is not simply connected (right) disconnected set

will briefly discuss an example of a non-connected group in the next chapter in example 5.2.1 so that you will get an idea of what is going on. The fact that $SU(2)$ is simply connected and $SO(3)$ only connected is related to the existence of fermions.

Simplicity A ***simple*** Lie group is a connected non-abelian Lie group which does not admit (non-trivial) connected normal subgroups. This connects to the definition of a simple group for finite groups. Similarly simple Lie groups are the atoms of Lie group since no (Lie) quotient groups can be defined. ***Semi-simple*** Lie groups are direct products of simple Lie groups. *Composite* Lie groups are composed of simple Lie groups through semi-direct products or more complicated operations and are beyond the scope of this course. We shall only be concerned with simple Lie groups[3] in this course.

Some examples of simple and compact Lie groups used in physics are:

Examples 5.1.3 ***Examples of finite, simple and compact Lie groups are:***

- ***Special orthogonal groups*** are real.[4]

$$SO(n) = \{O \in GL(n, \mathbb{R}) \mid O^T O = \mathbb{I}_n \, , \det O = 1\} \tag{5.3}$$

- ***Special unitary groups*** are complex.

$$SU(n) = \{U \in GL(n, \mathbb{C}) \mid U^\dagger U = \mathbb{I}_n \, , \det U = 1\} \tag{5.4}$$

- ***Symplectic groups*** are complex.

$$USp(2n) \equiv Sp(n) = \{S \in GL(2n, \mathbb{R}) \mid S^T J S = J \, , J = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}\} \tag{5.5}$$

  play a role in Hamiltonian dynamics. Note $J^2 = -\mathbb{I}_{2n}$ and hence J is in some sense a square root of minus one.

[3]Including the abelian Lie groups in the simple category.
[4]The S-denotes the determinant condition.

G	dim	rank
A_n $[SU(n+1)]$	$(n+1)^2-1$	n
B_n $[SO(2n+1)]$	$(2n+1)n$	n
C_n $[USp(2n)]$	$n(2n+1)$	n
D_n $[SO(2n)]$	$n(2n-1)$	n
E_6	78	6
E_7	133	7
E_8	248	8
F_4	52	4
G_2	14	2

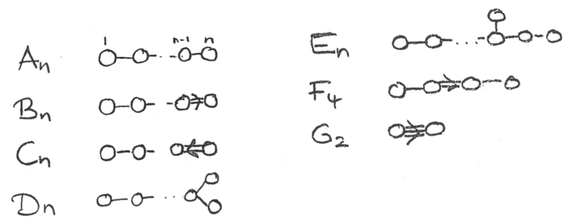

Table 5.1: Dimension and rank (which is the dimension of the Lie Algebra, c.f. section 5.3) of simple and compact Lie groups. The so-called exceptional groups $E_{6,7,8}$, F_4 and G_2 are mentioned for completeness only and do not play any rôle in the remainder of the course. They are called exceptional because they do not fall into any of four first series. The corresponding Dynkin diagrams are shown on the right for completeness only without any in depth explanation other than the remark that the number of circles corresponds to the rank. Symmetries can be inferred from the Dynkin diagrams. For example some types of low ranks are degenerate. E.g. $A_1 \simeq B_1$ ($SU(2) \simeq SO(3)$ which is relevant for the course). Or $D_2 \simeq A_1 \oplus A_1$ ($SO(4) \simeq SU(2) \otimes SU(2)$) which is the only example of a semi-simple type in this list.

The groups $U(n)$ and $O(n)$ are the same as $SU(n)$ and $SO(n)$ except that the determinant condition is not imposed. (The "S" stands for special). A relation between the two is given in (5.17) later on. The classification of the simple compact Lie groups has been done by Cartan and Dynkin and is astonishingly beautiful and simple. The study of which is beyond the scope of this course. In table 5.1 we have listed all groups for the sake of completeness. The dimension of the defining representations (5.3,5.4,5.5) is given in the middle column and the rank (to be defined in section 5.3) is given in the right column. For pedagogical reason let us give an example of a non-compact Lie group of importance in physics:

Examples 5.1.4 ***The Lorenzt-group*** $SO(3,1)$***: a non-compact Lie group***

$$SO(3,1) = \{\Lambda \in M_4(\mathbb{R}) \mid \Lambda^T \eta \Lambda = \eta \, , \det \Lambda = 1\} \, , \quad \eta = \mathrm{diag}(1,-1,-1,-1) \, . \tag{5.6}$$

A way to see that they are not compact is to note that they contain the boost to any reference frame of relative speed $v \in [0, c[$ where c is the speed of light. This is a non-compact interval and therefore $SO(3,1)$ cannot be a compact group.[5]

Much of the power of Lie groups relies on the realisation that, for many aspects, it is sufficient to linearise the problem in the parameters $\{\alpha\}$ mentioned earlier. By linearising one departs from the Lie group to the ***Lie Algebra*** which may be thought of as the tangent space of the manifold. By exponentiating the Lie Algebra elements one gets back the ***connected*** part of the Lie group (by which we mean the part connected to the identity). These concepts will be illustrated through the simple example of $U(1)$ in the next section. The end this introduction with the following mnemonic:

$$\begin{aligned} &\text{Linearisation: Lie group } \Rightarrow \text{ Lie Algebra ,} \\ &\text{Exponentiation: Lie Algebra} \Rightarrow \text{ connected part of Lie group} \end{aligned} \tag{5.7}$$

[5] One consequence of this fact is that there do not exist any finite dimensional unitary representations of the Lorentz group. (For the experts: this is why $\bar{\Psi} = \Psi^\dagger \gamma_0$ is necessary to define invariant of the form $\bar{\Psi}\Psi$ where Ψ is a spinor.)

Exercise 5.1.1 ***Generalities on compact simple Lie groups***

a) Check that the dimension of the matrices $SO(n)$ and $SU(n)$ are indeed n^2-1 and $n(n-1)/2$ in accordance with table 5.1.
Hint: Use the fact that $U = \exp iH$ and $O = \exp iA$ where H and A are hermitian traceless and antisymmetric respectively. Note an arbitrary $M_n(\mathbb{C})$-matrix has got $2n^2$ parameters.

b) Using the formula $\det(\exp(tr[A])) = \exp(tr[A])$ check that the determinant condition is satisfied if H from the previous exercise are traceless.

5.2 The two abelian groups: $U(1)$ and $SO(2)$

The groups $U(1)$ and $SO(2)$ can be defined as:

$$U(1) = \{z \in \mathbb{C} \mid |z| = 1\} \ , \quad SO(2) = \{O \in M_2(\mathbb{R}) \mid O^T O = \mathbb{I}_2, \ \det(O) = 1\} \ . \tag{5.8}$$

They are rather simple, as abelian groups tend to be, and in particular isomorphic to each other. They correspond to an angle of rotation around 2π. Representation matrices are given by

$$\rho_n(\alpha)|_{U(1)} = e^{in\alpha} \ , \quad \rho_n(\alpha)|_{SO(2)} = \begin{pmatrix} \cos(n\alpha) & \sin(n\alpha) \\ -\sin(n\alpha) & \cos(n\alpha) \end{pmatrix} \quad , n \in \mathbb{Z} \ , \quad \alpha \in [0, 2\pi[\tag{5.9}$$

which makes the isomorphism $U(1) \simeq SO(2)$ explicit. Intuitively they correspond to rotation around a single axis. All irreducible representations of $U(1)$ are given by one dimensional representation on $\mathbb{C}$. The irreducible representations of $SO(2)$ are given by two dimensional representations on $\mathbb{R}^2$. Note that the cyclic groups $\mathbb{Z}_k$ are finite subgroups of $U(1)$. The Kronecker product (Clebsch-Gordan series) of two representations is given by

$$U(1): \quad \rho_n \otimes \rho_m = \rho_{n+m} \tag{5.10}$$

for $U(1)$ (and therefore for $SO(2)$ as well.)

In principle we could end here but we consider it useful to illustrate the notions of exponentiation, Lie Algebra and connectedness in this simplified context. Let us make the following ansatz for $O \in SO(2)$[6]

$$O = \exp(i\alpha T) \ , \quad T \in M_2(\mathbb{R}) \ , \tag{5.11}$$

where the exponential map of matrices can be understood in the sense of a power series or any other well known limiting process

$$\exp(A) = \sum_{n \geq 0} \frac{A^n}{n!} = \lim_{n \to \infty} \left(1 + \frac{A}{n}\right)^n \ . \tag{5.12}$$

[6]Mathematicians often omit the factor of i in the exponential and absorb it into the generator. This is in some ways more convenient as you will notice. We will stick to the convention below which is the one adopted in the physics literature.

The exponential has the following property: $\det(\exp(A)) = \exp(\mathrm{Tr}(A))$ which can be shown by considering the eigenvalue expression of the determinant and the trace.

Note that the defining condition (5.9) (using $\exp(A)^T = \exp(A^T)$ which follows from (5.12))

$$\mathbb{I}_2 = O^T O = \exp(i\alpha T^T)\exp(i\alpha T)) = \mathbb{I}_2 + i\alpha(T^T + T) + \mathcal{O}(\alpha^2) \quad \Leftrightarrow \quad T^T = -T \ , \tag{5.13}$$

implies that T has got to be an antisymmetric matrix and this condition immediately generalises to $SO(n)$.[7] The following matrix,

$$T = -i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T^2 = \mathbb{I}_2, \quad e^{i\alpha T} = \mathbb{I}_2\cos(\alpha) + iT\sin(\alpha) = \rho(\alpha)_{SO(2)} \ , \tag{5.14}$$

leads to the representation (5.9). The generator T emerges from the linearisation of $e^{i\alpha T} = 1 + i\alpha T + \mathcal{O}(\alpha^2)$. In the case at hand the Lie Algebra is rather trivial since it has only a single element T and no further structure. Nevertheless it serves us well in illustrating or hinting at the connection with differential geometry in terms of manifolds (surfaces correspond to manifolds in two dimensions). In Fig. 5.2 we illustrate the manifold nature of $SO(2)$ and its tangent space. It is meant as an illustration only and is of no importance for the rest of the course other than broadening the viewpoint.

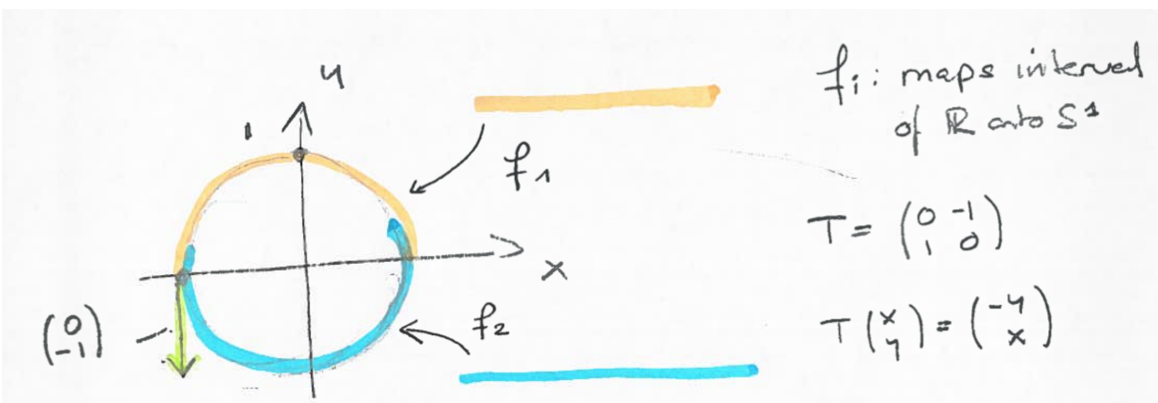

Figure 5.2: $SO(2)$-manifold S^1 embedded into $\mathbb{R}^2$. (S^1 corresponds to a circle). Shown are two maps from intervals on $\mathbb{R}$ which cover the manifolds. An explicit realisation of f_1 is given by $f_1(\theta) = (\cos(\theta), \sin(\theta)$ for example. It is explicitly seen that T, the (Lie Algebra) generator of $SO(2)$, corresponds to the tangent vector. Illustrated for $(x, y) = (-1, 0)$ (in green) which leads to $T(-1, 0) = (0, -1)$.

Another element which wish to illustrate is *(dis)connectedness*.

Examples 5.2.1 $O(2)$***: A disconnected Lie group.*** The manifold $SO(2)$ is connected as can be seen from the figure but $O(2)$ is not. Note that $\det(O) = \pm 1$ follows from $O^T O = \mathbb{I}_2$. Since O is differentiable, one *cannot* go in a continuous fashion from a point in the set $\det(O) = 1$ to a point of

[7]Note the last step on the left hand side in (5.13) is only valid if T commutes with its transpose as otherwise the Baker–Campbell–Hausdorff formula (to be discussed soon) has to be applied. Since T is an antisymmetric matrix it does commute with its transpose and so the steps are finally consistent.

the set $\det O = -1$ and hence $O(2)$ is not connected.[8] For example

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad P \in O(2) , \quad P \notin SO(2) \tag{5.15}$$

is an example of a matrix which is in $O(2)$ but not in $SO(2)$. One may think of P as a parity operator. In fact P generates the cyclic group $\mathbb{Z}_2$ which is a normal subgroup[9] of $O(2)$ and the following relation holds:

$$SO(2) \simeq O(2)/\mathbb{Z}_2 , \tag{5.16}$$

and slightly more generally

$$SO(n) \simeq O(n)/\mathbb{Z}_2 , \quad SU(n) \simeq U(n)/U(1) . \tag{5.17}$$

The symbol $\simeq$ stands for isomorphic.

It seems worthwhile to pause for a moment and try to reconnect to the chapter of finite groups. First we note that there ought to be, in general, an infinite amount of irreducible representations by virtue of a limiting process of the dimensionality theorem 4.1.2 ($|G| \to \infty$ implies infinitely many irreducible representations in Eq. (4.19)). For $U(1)$ the series is given in Eqs. (5.9). For $U(1)$ the Haar measure.[10] is rather simple. An inner product is defined as follow[11]:

$$\langle \chi_n | \chi_m \rangle |_{U(1)} \equiv \int_0^{2\pi} \frac{d\alpha}{2\pi} \chi(\rho_n(\alpha))^* \chi(\rho_m(\alpha)) = \int_0^{2\pi} \frac{d\alpha}{2\pi} e^{i\alpha(m-n)} = \delta_{mn} , \tag{5.18}$$

under which the χ_n are orthonormal. The characters will not play a pivotal rôle in the analysis of of the representations as we will find that the so-called highest weight representations of Lie Algebra to be a powerful tool to construct representations.

Exercise 5.2.1 *$U(1)$ **and** $SO(2)$*

a) Check that $\rho_n(\alpha_1)_{SO(2)}\rho_n(\alpha_2)_{SO(2)} = \rho_n(\alpha_1 + \alpha_2)_{SO(2)}$ using the representation (5.9). Trigonometric identities can easily derived using Euler's formula $e^{ia} = \cos(a) + i\sin(a)$.

b) Using Euler's formula verify (5.14).

c) Conceptual question: what would happen in case n is not chosen to be an integer in (5.9)?

[8] If one were to complexify the group which merely amounts to allow for complex $\{\alpha\}$ then the determinant would take on any complex value and one could go from any point to another in a continuous fashion. Hence the complexification of $O(2)$ is connected. This general knowledge is encapsulated in Jacques Hadamard's (1865-1963) famous quote: "the shortest path between two truths in the real domain passes through the complex domain".

[9] In particular any abelian subgroup is normal. Hence $\mathbb{Z}_2$, which is abelian, is normal.

[10] The Haar measure generalises $\frac{1}{|G|}\sum_{g\in G}$ from finite groups.

[11] The second orthogonality relation becomes $\langle \chi[\alpha] | \chi[\alpha'] \rangle_C |_{U(1)} = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} e^{in(\alpha-\alpha')} = \delta(\alpha - \alpha')$ for $\alpha, \alpha' \in [0, 2\pi[$.

5.3 Some generalities of (non-abelian) simple compact Lie groups

In this section we would like to clarify and elaborate on the aspect of the necessity of the exponential map and the structure of the Lie Algebra. The exponentiation of the latter gives rise to the group structure.

Theorem 5.3.1 ***exponential map***
Each one-parameter subgroup of $GL(n,\mathbb{C})$ is given by a matrix exponential $\rho_T(t) = \exp(tT)$ where $T \in M_n(\mathbb{C}) \neq 0$ is the generator.

Proof: Consider a one-parameter homomorphism $\rho_T(t) : \mathbb{R} \to GL(n,\mathbb{C})$ defined by:

$$\frac{d}{dt}\rho_T(t)|_{t=0} = T \ , \tag{5.19}$$

with boundary condition $\rho_T(0) = \mathbb{I}$. Since ρ_T is a group homomorphism

$$\rho_T(s+t) = \rho_T(s)\rho_T(t) \ . \tag{5.20}$$

Differentiating w.r.t. to s and setting $s = 0$, leads to

$$\frac{d}{dt}\rho_T(t) = T\rho_T(t) \quad \Rightarrow \quad \rho_T(t) = \exp(tT) \ , \tag{5.21}$$

a first order linear differential equation with unique solution as indicated q.e.d.

Next we want to explore what constraints the group axioms impose on the structure of the generator T in general. Higher dimensional groups are parameterised by several parameters as discussed at the beginning of the chapter (e.g. the three-dimensional rotation group $SO(3)$ is described by the three Euler angles.). Motivated by the exponential form we start with the following ansatz:

$$\begin{aligned}
\rho_1 &= \mathbb{I} + i\alpha^a T_a + \frac{1}{2}(i\alpha^a T_a)^2 + \mathcal{O}(\alpha^3) \ , \\
\rho_2 &= \mathbb{I} + i\beta^b T_b + \frac{1}{2}(i\beta^b T_b)^2 + \mathcal{O}(\beta^3) \ , \\
\rho_3 &= \mathbb{I} + i\gamma^c T_c + \frac{1}{2}(i\gamma^c T_c)^2 + \mathcal{O}(\gamma^3) \ ,
\end{aligned} \tag{5.22}$$

where summation over repeated indices a, b and c is understood $a = 1..d_{\mathrm{LA}}$. (In the case of $SO(3)$, α^a correspond to the three Euler angles.) Further we note that the precoefficient of the quadratic term follows from $\rho(\alpha)^2 = \rho(2\alpha)$ which in turn derives from (5.20) or generally from theorem 5.3.1. The matrices T_a are the ***Lie Algebra generators***.

Multiplication of two exponential of matrices is different from normal multiplication of two exponentials $\exp(a)\exp(b) = \exp(a+b)$ for $a, b \in \mathbb{C}$ in that the two matrices do not necessarily commute. The formula governing the arithmetics is the famous ***Baker–Campbell–Hausdorff formula***:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{3!}\frac{1}{2}([A,[A,B]]+[[A,B],B])+\cdots} \ , \tag{5.23}$$

where $[A, B] = AB - BA$ denotes the commutator. In order to assess the structure of the Lie Algebra we are going to investigate under what conditions

$$\rho_3 = \rho_1 \rho_2 \rho_1^{-1} \rho_2^{-1} \,, \tag{5.24}$$

is consistent with the linear ansatz in (5.22). Eq. (5.24) can be considered as the definition of the element ρ_3 for instance. In order to determine the conditions on the $\alpha^a, \beta^b, \gamma^c$'s we substitute (5.22) into (5.24) which leads to

$$[\alpha^a T_a, \beta^b T_b] = -i\gamma^c T_c + \mathcal{O}(\alpha^3, \beta^3, \gamma^3) \,, \tag{5.25}$$

where all lower order terms cancel. The solution is given by $\gamma^c = -f_{ab}{}^c \alpha^a \beta^b$ provided that the generators satisfy the so-called ***Lie Algebra***

$$\text{Generic Lie Algebra: } [T_a, T_b] = i f_{ab}{}^c T_c \,. \tag{5.26}$$

Summation over c is again understood and $f_{ab}{}^c$ are known as the ***structure constants*** of the Lie Algebra which are in particular antisymmetric in the indices a and b. This answers the question what kind of structure the generators have to satisfy in order to be compatible with the group structure. Crucially the Lie Algebra only depends on the commutator, as is inherent in the Baker–Campbell–Hausdorff formula (5.23), and not on the anti-commutator. The latter depends on the representation of the Lie Algebra.[12]

An important notation is the ***rank*** of a Lie Algebra , which is the maximal number of Lie Algebra generators that commute with each other (and not to be confused with the dimension of the Lie Algebra $d_{\text{LA}} \geq$ rank). For the simple compact groups this number is indicated in table 5.1 on the right hand side and corresponds to the number of dots in the Dynkin diagrams. Its instrumental in constructing the representations. Note, the rank of $SO(3)$ and $SU(2)$ is one. Hence they are the simplest non-abelian Lie groups but nevertheless rich enough to keep us busy until the end of the course! We note that the matrix T (5.14) plays, the slightly degenerate, rôle of the Lie Algebra of $SO(2)$.

Theorem 5.3.2 ***Lie Algebra $\leftrightarrow$ unique simply connected Lie group***
To each given Lie Algebra corresponds a unique Lie Group which is simply connected. This group is known as the ***universal covering group***.[13]

We will not present the proof. The impotent point is that for identical Lie Algrabra's correspond in general several Lie groups. Only under the restriction to simply connectedness the exponential map is 1-to-1 (bijective). The other not simply connected groups can be obtained as quotient groups.

Theorem 5.3.3 ***Lie Algebra generators of a compact group are hermitian***
For compact Lie groups there exist a Lie Algebra basis where the generators are hermitian ($T_a^\dagger = T_a$).

Proof: (sketch) The part connected to the identity matrix can be written as $\rho = \exp(i\alpha^a T_a)$ and since compact groups admit finite dimensional *unitary* representations,

$$\begin{aligned} \mathbb{I}_n &= \exp(i\alpha^a T_a)^\dagger \exp(i\alpha^a T_a) = \exp(-i\alpha^a T_a^\dagger) \exp(i\alpha^a T_a) && (5.27) \\ &= \mathbb{I}_n + i\alpha^a (T_a - T_a^\dagger) + \mathcal{O}(\alpha^2) \quad \Leftrightarrow \quad T^\dagger = T \,, && (5.28) \end{aligned}$$

expansion in the parameter α-reveals the condition.

[12] For $SU(2)$ the anti-commutator vanishes for all representations and is related to the fact that $SU(2)$ is of rank one.
[13] A term which derives from the field of topology.

Corollary 5.3.1 The ***Hilbert-Schmidt inner product*** on $M_n(\mathbb{C})$ assumes the form

$$\langle T_a | T_b \rangle \equiv \mathrm{Tr}[T_a^\dagger T_b] \overset{\text{thm } 5.3.3}{=} \mathrm{Tr}[T_a T_b] \ . \tag{5.29}$$

Theorem 5.3.4 For compact, semi-simple Lie algebras there exists an orthogonal basis,

$$\langle T_a | T_b \rangle = \mathrm{Tr}[T_a T_b] = 2k_R \delta_{ab} \ . \tag{5.30}$$

The number k_R is the called the ***Dynkin index*** and depends on the representation (hence the subscript R). In this basis (5.30) the structure constants ${f_{ab}}^c = f_{abc}$ are *completely antisymmetric* in all three indices (c.f. box below).

We shall omit the proof. Below we give a slightly more precise statement which is though beyond the scope of this course and targeted at the interested reader only. Yet we shall use the result as a matter of convenience to lower all indices.[14]

*for the interested reader (**not** part of the course)*

Theorem 5.3.5 ***Killing metric***
One can define a unique symmetric tensor (***Killing metric***)

$$\mathrm{Tr}[T_a T_b] = k_R k_{ab} \ , \tag{5.31}$$

invariant under the action of the Lie group

$$0 = \delta_{T_c} \mathrm{Tr}[T_a T_b] = \mathrm{Tr}[[T_c, T_a] T_b] + \mathrm{Tr}[T_b [T_c, T_b]] = 2ik_R({f_{ca}}^b + {f_{cb}}^a) \ . \tag{5.32}$$

The number k_R is the *Dynkin index* which is dependent on the representation. Note the (complete) antisymmetry of ${f_{ab}}^c$ follows from the invariance of the killing metric (5.32). For semi-simple Lie groups the killing metric is non-singular and if in addition the group is compact then there exists a basis where

$$k_{ab} = 2\delta_{ab} \ . \tag{5.33}$$

Proof omitted. Note the Killing metric can be used to lower and raise indices. Since for semi-simple groups the Killing metric is the identity matrix we shall take a rather relaxed attitude towards upper and lower indices.

Theorem 5.3.6 ***General properties of structure constants f_{abc} for Lie groups***
The structure constant satisfy the ***Jacobi identity***:

$$f_{aed} f_{bce} + f_{bed} f_{cae} + f_{ced} f_{abe} = 0 \ . \tag{5.34}$$

Proof: (sketch) Any three matrices A, B, C satisfy (exercises):

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \ , \tag{5.35}$$

[14] For semi-simple but non-compact groups the metric $\langle T_a | T_b \rangle = k_{ab}$ is non-degenerate and can be used to lower indices.

and so does

$$(T_a^{\rm adj})_{bc} = i f_{abc} \ , \tag{5.36}$$

interpreted as a matrix q.e.d.The label adj(oint) is explained in the preceding corollary.

Corollary 5.3.2 ***The adjoint representation is real***

a) Eq. (5.36) defines a representation called the ***adjoint representation***.[15]

b) The structure constants are real. ($f_{abc} \in \mathbb{R}$)

Proof: The show that (5.36) is a representation is left as an (optional) exercise. Reality of f_{abc} can be seen as follows. If T_a generates a representation of the Lie Algebra so does $(-T_a^*)$, the complex conjugate representation.[16] By Taking the complex conjugate of (5.26) one gets:

$$[-T_a^*, -T_b^*] = i f_{abc}^* (-T_c^*) \ . \tag{5.37}$$

Since the complex conjugate corresponds to the same group this implies $f_{abc}^* = f_{abc}$ q.e.d.

Exercise 5.3.1 ***Generalities on compact simple Lie groups***

a) Check that Eq. (5.35) is correct for any set of matrices.

b) Check that $\rho(\alpha)^2 = \rho(2\alpha)$ and $\rho(\alpha)^{-1} = \rho(-\alpha)$ using equations in the proof of theorem 5.3.1.

c) Verify Eq. (5.25).

d) (optional) Using the Jacobi identity (5.34) show that (5.36) is indeed a Lie Algebra representation of the form (5.26).

5.4 The two non abelian groups: $SU(2)$ and $SO(3)$

5.4.1 Lie Algebras of $SO(3)$ and $SU(2)$

After the preparatory steps of the preceding chapter it is not difficult to write down the Lie Algebra of $SO(3)$ (usually denoted by $so(3)$ lower case letters). From exercise 5.1.1 we know that any three linearly independent antisymmetric matrices will generate the algebra. A basis for which the structure constant is completely antisymmetric, which is a possibility according to theorem 5.3.4, is given by:

$$T_1 = -i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} , \qquad T_2 = -i \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \qquad T_3 = -i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{5.38}$$

[15] The adjoint representation is reminiscent of the regular representation in that the element of the set are declared to be linearly independent vectors defining a vector space. The adjoint representation acts on the vector space of the Lie Algebra as follows: $(T_a^{\rm adj})T_d = [T_a, T_d] = i f_{ade} T_e$.

[16] The minus sign originates from the factor i in the parameterisation $\exp(i\alpha^a T_a)$ of Lie group elements.

The first two indices of T_3 are equal to the $SO(2)$ rotation T in (5.14) and corresponds to a rotation around the z-axis. The remaining $T_{1,2}$ are just permutations thereof. The Lie Algebra is readily computed to be

$$\text{Lie Algebra:} \quad [T_a, T_b] = i\epsilon_{abc} T_c \ , \tag{5.39}$$

where the structure constant $f_{abc} = \epsilon_{abc}$ is the completely antisymmetric Levi-Civita tensor ($\epsilon_{123} = 1$). In accordance with theorem 5.3.6 the structure constant is real and totally antisymmetric as claimed above. From $\mathrm{Tr}[T_a T_b]_{(5.38)} = 2\delta_{ab}$, $(k_R)_{(5.38)} = 1$ by (5.30).

The Lie Algebra of $SU(2)$ does not follow, at least in the first instance, in such a geometric manner. We know though that the Lie Algebra generators of $SU(n)$ are hermitian and traceless matrices (according to exercise 5.1.1). The group $SU(2)$ is of dimension three and the three generators can be chosen to be

$$(T_a)_{SU(2)} = \frac{\sigma_a}{2} \ , \tag{5.40}$$

proportional to the famous Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{5.41}$$

This implies that (5.40) satisfies the Lie Algebra (5.39). Yet $\mathrm{Tr}[T_a T_b]_{(5.40)} = 1/2\delta_{ab}$ and hence the Dynkin index $(k_R)_{(5.40)} = 1/4$. *What is going on?* This means that the Lie groups $SU(2)$ and $SO(3)$ are locally isomorphic (same Lie Algebra) but are globally different (unequal Dynking index). This is indeed possible as they are of the same rank; namely of rank one. By virtue of theorem 5.3.2 there is a unique simply connected Lie group associated with each Lie Algebra. In the case at hand this is $SU(2)$ as we wish to demonstrate. Consider the mapping from $\mathbb{R}^4 \to SU(2)$:

$$\phi : (x_0, x_1, x_2, x_3) = x_0 \mathbb{I}_2 + x_1 i\sigma_1 + x_2 i\sigma_2 + x_3 i\sigma_3 = \begin{pmatrix} x_0 + ix_3 & ix_1 + x_2 \\ ix_1 - x_2 & x_0 - ix_3 \end{pmatrix} = U \tag{5.42}$$

The two defining conditions of $SU(2)$ lead to:

$$UU^\dagger = \mathbb{I}_2 \ , \quad \det U = 1 \quad \Leftrightarrow \quad x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 \ . \tag{5.43}$$

It follows that $SU(2)$ is topologically equivalent to the three sphere S^3. The latter is simply connected and therefore $SU(2)$ is *the unique* simply connected Lie group corresponding to the Lie Algebra (5.39). Hence if $SO(3)$ is not identical to $SU(2)$, which it is clearly not, then there must exist a n-to-1 homomorphism from $SU(2)$ to $SO(3)$. The latter is given by the so-called ***Weyl-homomorphism***

$$h(\vec{x}) = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 \ , \tag{5.44}$$

which obeys

$$U h(\vec{x}) U^\dagger = h(O\vec{x}) \ , \qquad U = e^{i\alpha^a (T_a)_{SU(2)}} \ , \quad O = e^{i\alpha^a (T_a)_{SO(3)}} \ . \tag{5.45}$$

Hence the Weyl homomorphism is a 2-to-1 map $\rho_{SU(2)} \to \rho_{SO(3)} : U \to O$. The kernel of this homomorphism is given by $\{\pm \mathbb{I}\} \simeq \mathbb{Z}_2$ and by virtue of the first isomorphism theorem 2.2.8 the following isomorphism holds

$$SO(3) \simeq SU(2)/\mathbb{Z}_2 \ . \tag{5.46}$$

We note that $SO(3)$ is not simply connected since $SU(2)$ is the unique simply connected Lie group associated to (5.39). This has profound implications and is related to the existence of fermions as we shall see.

Exercise 5.4.1 $SO(3)$ ***and*** $SU(2)$ ***Lie algebra***

a) Verify that (5.40), (5.38) satisfy the Lie Algebra (5.39).

b) Verify that the representations in a) satisfy (5.30) with $k_R = (1/4, 1)$ respectively.

5.4.2 Irreducible representations of $SU(2)$ (and hence $SO(3)$)

In view of the 2-to-1 Weyl homomorphism it is sufficient to construct the irreducible representations of $SU(2)$ from which the irreducible representations of $SO(3)$ will follow. We shall apply the very general method of *highest weight representation* which can be applied to any of the Lie Algebra's of the groups given in table 5.1. The complication raises with the rank. of the irreducible representations. In fact to each $r =$ rank is associated a so-called ***Casimir*** operator which is an operator constructed from the Lie Algebra element which commutes with the Lie Algebra. Irreducible representations are labeled by r labels.

The group $SU(2)$ is of rank one and has therefore got one Casimir operator. In this section we shall denote, because of reasons of familiarity, the generators of the $SU(2)$ Lie Algebra by

$$J_a \equiv T_a \text{ of Eq. (5.39)} \tag{5.47}$$

The quadratic Casimir operator, for any Lie group, is given by $T^2 \equiv k^{ab} T_a T_b$ (with summation over indices a and b understood); when applied to $SU(2)$ one gets:

$$J^2 = J_1^2 + J_2^2 + J_3^2 \ , \quad [J^2, J_i] = 0 \tag{5.48}$$

in the basis (5.31). The Casimir operator is of the form,

$$J^2 = C_2(R) \mathbb{I}_{\dim V_R} \ , \tag{5.49}$$

by virtue of Schur's lemma! The eigenvalue $C_2(R)$ (which will be identified by $j(j+1)$ latter on) serves to label the irreducible representation. As a set of maximally commuting operators we choose,[17]

$$\{J^2, J_3\} \ . \tag{5.50}$$

The eigenvalue of J_3, denoted by m, labels the states in a given irreducible representation:

$$J_3 e_m = m e_m \ , \quad m \in \mathbb{R} \ , \tag{5.51}$$

where e_m denotes a basis state in a given representation. From the Lie Algebra the following algebraic relations are immediate:

$$[J_3, J_\pm] = \pm J_\pm \ , \quad J_\pm \equiv J_1 \pm i J_2 \quad ((J_\pm)^\dagger = J_\mp) \tag{5.52}$$

$$J^2 = J_3^2 + J_3 + J_- J_+ \ , \tag{5.53}$$

$$J^2 = J_3^2 - J_3 + J_+ J_- \ . \tag{5.54}$$

[17] In a quantum system that exhibits rotational symmetry, one would add the Hamiltonian to this set of commuting observables. Here we try to extract the group theoretic aspects only.

One concludes that with $J_\pm$ one may construct,

$$J_3(J_\pm e_m) = (m \pm 1)(J_\pm e_m) \;, \tag{5.55}$$

and therefore $e_{m\pm 1} \propto (J_\pm e_m)$ is an orthogonal vector (since J_3 is a hermitian operator). That is to say provided that (5.55) does not yield zero. The operators $J_\pm$ are usually referred to as ***ladder operators*** for the obvious reason that they allow to go up and down in the representation space as illustrated in Fig. 5.3.

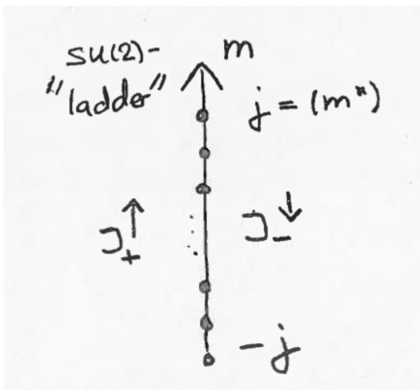

Figure 5.3: Graphic illustration of *highest weight representation* of $SU(2)$. The $J_\pm$ ladder operator allow to move up and down in representation space. Eq. (5.56) corresponds to the definition of the highest weight state $m = j(-m^*)$. Other states are obtained through Eq. (5.69) by applying J_-.

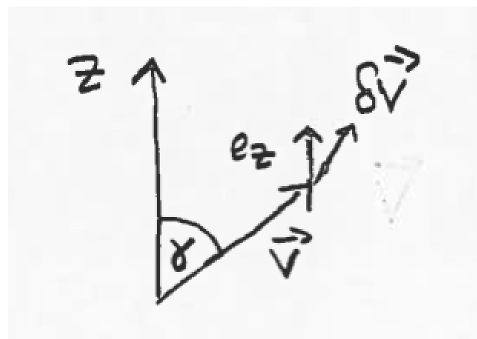

Figure 5.4: Geometric interpretation of angular momentum acting on a vector operator v_i^1 where the one stands for $j = 1$.

We have got all the tools to construct the highest weight representation. Our goal is to construct a finite dimensional representation. For the latter there exists a maximal m^* (top of the ladder) for which

$$J_+ e_{m^*} = 0 \;. \tag{5.56}$$

This state is called the highest weight state. Acting with (5.53) on this state one gets

$$J^2 e_{m^*} = m^*(m^*+1) e_{m^*} \;. \tag{5.57}$$

This relation remains true for all other states $e_{m^*-n} \propto (J_-)^n e_{m^*}$ since $[J^2, J_-] = 0$ as J^2 is a Casimir operator. Hence we write

$$J^2 e_{m^*-n} = m^*(m^*+1) e_{m^*-n} \;. \tag{5.58}$$

Since the representation is finite dimensional there must exists a positive integer n^* for which

$$J_- e_{m^*-n^*} = 0 \;. \tag{5.59}$$

Applying J^2 (5.54) on this state and using the fact that

$$\begin{aligned} J^2 e_{m^*-n^*} &\overset{(5.54)}{=} (m^*-n^*)((m^*-n^*)-1) e_{m^*-n^*} \;, \\ J^2 e_{m^*-n^*} &\overset{(5.58)}{=} m^*(m^*+1) e_{m^*-n^*} \;, \end{aligned} \tag{5.60}$$

once concludes that

$$2m^* = n^* \in \mathbb{Z}^+ \,, \tag{5.61}$$

the irreducible representations are labelled by *half-integer*.

Going over to more standard notation we choose $j \equiv m^*$ and denote the state vectors by

$$e_m \to |j, m\rangle \,, \tag{5.62}$$

where we have introduced in addition the letter j which labels the irreducible represenation. In summary the irreducible representations of $SU(2)$ are labelled by half integer j and they are $(2j+1)$ dimensional with m ranging from $m = \{-j, -(j-1), .., (j-1), j\}$.

In particular we conclude that the $SO(3)$ representation (5.38) corresponds to $j = 1$ (note $(2 \cdot (j = 1) + 1) = 3$ is three dimensional) and the $SU(2)$ representation (5.40) corresponds to $j = 1/2$. Hence (5.40) is the smallest irreducible representation for $SU(2)$ and (5.38) is the smallest irreducible representation of $SO(3)$. The matrix elements of the highest weight representation are given by:

$$\left(J_3^{(j)}\right)_{mm'} = \langle j, m|J_3|j, m'\rangle = m\delta_{mm'} \,, \tag{5.63}$$

$$\left(J_\pm^{(j)}\right)_{mm'} = \langle j, m|J_\pm|j, m'\rangle = N_\pm(j, m)\delta_{m,m'\pm 1} \,, \tag{5.64}$$

with normalisation factor

$$N_\pm(j, m) \equiv [(j \mp m)(j \pm m + 1)]^{1/2} = [j(j+1) - m(m \pm 1)]^{1/2} \,. \tag{5.65}$$

The equations above fully specify the representations of the algebra. The representation ρ themselves are given by

$$\rho_j(\vec{\alpha})_{mm'} = \langle j, m|e^{i\alpha^a T_a}|j, m'\rangle \,, \tag{5.66}$$

which connects us back to the way we have been handling representations for finite groups. Note j denotes the representation, α "enumerates" the group elements and m, m' parameterise the $(2j+1) \times (2j+1)$-dimensional matrix in the so-called highest weight representation. A specific parameterisation,

$$D^j(\alpha, \beta, \gamma)_{mm'} = \langle j, m|e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z}|j, m'\rangle \equiv e^{-i(m\alpha + m'\gamma)} d^j(\beta)_{mm'} \tag{5.67}$$

is known as the ***Wigner D-matrix*** ($d^j_{mm'}(\beta)$ is sometimes called the little Wigner D-matrix). It is widely used in the literature and ties in nicely with calculation of angular decay distribution amplitudes. Note J_x does not need to be parameterised since it is implicitly generated through commutators of J_y and J_z.

The quadratic Casimir and the Dynkin index are given by:

$$C_2(j) = j(j+1) \,, \quad k_j = j(j+1)\frac{(2j+1)}{6} \,, \tag{5.68}$$

where the first equation follows from (5.58) with $m^* = j$ and the second one is left as an exercise.

Exercise 5.4.2 ***Irreps of*** $SU(2)$

a) Derive Eqs. (5.52,5.53,5.54) using the commutation relations (5.39).

d) Verify the eigenvalue equation for the ladder states in Eq. (5.55).

c) Prove that (5.65) is the correct normalisation factor of

$$J_{\pm}|j,m\rangle = N_{\pm}(j,m)|j,m \pm 1\rangle \,, \tag{5.69}$$

up to an absolute value. *Hint:* You may want to use: (5.53) or (5.54) in doing so as well as the completeness relation in $\mathbb{I}_{2j+1} = \sum_m |j,m\rangle\langle j,m|$.

d) Obtain $J_{x,y,z}$ from Eqs. (5.63,5.64) for $j = 1$. Check that $[J_x, J_y] = iJ_z$ by explicit matrix multiplication. How do the rotation matrices compare with (5.38)?

e) (optional)

- Take the trace of the Casimir

$$T^2 = \delta_{ab} T^a T^b = C_2(R) \mathbb{I}_{\dim V_R} \tag{5.70}$$

to establish[18]

$$2k_R \dim V_{\text{adj}} = C_2(R) \dim V_R \,. \tag{5.71}$$

- For $SU(2)$ establish (5.68):

$$k_j = j(j+1)\frac{(2j+1)}{6} \tag{5.72}$$

and hence $(k_{1/2}, k_1, k_{3/2}, ..) = (1/4, 1, 3/2, ..)$. *Hint:* $C_2(j) = j(j+1)$, $\dim V_j = (2j+1)$, $\dim_{V_{\text{adj}}} = \dim_{V_1} = 3$. Make sure you understand all the equations in the hint.

5.4.3 Clebsch-Gordan series of $SU(2)$

From your course on quantum mechanics you know that when two particles of angular momentum j_1 and j_2 are combined then the total angular momentum ranges from $|j_1 - j_2|$ to $j_1 + j_2$ in integer steps. The addition of angular momentum corresponds to taking tensor product representations of states[19]. The product on the level of the group corresponds by virtue of the exponential map to addition at the level of the Lie Algebra.

Theorem 5.4.1 The ***Clebsch-Gordan series for*** $SU(2)$ (or Kronecker product) is given by

$$\rho_{j_1} \otimes \rho_{j_2} = \rho_{|j_1-j_2|} \oplus \rho_{|j_1-j_2|+1} \oplus ... \oplus \rho_{j_1+j_2} = \bigoplus_{J=|j_1-j_2|}^{j_1+j_2} \rho_J \,. \tag{5.73}$$

where two representations ρ_j are in an irreducible representation with Casimir value $j(j+1)$.

[19] Strictly speaking one assumes, in this simplified line of argument, that the mutual interaction will not change this picture. In fact it doesn't. The explanation of which is beyond the scope of this course.

A few explanations are in order. We see that the multiplicity of each representation is one which implies that $SU(2)$ is a simply reducible group (c.f. definition 6.3.2) as stated earlier on. A non-trivial check of this formula is given by matching up the dimensions,

$$(2j_1+1)(2j_2+1) = (2|j_1-j_2|+1) + (2(|j_1-j_2|+1)+1) + ...(2(j_1+j_2)+1) \,, \tag{5.74}$$

which is left as an exercise. A more substantial proof, from which we will use some results later on, is given below.

Proof: Let us compute the character of a Wigner rotation matrix. First consider a rotation around a single angle. This is in fact sufficient since any other rotation is equivalent by similarity transformation and has therefore identical character. Let us choose $(\alpha, \beta, \gamma) = (\theta, 0, 0)$ for which the rotation matrix (5.67) is diagonal:

$$D^j(\theta,0,0)_{mm'} = e^{-im\theta}\delta_{mm'} \,. \tag{5.75}$$

From (5.75) one obtains (last step is left as an exercise):

$$\chi(\rho_j(\theta)) \equiv \chi_j(\theta) = \sum_{m=-j}^{j} \left(D^j(\theta,0,0)\right)_{mm} = \sum_{m=-j}^{j} e^{im\theta} = e^{-ij\theta}\sum_{m=0}^{2j}\left(e^{i\theta}\right)^m = \frac{\sin(j+1/2)\theta}{\sin\theta/2} \,, \tag{5.76}$$

where

$$\chi_{1/2}(\theta) = 2\cos(\theta/2) \,, \quad \chi_1(\theta) = 2\cos\theta + 1 \,, \tag{5.77}$$

are special cases. Using Eq. (5.76), we can prove the following identity:

$$\chi_{j_1}(\theta)\chi_{j_2}(\theta) = \chi_{j_1+j_2}(\theta) + \chi_{j_1-1/2}(\theta)\chi_{j_2-1/2}(\theta) \,. \tag{5.78}$$

Without loss of generality we mat assume that $j_1 \geq j_2$. Then this equation can be iterated $2j_2 - 1$ more times to

$$\chi_{j_1}\chi_{j_2} = \chi_{j_1+j_2} + \chi_{j_1+j_2-1} + \ldots + \chi_{j_1-j_2} \,, \tag{5.79}$$

and this shows that (5.73) is correct and therefore proves the theorem. q.e.d.

Corollary 5.4.1 ***Fundamental representation***
From the representations $j = 1/2$ (5.40) [and $j = 1$ (5.38)], known as ***fundamental representations***[20] of $SU(2)$ [and $SO(3)$ respectively], all other finite dimensional irreducible representations can be generated through Kronecker products (direct product representations).

The proof of this corollary is given by Eq. (5.73). The corollary makes it clear that the irreducible representations ($j = 0, 1, 2, ..$) of $SO(3)$ are just a sub-series of irreducible representations ($j = 0, 1/2, 1, 3/2, 2, ..$) of $SU(2)$. The identification follows through the representation dependent Casimir operator and or the Dynkin index (5.68).

A curious fact is that in particle physics the smallest constituent of matter are all in fundamental representations. Surely when particles organise themselves in representations other than the

[20]The term fundamental representations originates from the Dynkin-Cartan classification. A fundamental representation is a representation whose highest weight is a fundamental weight. For some groups there can be more than one representation which is fundamental e.g. for $SO(2n)$. For $SU(2)$ and $SO(3)$ there is a unique one.

fundamental ones then there is a good chance that they are made up of constituents of smaller representations. For example the particles π^+, π^0, π^- are in the $j = 1$ representation of $SU(2)$. Indeed they are made out of more fundamental particles called quark for which the up and down quarks are in the fundamental representation. There are also physicists extending this pattern to "real" (i.e. space-time) spin in saying that the Higgs boson, discovered at the LHC, might not be a fundamental particle since it is a scalar particle $J = 0$. It could be made out of two spin 1/2 (fundamental representation) constituents much alike the pions.

The representations of $SO(3)$ are manifestly real but the $SU(2)$ representations aren't. As hinted at earlier on they are pseudo-real (quaternionic).

The essence of pseudo-real is that the complex conjugate representation is not an inequivalent representation and yet no basis can be found where the representation matrices are real. This in particular implies that:

$$\rho_{j_1} \otimes \bar{\rho}_{j_2} \simeq \rho_{j_1} \otimes \rho_{j_2} \,, \tag{5.80}$$

and hence we do not need to consider the complex conjugate representation of $SU(2)$. Stated more precisely:

Theorem 5.4.2 ***Pseudo-reality of*** $SU(2)$
The representation (5.40) is pseudo real. I.e. the complex conjugate representation is equivalent through conjugation At the level of the Lie Algebra the equivalence relation is given by:

$$(-T_i^*) = S T_i S^{-1} \,, \quad S = \sigma_2 = 2T_2 \,. \tag{5.81}$$

The proof is left as an exercise. Note the minus sign on the left hand side is due to the convention $\exp(i\alpha T)$.

Exercise 5.4.3 ***Irreps of*** $SU(2)$

a) Verify the correctness of Eq. (5.74). *Hint:* without loss of generality assume that $j_1 \geq j_2$, as then you can work without absolute values.

b) Verify the steps in Eq. (5.76) using the geometric progression formula: $\sum_{i=0}^{N} z^i = (1 - z^{N+1})/(1-z)$.

c) Show that Eq. (5.81) holds.

d) Investigate the limit $\theta \to 0$ of Eq. (5.76) and give an interpretation of your result. Stated in other words: could you have predicted the result?

Chapter 6

Applications to quantum mechanics

6.1 A short synopsis of quantum mechanics

Quantum mechanics was developed, in its current form, from 1925 onwards within a short amount of time. Different versions exists such as the Schrödinger equation (an equation based on energy conservation) and Heisenberg's abstract matrix mechanics. Dirac showed that (the) different representations are connected by unitary transformations.

Abstract rules

A quantum mechanical state is described by a state in a Hilbert space (linear space with an inner product as described in section 1.3) for which we use the ket notation:

$$\text{Physical state: } |\Psi\rangle \tag{6.1}$$

The latter corresponds to some physical situation which is described by a dynamical equation. For example the Schrödinger equation which is an eigenvalue equation for the Hamiltonian:

$$H|\Psi\rangle = E_\Psi|\Psi\rangle \tag{6.2}$$

The Hamiltonian *(as all observables) are hermitian operators* $H^\dagger = H$, and E_Ψ are therefore a real number since:

$$E_\Psi\langle\Psi|\Psi\rangle = \langle\Psi|H|\Psi\rangle = \langle\Psi|H^\dagger|\Psi\rangle = \langle\Psi|H|\Psi\rangle^* = (E_\Psi)^*\langle\Psi|\Psi\rangle^* = (E_\Psi)^*\langle\Psi|\Psi\rangle \;, \tag{6.3}$$

which implies $E_\Psi = (E_\Psi)^* \in \mathbb{R}$. Furthermore states with different eigenvalues are orthogonal to each other since:

$$E_\Psi\langle\Psi'|\Psi\rangle = \langle\Psi'|H|\Psi\rangle = \langle\Psi'|H^\dagger|\Psi\rangle = \langle\Psi|H|\Psi'\rangle = E_{\Psi'}\langle\Psi|\Psi'\rangle \;, \tag{6.4}$$

which implies $\langle\Psi|\Psi'\rangle = 0$ if $E_\Psi \neq E_{\Psi'}$ (with both of them being real). The basic rule of quantum mechanics is that everything is determined up to a probabilities only. The probability of Ψ becoming Ψ' is given by:

$$P = |A_{\Psi\to\Psi'}|^2 \;, \qquad A_{\Psi\to\Psi'} \equiv \langle\Psi'|\Psi\rangle \;. \tag{6.5}$$

For the probability to be properly normalised one chooses $\langle\Psi|\Psi\rangle = \langle\Psi'|\Psi'\rangle = 1$.

From Hamiltonian to the Schrödinger equation in coordinate space

The Hamiltonian, in many cases, can be obtained from classical mechanics up to ***ambiguities of ordering***. In classical mechanics quantities like position and momentum can be known to an arbitrary precision whereas in quantum mechanics this os not the case as encapsulated in the commutation relations of the operators[1], and described by the uncertainty relations. The most famous ones are the ***Heisenberg commutation relations*** for the position operator $\hat{x}$ and the momentum operator $\hat{p}$,

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij} \;, \quad [\hat{x}_i, \hat{x}_j] = 0 \;, \quad [\hat{p}_i, \hat{p}_j] = 0 \;, \tag{6.6}$$

where $[A, B] = AB - BA$, $i, j = 1..3$ and $\hbar \equiv h/(2\pi) = 1.055 \cdot 10^{-34} Js$. Let us note that from these relations one can derive the famous Heisenberg uncertainty relations. We shall not do so as this is not central to the course.

The Schrödinger equation is an equation for a single non-relativistic particle of mass m subdue to a potential $V(x)$

$$H = T_{\text{kin}} + V(\hat{x}) = \frac{\hat{p}^2}{2m} + V(\hat{x})$$

The question on how this is turned into a computational scheme is answered by going to a certain representation which constitutes a set of maximally commuting operators. The canonical choice is the $|x\rangle$ representation for which:

$$\hat{x_i}|x\rangle = x_i|x\rangle \tag{6.7}$$

and

$$\mathbb{I}_x = \int d^3x |x\rangle\langle x| \;, \quad \langle x|x'\rangle = \delta^{(3)}(\vec{x} - \vec{x}') \;. \tag{6.8}$$

Most importantly the substation for the canonically conjugate momentum operator is as follows:

$$\hat{p}_j|_{x-space} \to -i\hbar\partial_j \;, \qquad \partial_j \equiv \frac{\partial}{\partial x_j} \;. \tag{6.9}$$

Now we have all the tools for bridging the gap between the abstract statement of the energy equation (6.2) and the Schrödinger equation in its conventional form:

$$\int d^3x' \langle x|H|x'\rangle\langle x'|\Psi\rangle = E\langle x|\Psi\rangle \;. \tag{6.10}$$

The latter is equivalent to the ***time independent Schrödinger equation***[2]:

$$\left(-\frac{\hbar^2}{2m}\Delta + V(x) \right) \Psi(x) = E\Psi(x) \;, \qquad \Psi(x) \equiv \langle x|\Psi\rangle \;, \tag{6.11}$$

with $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ upon using Eqs.(6.8) and (6.9).

A typical range of our application involves the transitions matrix elements of the form:

$$A_{\Psi \to \Psi'} = \langle \Psi'|H_{\text{trans}}|\Psi\rangle \;, \tag{6.12}$$

where H_{trans} is some transition Hamiltonian which are prone to a application of the Wigner-Eckart theorem. In the case where $H = H_0 + H_{\text{trans}}$ and Ψ and Ψ' are eigenstates of H_0, then (6.12) describes the first order effect and is known as ***Fermi's golden rule***. It is self-understood that the formula gives good results in the case where the perturbation can be considered to be small.

[1] The latter can be abstracted from the Poisson brackets of Hamiltonian dynamics as emphasised by Dirac in his transformation theory approach.

[2] To get the time dependent Schrödinger equation one substitutes $E \to -i\hbar\partial_t$

Symmetries and conservation laws

From course on Hamiltonian dynamics (known as the Noether theorem) you might be familiar with the concept that symmetries lead to conservations laws and vice versa. The same connection is observed in quantum mechanics.

Suppose we have a system which is governed by a Hamiltonian H. The transition matrix elements (6.12) involve the Hamiltonian and if the system is to be invariant under a symmetry $|\Psi\rangle \to U|\Psi\rangle$ then this implies,

$$\langle\Psi'|H|\Psi\rangle = \langle U\Psi'|H|\Psi\rangle = \langle\Psi'|U^\dagger HU|U\Psi\rangle \quad \Leftrightarrow \quad [U,H] = 0 \; , \tag{6.13}$$

that the unitary symmetry transformation[3] commutes with the Hamiltonian H. The unitary transformation written in terms of a generator G as $U = \exp(i\alpha G)$ then implies that the generator has to commute with the Hamiltonian, $[H,G] = 0$.

Furthermore the Heisenberg equation of motion of an operator G, independent of time ($\partial_t G = 0$), reads:

$$\frac{d}{dt}\langle G\rangle = i\langle[H,G]\rangle \; , \tag{6.14}$$

where $\langle..\rangle$ denotes the expectation value on any state. Hence the quantity $\langle G\rangle$ is a conserved quantity if it commutes with the Hamiltonian. Therefore:

$$\text{symmetry generated by } G \quad \Leftrightarrow \quad \text{conservation law for } G \tag{6.15}$$

A list of symmetries and conservations laws are given below:

- Space-time symmetry: The Poincaré group[4] $SO(3,1) \ltimes \mathbb{R}^4$ which consists of the Lorentz group $SO(3,1)$ and space-time translations. We shall discuss them one by one:
  - In a rest-frame this reduces to: Rotation group $SO(3)$ (or $SU(2)$) and time and space translation
    - $SO(3)$ rotational symmetry $\leftrightarrow$ conservation of angular momentum
    - space translations $\leftrightarrow$ conservation of momentum
    - time translation $\leftrightarrow$ conservation of energy
  - When one considers different inertial frames then there are in addition rotations, so to say, between the three space and the time components. Thos three additional symmetries correspond to boosts between in the three space directions. Note $\dim SO(3) = 3$ and $\dim SO(3,1) = 6$ and this explains the additional three symmetries.
- Extension of space-time symmetries are C, P and T:
  - parity symmetry P: $(x,y,z,t) \to (-x,-y,-z,t)$ conserved up to the weak interactions.
  - time symmetry T: $(x,y,z,t) \to (x,y,z,-t)$ small violations in weak interactions (presumably responsible for matter anti-matter asymmetry in the Universe)

[3]Besides unitarity symmetry transformation there is second possibility according to a famous result by Wigner: anti-unitary transformations for which $\langle A\Psi|A\Psi'\rangle = \langle\Psi'|\Psi\rangle$. We shall not discuss them in this course but mention that the time inversion operator is an anti-unitary transformation.

[4]The Poincaré group is a group of rank two. Its Casimir operators correspond to the spin and the mass of a particle which are therefore, besides the interaction, the primary characterisation of a particle.

  - charge symmetry C: exchange of particle and anti-particles conserved up to weak interactions
  - the combined symmetry of CPT is within the foundation of quantum field theory. Violations thereof would lead to a revolution in the field. In this sense violation of T-symmetry is the same as violation of the combined CP-symmetry.

- Gauge symmetries (local symmetries) of Lie groups lead to conservation of charges. The Standard Model exhibits, at high energy, a $SU(3)_c \times SU(2)_L \times U(1)_Y$ symmetry. The c stands for the colour charge, L for left handed and Y for something called hypercharge. At low energies the group reduces, through the Higgs mechanism, the $SU(3)_c \times U(1)_{\rm em}$. The $U(1)_{\rm em}$ formally describes the conservation of electric charge (and em stand for the electromagnetic).

- Global flavour symmetries such as described in exercise 6.7.1 a) which help to classify bound states of the strong interactions.

Exercise 6.1.1 ***Basic quantum mechanics***

a) Verify the Heisenberg commutation rules in x-space by using the substitution rule (6.9) for the momentum.

b) Complete the steps from (6.10) to (6.11).

6.2 $SU(2)$-group theory and the physics of angular momentum

The goal of this section is to connect results on $SU(2)$-group theory with the physics of angular momentum. The bottom line is as follows: According to the principle of the previous section (6.15), rotational symmetry implies conservation of angular momentum and vice versa. More precisely the Casimir number $C_2(j) = j(j+1)$ (or equivalently j) as well as the projection on one component, say m (magnetic quantum number J_z), are conserved. The former being independent of the frame whereas the latter is frame dependent (since J_z is not a Casimir operator).

Before we outline a few important points below let us be clear about the notation used throughout.

- $\vec{L} = \vec{x} \times \vec{p}$ (or L_a in components) denotes the ***orbital angular momentum***.
- $\vec{S}$ denotes the ***intrinsic angular momentum*** also known as ***spin***.[5]
- $\vec{J}$ denotes the ***total angular momentum***, i.e. the sum of all angular moment involved in the physical system. Note, if there is either no $\vec{L}$ or no $\vec{S}$ involved then $\vec{J}$ can also stand for $\vec{S}$ or $\vec{L}$ alone.

[5] The spin is a relativistic concept and entered quantum mechanics, at the formal level, through Dirac's relativistic equation. One can nevertheless incorporate it into the non-relativistic Schrödinger equation in a heuristic manner.

i) *Can we make the symmetry conservation principle* (6.15) *more concrete?*
One may simply observe that the quantum mechanical angular momentum,[6]

$$L_a = (\vec{x} \times \vec{p})_a = \epsilon_{abc}\hat{x}_b\hat{p}_c|_{x-space} \to -i\hbar\epsilon_{abc}x_b\partial_c \; , \tag{6.16}$$

satisfies the $SU(2)$ Lie Algebra (5.39) with an additional, but trivial, extra factor $\hbar$ on the right hand side.

ii) *What is the meaning of angular momentum j?*
A system with angular momentum j has to be rotated by $360°/j$ in order to be identical to its initial configuration. This follows from (5.75) $D^j(\theta, 0, 0)_{mm'} = e^{-im\theta}\delta_{mm'}$ and remembering that $|m|_{\max} = j$, and hence $D^j(2\pi/j, 0, 0) = \mathbb{I}_{2j+1}$. We observe that:

$$D^j(2\pi, 0, 0)_{mm'} = (-1)^{2m} = (-1)^{2j} = \begin{cases} 1 & j \quad \text{integer} \quad \text{bosons} \\ -1 & j \text{ half-integer} \quad \text{fermions} \end{cases} . \tag{6.17}$$

Hence if a fermion is rotated by $360°$-degrees then it picks up a minus sign. This seems odd at first but in fact this sign does disappear when one considers matrix elements since the fermion number is a superselection rule (c.f. also section 6.6) by virtue the nature $SU(2)$ Clebsch-Gordan series. The crucial point is that in order to obtain a scalar (invariant) one needs an even number of fermionic objects. Any Kronecker product of an odd number of half-integer representations does not give rise $j = 0$ object. To be discussed in section 6.6. The crucial differences between bosons and fermions have been briefly described by in section 4.5.2. It seems remarkable that the fact that $SO(3)$ (the rotation group of our 3D-space) is not simply connected opens the gate to half-integer spin and fermions (through the universal covering group $SU(2)$).

iii) *Total angular momentum:* ***addition of angular momentum***
Suppose there is a quantum mechanical system with many subsystems then a change of coordinate frame e.g. rotation must affect all its constituents. Hence the generator G in (and around) (6.13) corresponds to the total angular momentum.[7] This applies at the level of two particles which form a bound state $\vec{J}_{\text{tot}} = \vec{J}_1 + \vec{J}_2$ or for a single particle with orbital angular momentum $\vec{L}$ and spin $\vec{S}$ to $\vec{J}_{\text{tot}} = \vec{L} + \vec{S}$. Furthermore: unless otherwise stated, two angular momenta $J_i^{(a)}$ and $J_i^{(b)}$ are compatible in the sense that $[J_i^{(a)}, J_j^{(b)}] = 0$ for $(a \neq b)$.

An example: a total wave function could be made out of spatial wave function Ψ_l which is in a $l = 1$ state and a spin $s = 1/2$ state. Schematically and according theorem 5.4.1,

$$|\Psi_{l=1}\rangle \otimes |s = 1/2\rangle = |j = 1/2\rangle \oplus |j = 3/2\rangle \; . \tag{6.18}$$

Crucially there is no $j = 1$ by virtue of the superselection rule for bosons and fermions.

Exercise 6.2.1 ***The physics of angular momentum.***

[6] Note, there's no problem of normal ordering by going to the quantum system.

[7] It's the same pattern with energy and momentum. The total sum of energy and momentum, in a specific process, is conserved and not the individual momentum of say one particle. Otherwise the world would not be as interesting as it is!

a) Verify that the representation of the angular momentum (6.16) satisfies the SU(2) Lie Algebra (5.39) . Do this by considering just $[L_1, L_2] = i\epsilon_{123}L_3$ commutator. The general derivation, handling the Levi-Civita tensor, is trickier than it looks and I do not recommend to do it unless you know the right kind of identities.

b) Consider $j = 1$ representation. The state space is spanned by $|1,-1\rangle$, $|1,0\rangle$ and $|1,1\rangle$. These states will mix into each other under a generic rotation. First state why this has to happen on physical grounds. Second provide a more formal argument why it happens. Should you find it difficult to disentangle the two points then that is fine as it could mean that you have already internalised the notation!

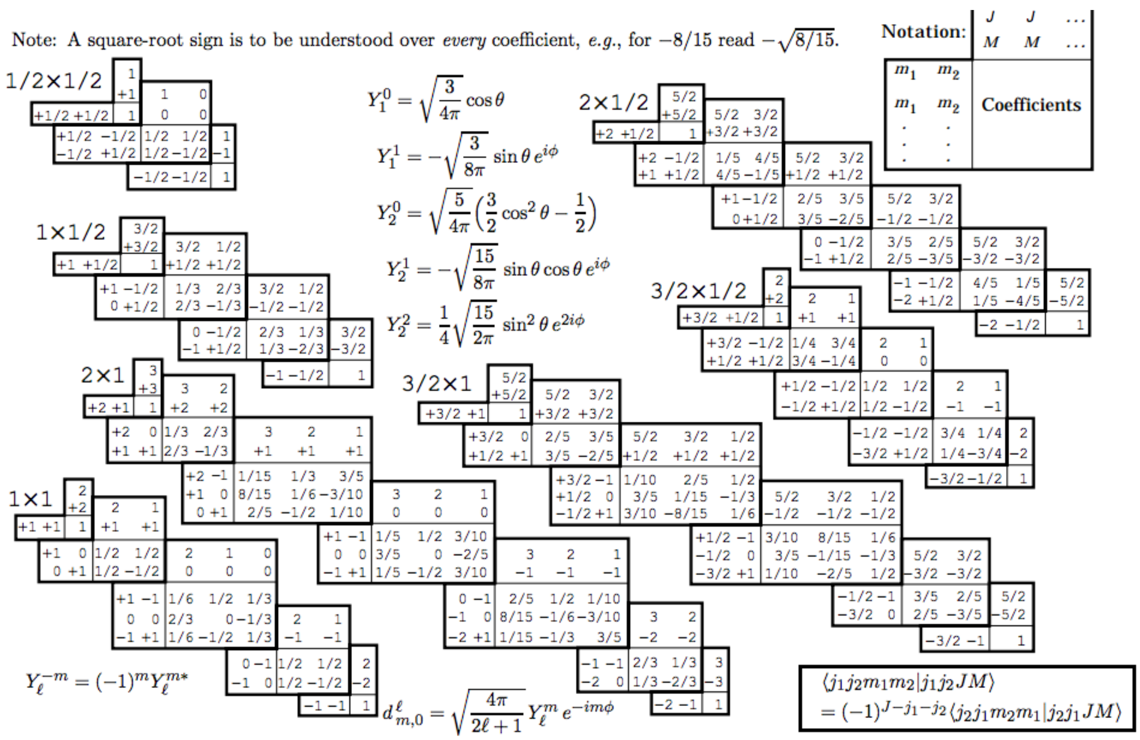

Figure 6.1: Taken from the Particle Data Group Book (booklets available for free) from http://pdg.lbl.gov/. In the Condon Shortly phase convention the coefficients are real so that one does not need to worry about complex conjugation.

6.3 Selection rules I: Wigner-Eckart theorem for $SU(2)$ (and $SO(3)$)

In order to state the celebrated Wigner Eckart theorem we have to introduce the definition of tensor operator and Clebsch-Gordan coefficient. After stating the theorem for $SU(2)$ we shall give some general remarks on the general version which mainly has to do with the case when the group is not simply reducible.

Definition 6.3.1 ***Tensor operator:*** A ***tensor operator***[8] is an operator that transforms under an irreducible representation of a group G. More specifically let $\rho(g)$ be a representation on the vector space under consideration then $T^c_{m_c}$ is a tensor operator in the irreducible representation c if it transforms as follow,

$$\rho(g) T^c_{m_c} \rho(g)^\dagger = (\rho_c(g))_{m_c m'_c} T^c_{m'_c} \, , \tag{6.19}$$

with summation over the index m'_c implied.

An example of a tensor operator is given by the momentum $\vec{p}$ which transforms under the vector representation of the $SO(3)$. The type of transformation law in Eq. (6.19) is also known as the operator or quantum mechanical implementation of a symmetry transformation.

Definition 6.3.2 ***Clebsch-Gordan coefficient*** Since the direct product of irreducible representations decomposes into a direct sum of irreducible representations (4.52), the following relations must exist,

$$|a, m_a \otimes b, m_b\rangle = \sum_{c, m_c, i} \underbrace{\langle c(i), m_c | a, m_a \otimes b, m_b\rangle}_{\mathcal{C}^{c(i)ab}_{m_c m_a m_b}} |c(i), m_c\rangle \, , \tag{6.20}$$

with $i = 1..m_{ab^c}$ denoting the multiplicity. Groups for which $m^c_{ab} = 0, 1$ are called ***simply reducible*** and for the latter we may write:

$$\text{simply reducible:} \quad |a, m_a \otimes b, m_b\rangle = \sum_{c, m_c} \underbrace{\langle c, m_c | a, m_a \otimes b, m_b\rangle}_{\mathcal{C}^{cab}_{m_c m_a m_b}} |c, m_c\rangle \, . \tag{6.21}$$

The groups $SO(3)$ and $SU(2)$ are simply reducible. An efficient way to picture Eqs. (6.20,6.21) is to note that:

$$\mathbb{I}_{a\otimes b} = \sum_{c, m_c, i} |c(i), m_c\rangle\langle c(i), m_c| \, . \tag{6.22}$$

The matrix element $\mathcal{C}^{c(i)ab}_{m_c m_a m_b}$ is known as a ***Clebsch-Gordan coefficient***. For fixed $a, b, c, i, m_a, m_b, m_c$ it is indeed a coefficient. In this course we shall be concerned with simple reducible cases only.

Theorem 6.3.1 ***Wigner-Eckart theorem (1930-1931)*** (for simply reducible groups)
The Wigner-Eckart theorem states that the matrix element of two states and a tensor operator is proportional to the Clebsch-Gordan coefficient,

$$\langle c, m_c | T^a_{m_a} | b, m_b\rangle = \mathcal{C}^{cab}_{m_c m_a m_b} \, \langle c || T^a || b\rangle \tag{6.23}$$

times a ***reduced matrix element*** $\langle c || T^a || b\rangle$ which is independent of the directions m_{abc}.

[8]The word tensor, presumably, originates from the Cauchy stress tensor which is a tensor with two vector indices under $SO(3)$.

Proof: (sketch) The state $T^a_{m_a}|b, m_b\rangle$ transforms like the direct product representation. This can be seen as follows:

$$\rho(g)T^a_{m_a}|b, m_b\rangle = \rho(g)T^a_{m_a}\rho(g)^\dagger \rho(g)|b, m_b\rangle \overset{(6.19)}{=} \rho_a(g)_{m_a m'_a}\rho_b(g)_{m_b m'_b} T^a_{m'_a}|b, m'_b\rangle \ . \tag{6.24}$$

Therefore we may use (6.21) to deduce:

$$T^a_{m_a}|b, m_b\rangle \propto \sum_{c', m'_c} \mathcal{C}^{c'ab}_{m'_c m_a m_b}|c', m'_c\rangle \ . \tag{6.25}$$

Eq. (6.23) follows by applying a bra state $\langle c, m_c|$ and using $\langle j, m|j', m'\rangle = \delta_{jj'}\delta_{mm'}$ q.e.d.

A few remarks:

- The moral of the Wigner-Eckart theorem is that for a given representation one only needs to compute one matrix element. The rest of them follows from group theory only; namely the Clebsch-Gordan coefficients. The Clebsch-Gordan coefficients are standard devices which are tabulated in books. One may not even want or be able to compute the reduced matrix element but parameterise it by a number and consider ratios of matrix elements only. *The latter gives relative decay times in the case the particles are part of a representation of an internal symmetry group. This is the case for the so-called $SU(2)$ isospin symmetry (exchange of up and down quarks) for the pions. Be warned that the phases of the Clebsch-Gordan coefficients are dependent on the phase convention of $|a, m_a\rangle$. The Wigner-Eckart theorem is a device that gives immediate access to so-called* ***selection rules****. By selection rules physicists mean that certain class of matrix elements are zero which could be related to certain atomic or decay transitions being forbidden (by symmetry).*

- We consider it worthwhile to briefly restate the essence in a colloquial style. The Wigner-Eckart theorem and Clebsch-Gordan coefficient construction states: i) that a matrix element $\langle c, m_c|T^a_{m_a}|b, m_b\rangle \neq 0$ only if the irreducible representation c appears in the Clebsch-Gordan series of $a \otimes b\,(= m^c_{ab} c \oplus ...$ with $m^c_{ab} \neq 0)$ ii) the direction of the matrix element is completely determined through Clebsch-Gordan coefficient. I have tried to summarise the essence of all this in the diagram in Fig. 6.2.

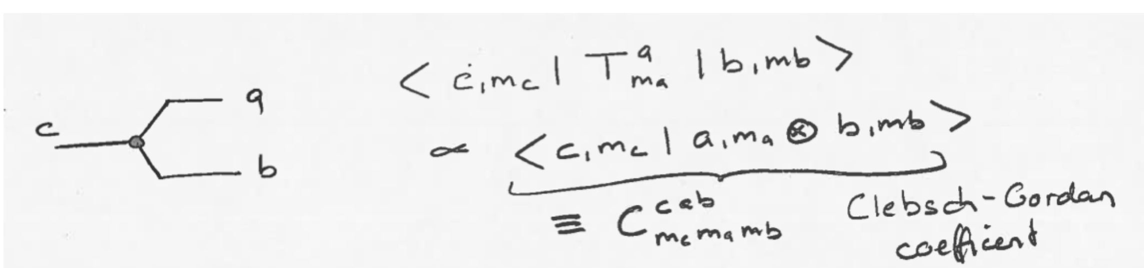

Figure 6.2: The Wigner-Eckart theorem illustrated in a figure.

The term ***selection rule*** refers to matrix elements which are zero on grounds of vanishing Clebsch-Gordan coefficient. The term originates from looking at atomic transitions and the observation that the symmetry *selects* only certain transitions. Disregarding specific selection rules on the coefficients m_a, m_b and m_c, a first selection rule is provided by the necessity of $m_{ab}^c \neq 0$: i.e. the Clebsch-Gordan series of $\rho_a \otimes \rho_b$ must contain ρ_c at least once as otherwise the $C^{c(t)ab}_{m_c m_a m_b} = 0$. We will discuss applications of the Wigner-Eckart theorem towards the end of the course in the context of $SO(3)$.

- It ought to be clear that none of the arguments above rely on the $SU(2)$-representation theory. Thus the Wigner-Eckart theorem applies to any group with finite dimensional representation straightforwardly. In the case where the group is not simply reducible the general Clebsch-Gordan coefficients have to be employed and the formulae complicate but no new ideas are needed.

We consider it worthwhile to restate it here in the notation used for $SU(2)$ because of its numerous applications. Furthermore we will restate the tensor operator condition (for $l = 1$) at the level of the Lie Algebra.

The *Wigner-Eckart theorem for SU(2)-notation* reads:

$$\langle J, M | T^{j_1}_{m_1} | j_2, m_2 \rangle = \mathcal{C}^{J j_1 j_2}_{M m_1 m_2} \langle J || T^{j_1} || j_2 \rangle \tag{6.26}$$

with Clebsch-Gordan coefficient $C^{J j_1 j_2}_{M m_1 m_2} \equiv \langle J, M | j_1, m_1 \otimes j_2, m_2 \rangle$ explicitly given in Fig. 6.1 (borrowed from the Particle Data Group book) in the so-called Condon-Shortly convention which is rather standard throughout the literature. Equation (6.26), unmasks the selection rules imposed by the Wigner-Eckart theorem on $SU(2)$, *as* conservation of the J^2_{tot} and $(J_{\text{tot}})_z$ quantum numbers

$$\textbf{selection rules}: \quad j_1 + j_2 \geq J \geq |j_1 - j_2| \;, \quad M = m_1 + m_2 \;. \tag{6.27}$$

Theorem 6.3.2 The definition 6.3.1 of a tensor operator $\rho(g) T^j_m \rho(g)^\dagger = (\rho_j(g))_{mm'} T^j_{m'}$ (6.19) is written in infinitesimal form as:

$$[J_a, T^1_b] = i\epsilon_{abc} T^1_c \;, \tag{6.28}$$

with summation over repeated indices understood. Note in particular this shows that J_a itself is a tensor operator $j = 1$, i.e. a vector, since $T^1_c = J_c$ is obeys (6.28) by virtue of the $SU(2)$ Lie Algebra equation. Angular momentum is a vector! The proof is left as an exercise.

Clebsch-Gordan coefficients in direct products In practice a direct product of two $SU(2)$ representations can be expressed as follows:

$$|J, M\rangle = \sum_{j_1, m_1, j_2, m_2} (C^{J j_1 j_2}_{M m_1 m_2})^* | j_1, m_1 \otimes j_2, m_2 \rangle \tag{6.29}$$

using the fact that

$$\mathbb{I}_{j_1 \otimes j_2} = \sum_{j_1, m_1, j_2, m_2} | j_1, m_1 \otimes j_2, m_2 \rangle \langle j_1, m_1 \otimes j_2, m_2 | \;. \tag{6.30}$$

As an example of $j = 1/2, 3/2$ state on the right hand side of (6.18) let us quote:

$$\begin{aligned} |3/2, 1/2\rangle &= \sqrt{\frac{1}{3}}|1,1\rangle \otimes |1/2,-1/2\rangle + \sqrt{\frac{2}{3}}|1,0\rangle \otimes |1/2,1/2\rangle \ , \\ |1/2, 1/2\rangle &= \sqrt{\frac{2}{3}}|1,1\rangle \otimes |1/2,-1/2\rangle - \sqrt{\frac{1}{3}}|1,0\rangle \otimes |1/2,1/2\rangle \ , \end{aligned} \tag{6.31}$$

where the Clebsch-Gordan coefficients are taken from Fig. 6.1. The verification of Eq. (6.31) is left as an exercise.

Geometric interpretation of infinitesimal version of the $j = 1$ representation Expanding the left hand side of (6.19) (left as an exercise) to first order one gets

$$e^{i\alpha_a J_a} T^1_k e^{-i\alpha_b J_b} = T^1_k + \epsilon_{alk}\alpha_a T^1_l + \mathcal{O}(\alpha^2) \ . \tag{6.32}$$

The geometric interpretation of the $\mathcal{O}(\alpha)$-term in (6.32) is:

$$\epsilon_{alk}\alpha_a T^1_l = (\vec{\alpha} \times \vec{T})_k = \sin(\gamma)|\vec{\alpha}||\vec{T}|\vec{e}_k \ . \tag{6.33}$$

Above $\vec{e}_k$ is a unit vector orthogonal to the plane spanned by $\vec{\alpha}$ and $\vec{T}$. It is nothing but the *cross product* (familiar from electrodynamics for example) which is the infinitesimal version of a rotation. The reader is encouraged to consider Fig. 5.4 to convince him or herself.

Exercise 6.3.1 ***Wigner-Eckart theorem (The proof of theorem 6.3.2)***

a) Using the commutator (6.28) derive (6.32).

b) Let us consider $\alpha_3 = \alpha$ and $\alpha_{1,2} = 0$ in which case the rotation matrix $j = 1$ assumes the following form:

$$\rho_{j=1}(\alpha) \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{I}_3 + \begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + O(\alpha^2) \tag{6.34}$$

Show that (6.32) and (6.70) are consistent with the definition of a tensor operator

$$\rho(\alpha) T^j_m \rho(\alpha)^\dagger = (\rho_j(\alpha))_{mm'} T^j_{m'} \ .$$

c) Verify Eq. (6.31) using the table in Fig. 6.1.

d) Check in addition that the two states Eq. (6.31) are orthogonal to each other.

e) Verify (6.29). All the necessary equations are given in the corresponding section above.

6.4 Wigner-Eckart applied: *the Landé g-factor*

The H-atom with with Coulomb potential exhibits spherical symmetry. This symmetry is lifted by the application of a homogeneous magnetic field. The relevant part of the perturbing Hamiltonian reads[9]

$$H' = x(L_3 + 2S_3) = x(J_3 + S_3) \ , \quad x \equiv -\frac{eB}{2m_e c} \ , \tag{6.35}$$

with B the magnetic field in 3-direction, m_e, e the electron mass and charge and c the speed of light. The operators J_3 and S_3 are the total angular momentum and (intrinsic) spin operator in the 3-direction. We assume to know the states of the hydrogen atom including spin-orbit coupling in terms of $|n,l,j,m\rangle$ quantum numbers. Using first order perturbation theory[10] the in H' leads to a correction energy of the following form:

$$E' = \langle n,l,j,m|H'|n,l,j,m\rangle = x(\hbar m + \langle n,l,j,m|S_3|n,l,j,m\rangle) \ . \tag{6.36}$$

Note that m is the magnetic quantum number of the total angular momentum $\vec{J}$. Hence the task is to compute the matrix element on the right hand side with the complication that the state $|n,l,j,m\rangle$ is not obviously related to the spin operator S_3. The solution of this problem goes with a classic application of the Wigner-Eckart theorem.

In the derivation below the quantum numbers nlj are kept fixed and we shall use the condensed notation $|n,l,j,m\rangle \to |m\rangle$. By virtue of the Wigner-Eckart theorem we may write,

$$\langle m|S_i|m'\rangle = a\langle m|J_i|m'\rangle \tag{6.37}$$

since both $\vec{S}$ and $\vec{J}$ are vector operators. The symbol a is the ratio of reduced matrix elements which we do not need to specify any further.[11] Note, once a is known, E' is known and the problem is solved.

The rest of the derivation involves the combination of three equations. First the product $\vec{J}\cdot\vec{S}$ can be written in terms of three Casimir operators

$$-2\vec{J}\cdot\vec{S} = (\vec{J}-\vec{S})^2 - \vec{J}^2 - \vec{S}^2 = \vec{L}^2 - \vec{J}^2 - \vec{S}^2 \ . \tag{6.38}$$

Second one may write,

$$\begin{aligned}\langle m|\vec{J}^2|m\rangle &= \sum_{m'}\langle m|J_i|m'\rangle\langle m'|J_i|m\rangle \ , \\ \langle m|\vec{J}\cdot\vec{S}|m\rangle &= \sum_{m'}\langle m|J_i|m'\rangle\langle m'|S_i|m\rangle \end{aligned} \tag{6.39}$$

[9]This derives from the magnetic moment interaction with the magnetic field $H = -\vec{\mu}\cdot\vec{B}$ with $\vec{\mu}|_{\text{orbital}} = \frac{e\hbar}{2m_e c}\vec{L}$ and $\vec{\mu}|_{\text{spin}} = \frac{e\hbar}{m_e c}\vec{S}$. The latter differs by a factor of 2 which can be derived from the Dirac equation and was a bit of a surprise. The spin is a relativistic effect and has to be introduced into a non-relativistic framework like Schrödinger's equation in an ad-hoc way.

[10]If higher orders were to be considered then we would need to take into account the backreaction of H' on the states $|n,l,j,m\rangle$ amongst other effects.

[11]More precisely $a = \langle||S_i||\rangle/\langle||J_i||\rangle$ in our condensed notation. The ratio of Clebsch-Gordan coefficients is one and therefore disappears.

with summation over i implied. Now,

$$\begin{aligned}\langle m|\vec{J}\cdot\vec{S}|m\rangle &\overset{(6.37)}{=} a\langle m|\vec{J}^2|m\rangle = a\hbar^2 j(j+1)\ ,\\ \langle m|\vec{J}\cdot\vec{S}|m\rangle &\overset{(6.38)}{=} -\frac{\hbar^2}{2}\left(l(l+1)-j(j+1)-s(s+1)\right)\ ,\end{aligned} \tag{6.40}$$

where $S^2 = s(s+1)$ etc. Now, we can solve for a in (6.37) and obtain E' as follows,

$$E'_{jlsm} = \frac{-e\hbar m B}{2m_e c} g\ , \qquad g = 1 + \frac{1}{2}\frac{1}{j(j+1)}\left(j(j+1)+s(s+1)-l(l+1)\right)\ , \tag{6.41}$$

where g is called the Landé g-factor. The energy E' is a shift in energy that should be added to the main part and it resolves the degeneracy in the spectrum. Such tests were of utmost importance in developing an understanding of quantum mechanics.

Exercise 6.4.1 ***Applications of the Wigner-Eckart theorem***

a) Wigner-Eckart theorem and *nucleon-nucleon scattering*.
This exercise is basically solved for as you read along. You should try to understand the intermediate steps and provide the final step for completion.
Let us consider the following three processes:

$$\begin{aligned}&(1) \quad p + p \to deut + \pi^+\ ,\\ &(2) \quad p + n \to deut + \pi^0\ ,\\ &(3) \quad n + n \to deut + \pi^-\ ,\end{aligned} \tag{6.42}$$

where *deut*, p and n are deuteron (pn), proton and neutron respectively. The $\pi^{\pm,0}$ are pions of different charges as indicated. The processes in (6.42) are governed by the strong force. The latter does not distinguish between so-called up (u) and down (d) quarks. This can be formalised into a so-called $SU(2)$-isospin symmetry which acts on the vector (u, d). Instead of $|j, m\rangle$ one uses the symbols $|I, I_z\rangle$ where I stands for *Isospin*. (Isospin should not be confused with spin. The spin in isospin is borrowed because of the analogy that arises through $SU(2)$ in the mathematical sense.) The up and down quarks are associated with $|I, I_z\rangle$-states as follows:

$$u \sim |1/2, 1/2\rangle\ , \quad d \sim |1/2, -1/2\rangle$$

- Convince yourself that charge conservation is obeyed in Eq. (6.42). This is (very) easy if you know how it works.
- The protons and neutrons are made out of *uud* and *udd* quarks respectively. They are associated with the following isospin states:

$$p \sim |1/2, 1/2\rangle\ , \quad n \sim |1/2, -1/2\rangle\ . \tag{6.43}$$

Out of three isospin $I = 1/2$ states one can either form a $I = 1/2$ or $I = 3/2$ state. (The $I = 3/2$ states corresponds to the four Δ^{++}, Δ^+, Δ^0 and Δ^- which are degenerate in mass (up to small electromagnetic corrections) and weigh about 1.2 the proton and neutron mass.) Can you guess why the proton and neutron do not correspond to $I = 3/2$ states?

– Out of protons and neutrons (6.43) four states can be formed,

$$|1,1\rangle \sim |p\otimes p\rangle \;, \qquad |1,0\rangle \sim 1/\sqrt{2}(|p\otimes n\rangle + |n\otimes p\rangle) \;,$$
$$|1,-1\rangle \sim |n\otimes n\rangle \;, \qquad |0,0\rangle \sim 1/\sqrt{2}(|p\otimes n\rangle - |n\otimes p\rangle) \;, \tag{6.44}$$

one of isospin $I = 0$ (singlet) and three of isospin $I = 1$ (triplet). Show that the relations are correct by using (6.29) and table in Fig. 6.1 .

– As a matter of fact we know that the deuteron corresponds to singlet state. In addition the pions are made out of up and down quarks and corresponds to the following isospin associations:

$$|1,1\rangle \sim \pi^+ \;, \qquad |1,0\rangle \sim \pi_0 \;,$$
$$|1,-1\rangle \sim \pi^i \;. \tag{6.45}$$

From Eq.(6.44) we obtain $|p\otimes n\rangle \sim 1/\sqrt{2}(|0,0\rangle + |1,0\rangle)$. Show that the relative size of the amplitudes, $\mathcal{A}_1 = \langle deut\,\pi^+|p\otimes p\rangle$ (and analogous for $\mathcal{A}_{2,3}$), is given by:

$$\mathcal{A}_1 : \mathcal{A}_2 : \mathcal{A}_3 = 1 : 1/\sqrt{2} : 1 \;. \tag{6.46}$$

This implies that the cross section, which are proportional to the absolute value of the amplitudes squared, obey the ratios: $\sigma_1 : \sigma_2 : \sigma_3 = 2 : 1 : 2$ as they indeed do! Historically such processes go the other way around. A pattern in rates and cross section is observed and then possible symmetries of the underlying constituents are identified.
Hint: You can ignore the deuteron as it is a isospin singlet. Hence the right hand side is given in Eq. (6.45). In fact we have also obtained the left hand side completely (in terms of isospin states).

b) Which quantum number in E'_{jlsm} (6.41) indicates that spherical symmetry is broken?

c) (optional on popular demand) The usefulness of branching rules! (good revision exercise) The energy levels of a free atom, which enjoys spherical symmetry, arrange themselves in multiplets of the rotational symmetry. The s–wave ($l = 0$) corresponds to a 1–fold degenerate state, the p–wave ($l = 1$) to a 3–fold degenerate level, and finally the d–wave ($l = 2$) yields a 5–fold degenerate level.

Consider an atom in a crystal which has A_4 (tetrahedral) symmetry. The symmetry becomes more constraining and the mutiplets branch into smaller representations. By using the A_4 ($|A_4| = 12$) character table ($\omega = (\sqrt{3}-1)/2$):

A_4 class	()	(12)(34)	(123)	(132)
1	1	1	1	1
1′	1	1	ω	$-1-\omega$
1″	1	1	$-1-\omega$	ω
3	3	-1	0	0

Show that:

1. the 1–fold degeneracy of the s–wave is not lifted;
2. the 3–fold degeneracy of the p–wave is not lifted;
3. the 5–fold degeneracy of the p–wave is lifted , and we obtain three different eigenvalues, which are 1–fold, 1–fold, and 3–fold degenerate respectively.

Hint: You need to construct the $SO(3)$ character table restricted to the classes for A_4 for $l = 0, 1, 2$. In order to do this you need to identify the order of the class form which you will get the rotation angle and then you can use the general formula $\chi_l(\theta)$ obtained in the lecture. In a second step you then form the scalar products from which you can read off the branching rules: $\rho_l|_{SO(3)} \to \oplus_i m_i \rho_i|_{A_4}$ with i summing over the irreducible representations of A_4 and m_i being the multiplicities.

6.5 Spherical harmonics

The Heisenberg commutation relations (6.6) indicate that the x-space representation is infinite dimensional since for any finite dimensional representation $\mathrm{Tr}[A, B] = 0$ by the cyclicity of the trace. How come the reprentations $|l, m\rangle$ are all finite dimensional. The resolution lies in the fact that the Hilbert space on S^2, $\mathcal{H}_{S_2} \simeq \oplus_{l\geq 0}\mathcal{H}_l$, decomposes into an infinite sum of the finite dimensional Hilbert spaces $\mathcal{H}_{l,m}$ ($\dim \mathcal{H}_l = (2l + 1)$). The group theoretic study of the spherical harmonics, which is goal of this chapter, should clarify this statement.

Let us consider S^2 parameterised by the usual variables ϕ and θ as shown in Fig. 6.3. A state $|\theta, \phi\rangle$ obeys the following completeness and orthogonality relation, analogous to (6.8) for $|x\rangle$,

$$\begin{aligned} \mathbb{I}_{S^2} &= \int d\Omega |\theta, \phi\rangle\langle\theta, \phi| \quad \leftrightarrow \quad && \langle\theta, \phi|\theta', \phi'\rangle = \delta(\phi - \phi')\delta(\cos\theta - \cos\theta') \ , \\ &= \sum_{l\geq 0}\sum_{m=-l} |l, m\rangle\langle l, m| && \langle l, m||l, m\rangle = \delta_{ll'}\delta_{mm'} \end{aligned} \tag{6.47}$$

where

$$\int d\Omega = \int_0^{2\pi} d\phi \int_{-1}^{1} dcos\theta \ , \tag{6.48}$$

is the usual integration over the solid angle. The ***spherical harmonics*** Y_{lm} are the projections of the state $|l, m\rangle$ on $|\theta, \phi\rangle$:

$$Y_{lm}(\theta, \phi) \equiv \langle\theta, \phi|l, m\rangle \ . \tag{6.49}$$

From (6.47) the following orthogonality and completeness relation for the Y_{lm} are immediate:

$$\int d\Omega \, Y^*_{l'm'}(\theta, \phi)Y_{lm}(\theta, \phi) = \delta_{ll'}\delta_{mm'} \ , \tag{6.50}$$

$$\sum_{l\geq 0}\sum_{m=-l}^{l} Y_{lm}(\theta, \phi)Y^*_{lm}(\theta', \phi') = \delta(\phi - \phi')\delta(\cos\theta - \cos\theta') \ . \tag{6.51}$$

The Casimir operator L^2 on the two sphere S^2 is given by,

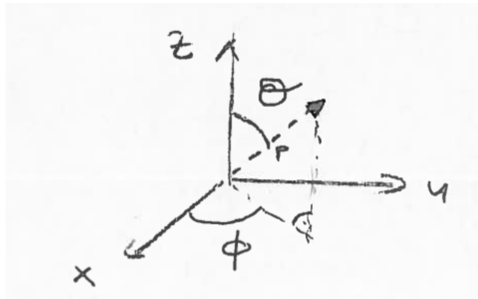

Figure 6.3: Coordinates on S^2 the unit sphere in three dimensions.

$$\Delta_{S^2} = \int d\Omega \langle \theta', \phi' | (-L^2/\hbar^2) | \theta, \phi \rangle = \frac{1}{\sin\theta^2} \partial_\phi^2 + \frac{1}{\sin\theta} \partial_\theta (\sin\theta \partial_\theta) \ , \tag{6.52}$$

and corresponds to the Laplacian on S^2. Hence the Y_{lm} satisfy,

$$\Delta_{S^2} Y_{lm} = -L^2/\hbar^2 Y_{lm} = -l(l+1) Y_{lm} \ , \tag{6.53}$$

the harmonic equation and is why they are called spherical harmonics.[12] After all a Laplacian is invariant (under rotation) and of second order and since there is only one Casimir the connection is a necessity rather than an accident!

Application to $\mathbb{R}^3 \simeq S^2 \otimes \mathbb{R}^+$

The three dimensional Euclidian space $\mathbb{R}^3$ is decomposed by polar coordinates as follows:

$$\mathbb{R}^3 \simeq S^2 \otimes \mathbb{R}^+ \ , \tag{6.54}$$

where $\mathbb{R}^+$ parameterises the radial coordinate r. Since the Y_{lm} are a complete set of functions on S^2 (6.51), any function $f(x, y, z)$ on (6.54) may be written as:

$$f(x, y, z) = \sum_n \sum_{l \geq 0} \sum_{m=-l}^{l} u_{nlm}(r) Y_{lm}(\theta, \phi) \ , \tag{6.55}$$

where $u_{nlm}(r) \in \mathbb{C}$ is any complete set of functions on $\mathbb{R}^+$. The Laplacian on $\mathbb{R}^3$ in polar coordinates is given by:

$$\Delta_{\mathbb{R}^3} = \frac{1}{r^2} (\partial_r r^2 \partial_r + \Delta_{S^2}) \ . \tag{6.56}$$

Hence we can write the Schrödinger equation (6.11) in terms of polar coordinates. We note that the so-called ***solid harmonics*** $\mathcal{Y}_{lm} = r^l Y_{lm}$ satisfy Laplace equation in spherical coordinates $\Delta_{\mathbb{R}^3} \mathcal{Y}_{lm} = 0$.

[12]The harmonics on euclidian space are sin and cos, as you know.

This becomes particularly useful in when the potential exhibits ***spherical symmetry*** $V(x,y,z) = V(r)$. In this case the solution does not depend on the magnetic quantum number m as the latter singles out a direction in space. Hence $u_{nlm}(r) \to u_{nl}(r)$ in Eq. (6.55) and the Schrödinger equation becomes an equation of one variable:

$$\left(-\frac{\hbar^2}{2m}\frac{1}{r^2}\partial_r r^2 \partial_r + V_{\rm eff}^{(l)}(r)\right) u_{nl}(r) = E_{nl} u_{nl}(r) \;, \tag{6.57}$$

with

$$V_{\rm eff}^{(l)}(r) = V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} \;, \tag{6.58}$$

and $u_{nl}(r)$ being the eigenfunction of the operator on the left hand side rather than an arbitrary set of complete function on $\mathbb{R}^+$ as previously described.

*for the interested reader (**not** part of the course)*

In this small mathematical aside we want to clarify why it is S^2 that appears as the natural manifold of the $SO(3)$ Lie group. Consider an arbitrary vector in $\mathbb{R}^3$ which we may well choose to be $v = (1,0,0)$. This vector has $SO(2)$ acting on the coordinates y and z as a stabiliser group. This signals redundancy of the problem and has to be divided out. The relevant manifold is then indeed,

$$S^2 = SO(3)/SO(2) \;. \tag{6.59}$$

Note, this is not a group manifold since $SO(2)$ is not normal in $SO(3)$! Some of you might already have been convinced by the parameterisation in Fig. 6.3 and it is indeed an equivalent way to look at it for $SO(3)$. One should be aware though that for groups which do not permit such a simple graphical illustration one has to go the algebraic way as outlined in this small note.

Exercise 6.5.1 ***Spherical harmonics***

a) In this exercise we intend to clarify to what extent the highest weight representation technique applies to the spherical harmonics or solid harmonics respectively. In fact the most efficient way to proceed is through cartesian and not spherical coordinates. The solid harmonics for $l = 1$ are given by:

$$\mathcal{Y}_{11} = \mathcal{Y}^*_{1-1} = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}(x+iy) \;, \quad \mathcal{Y}_{10} = \frac{1}{2}\sqrt{\frac{3}{\pi}} z \;. \tag{6.60}$$

First obtain $L_\pm = (L_x \pm iL_y)$ from

$$L_x = -i(y\partial_z - z\partial_y) \;, \quad L_y = -i(z\partial_x - x\partial_z) \;,$$

then check that i) $L_+\mathcal{Y}_{11} = 0$ (i.e. (5.56)) and ii) obtain $\mathcal{Y}_{10} \propto L_-\mathcal{Y}_{11}$ (expression just below (5.57)). Note the factor of proportionality is given by the normalisation factor $N_-(l = 1, m = 1)$.

b) Verify that the solid harmonics (see text) satisfy the Laplace equation $\Delta_{\mathbb{R}^3}\mathcal{Y}_{lm} = 0$ with $\Delta_{\mathbb{R}^3}$ as in Eq. (6.56).

c) Verify the orthogonality and completeness relation (6.50) and (6.51) by using the definitions given in the corresponding section.

6.6 Selection rules II: parity and superselection

Parity selection rule

Parity symmetry P, which was briefly mentioned in section 6.1, is the reflection of space (mirror):

$$P \circ (x, y, z, t) \to (-x, -y, -z, t) \; . \tag{6.61}$$

It was believed to be a strictly conserved quantum number of nature until 1956 when theorists (T.D.Lee and Yang) reexamined the grounds for this believe; coming to the conclusions that parity might be violated (in the weak interactions). Indeed it was found that parity is violated, even maximally in the weak interactions. This led Pauli's to one of his most famous quotes (alluding to the fact that weak interactions couple to left handed fermions in a definite way): "I can't believe that god is a weak left-hander". Yet the weak interactions are weak and can be neglected when the transition is induced by the strong interactions. Therefore parity, in practice, is a good quantum number as long as the weak force does not play a major rôle in the decay.

By including parity we effectively extend the discussion from $SO(3)$ to $O(3)$ since every $O(3)$-element can be written as a reflection times an $SO(3)$-element. The implementation of the parity symmetry is unitary. Furthermore $U(P)^2 = e^{i\eta}\mathbb{I}$[13] is identical to the identity times a phase which is conveniently chosen to be zero. Hence $U(P) = \pm\mathbb{I}$.

The parity transformation of the state are given by,

$$U(P)|1/2, m\rangle = |1/2, m\rangle \; , \quad U(P)|l, m\rangle = (-)^l |l, m\rangle \; , \quad l \in \mathbb{Z}^+ \; , \tag{6.62}$$

where the later can be inferred from the transformation properties of the Y_{lm}:

$$P \circ Y_{lm}(\theta, \phi) = Y_{lm}(\pi - \theta, \phi + \pi) = (-1)^l Y_{lm}(\theta, \phi) \; . \tag{6.63}$$

The transformation of the angles can be inferred from Fig. 6.3. Let us suppose that a transition is governed by an operator X with parity $U(P)XU(P)^\dagger = (-1)^{\eta_X} X$. Then the following selection rule applies

$$\langle lm|X|l'm'\rangle \neq 0 \quad \Rightarrow \quad l + l' + \eta_X \text{ even} \; . \tag{6.64}$$

This is a selection rule on top of the ones quoted in (6.27) (the latter applies provided X is a tensor operator).

Examples of operators with definite transformation properties are the coordinate $\vec{x}$, momenta $\vec{p}$, electric $\vec{E}$ and magnetic $\vec{B}$ field as well as angular momenta $\vec{L}$:

$$\eta_{\vec{x}} = \eta_{\vec{p}} = \eta_{\vec{E}} = 1 \; , \quad \eta_{\vec{L}} = \eta_{\vec{B}} = 0 \; . \tag{6.65}$$

[13] The very careful reader might notice that the phase can differ on each superselection sector.

The operators $\vec{L}$ and $\vec{B}$ are even vectors under parity and are also known as ***pseudo vectors***. For the angular momentum this comes from $P \circ \vec{L} = P \circ (\vec{x} \times \vec{p}) = P \circ \vec{x} \times P \circ \vec{p} = -\vec{x} \times (-\vec{p}) = \vec{L}$.

At last we would like to mention that there is a subtlety that we have not discussed. There is something called ***intrinsic parity*** which cannot be seen from l and s alone. As much as the $\vec{B}$ and $\vec{L}$ are pseudo vectors particles can also have this pseudo-quality. For example the ρ-meson and the A_1-meson are both vectors $S = J = 1$ but the former has minus parity and the other has plus parity. Hence A_1 is a pseudo vector particle in some sense. When you compute transitions between alike particles then the intrinsic parity cancels. This is often the case in quantum mechanics which is the scope of this course.

Superselection rules

Superselection rules were introduced by Wick, Wightman and Wigner in 1952 as a generalisation of selection rules. Selection rules refer to a a certain Hamiltonian which only allows transitions between states of restricted quantum numbers as formalised by the Wigner-Eckart theorem. Superselection rules state that certain states vanish as matrix elements on *all observables.* I.e. the states $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are said to be separated by a ***superselection rule*** if for all observables O

$$\langle \Psi_1 | O | \Psi_2 \rangle = 0 \,, \quad O \text{ any observale} \,. \tag{6.66}$$

This means that the relative phase between $|\Psi_1\rangle$ and $|\Psi_2\rangle$ is not observable and hence no coherent state of the form $a|\Psi_1\rangle + b|\Psi_2\rangle$ can be prepared (or exists). Hence the space of observables is smaller than the space of hermitian operators ($X = |\Psi_1\rangle\langle\Psi_2| + |\Psi_2\rangle\langle\Psi_1| = X^\dagger$ and $\langle \Psi_1 | X | \Psi_2 \rangle \neq 0$. X is hermitian but not an observable as it contradicts (6.66).)

An example is given by the fermion boson quantum number. Suppose you had a *coherent* quantum superposition of the form $|\psi\rangle = a|j = 1/2\rangle + b|j = 1\rangle$ and subject it to 360°-rotation. This leads to an expression,

$$U(360^\circ)|\psi\rangle \stackrel{(6.17)}{=} -a|j = 1/2\rangle + b|j = 1\rangle \,, \tag{6.67}$$

which is clearly non-sense, in the sense that the relative phase between a and b is unobservable. The same happens for conserved charges; e.g. the electric charge for example. It does not make sense to write $|\psi\rangle = |e^+\rangle + |e^-\rangle$ as they transform in complex conjugate representations which acquire different phases under rotation of the group (in this case $U(1)$) associated with the conserved charge. What about particles which transform under a real representations? Well they have no charges as previously stated!

6.7 Wigner-Eckart applied: electric dipole selection rules

From electrodynamics we know that the (dominant mode) of interaction between an electric field and a charge distribution is the dipole interaction $H' = -\vec{d}_e \cdot \vec{E}$. The electric dipole operator is the first second term in the multiple interaction and defined as $\vec{d}_e = \int d^3x \rho(x) \vec{x}$ (with ρ being the charge density) and has the same transformation properties as $\vec{x}$. Hence an applied external electric field allows for new types of transitions. The $\vec{x}$ is clearly a $j = 1$ tensor operator and hence the Wigner-Eckart selection rules apply.

Let us assume that the electric field is on the 3-direction which implies that $H' \sim z$. We may write the selection rules by writing

$$H' \sim z \stackrel{(6.60)}{\sim} \mathcal{Y}_{10} \tag{6.68}$$

and hence by virtue of the Wigner-Eckart theorem (6.26)

$$\langle lm|H'|l'm\rangle \propto \langle lm|\mathcal{Y}_{10}|l'm'\rangle = C^{l1l'}_{m0m'}\langle l||\mathcal{Y}_1||l'\rangle \ . \tag{6.69}$$

Hence we have the following selection rules by virtue of Eq. (6.27):

$$m = m' \ , \qquad \Delta l \equiv |l-l'| \leq 1 \ , \tag{6.70}$$

and in addition no transition if $l = l' = 0$.

Eq. (6.70) can be refined by invoking the parity selection rule of the last section. By virtue of $\eta_{\vec{x}} = -1$ (6.65) and (6.63) this implies this forbids $\Delta l = 0$ (i.e.$\Delta l \neq 0$) and hence $\Delta l \leq 1 \to \Delta l = 1$ under parity selection. This is in accordance with (6.64).

Exercise 6.7.1 ***Electric dipole moments***

a) Suppose $\vec{x}$ was a pseudo vector, how would the selection rule change?

6.8 Pauli's Hydrogen atom and $SU(2) \otimes SU(2)$-symmetry

The quantum mechanical solution of the hydrogen atom with Coulomb potential, in units where $\hbar = c = 1$, is given by

$$E_n = -\frac{m_e e^4}{2n^2} \ , \tag{6.71}$$

where n the principal quantum number. More precisely to each fixed n there corresponds $l = 0, 1, 2..(n-1)$ rotation representations (also called mutliplets). That is to say the total degeneracy is

$$\sum_{l=0}^{n-1}(2l+1) = \frac{2(n-1)}{2}n + n = n^2 \ . \tag{6.72}$$

What can be learned from this? Clearly we expect the states of fixed l to be $(2l+1)$-degenerate e.g. $l = 1$ $m = 1, 0, -1$ states. Yet for $n = 2$ there is $l = 1$ and $l = 0$ state and this should be taken as a sign that there is more symmetry than meets the eye. In fact this has been known for a long time (Kepler's problem) that there is an additional conserved vector (Laplace, Runge and Lenz) $A_i = \epsilon_{ijk}p_j L_k - me^2\frac{x_i}{r}$. For use in quantum mechanics the operator is symmetrised and becomes

$$A_i = \epsilon_{ijk}p_j L_k - me^2\frac{x_i}{r} - ip_i \ . \tag{6.73}$$

The following relations are readily verified:

$$\vec{L}\cdot\vec{A} = \vec{A}\cdot\vec{L} = 0 \tag{6.74}$$

$$[L_i, A_j] = i\epsilon_{ijk}A_k \tag{6.75}$$

$$[H, A_i] = 0 \ , \quad H = \frac{p_i p_i}{2m} - \frac{e^2}{r} \tag{6.76}$$

where Eq. (6.76)(6.75) mean that $\vec{A}$ is a conserved vector by virtue of Eqs.(6.14) and (6.28) respectively. Hence a ***hidden symmetry*** has been identified!

Redefining[14] $\tilde{A}_i = A_i(-2mH)^{-1/2}$ one gets

$$[\tilde{A}_i, \tilde{A}_j] = i\epsilon_{ijk}\tilde{A}_k \ , \tag{6.77}$$

which suggests to use,

$$X_i^{\pm} = \frac{1}{2}(L_i \pm \tilde{A}_i) \ , \tag{6.78}$$

since

$$[X_i^{\pm}, X_j^{\pm}] = i\epsilon_{ijk}X_k^{\pm} \ . \tag{6.79}$$

This shows that there is a $SU(2) \otimes SU(2)$-symmetry. An immediate question is what the two corresponding Casimir operators are. It turns out that they are the same by virtue of Eq. (6.74):

$$C_2^+ = (X^+)^2 = (X^-)^2 = C_2^- \tag{6.80}$$

and we shall parameterise it by $C_2 \equiv C_2^{\pm} = x(x+1)$ with $x = 0, 1/2, 1, 3/2, ...$

A rather lengthy, but in principle straightforward computation yields the following relation:

$$A_iA_i = 2mH(L_iL_i + 1) + m^2e^4 \tag{6.81}$$

Dividing both sides by $(-2mH)$ yields

$$H = -\frac{me^4/2}{L_iL_i + \tilde{A}_i\tilde{A}_i + 1} = -\frac{me^4/2}{4C_2 + 1} = -\frac{me^4/2}{(2x+1)^2} \ , \quad x = 0, 1/2, 1, 3/2, .. \ . \tag{6.82}$$

The orbital angular momentum is

$$L_i = X_i^+ + X_i^-$$

and counting the degeneracy therefore amount to counting the states in the direct product representation $\rho_{j=x} \otimes \rho_{j=x}$ which is $(2x+1)^2$ Comparing with (6.71) suggest the following identification with the principal quantum number n

$$n = (2x+1) \ . \tag{6.83}$$

Let us briefly pause and reflect what Pauli has done. He has solved the Hydrogen atom with group theory alone. He has done this by rewriting the Hamiltonian, purely, in terms of the two Casimir operators. Is this an accidental or not? Presumably not. In the absence of spin a single particle state is described by three quantum numbers e.g. $\{x_1, x_2, x_3\}$, $\{p_1, p_2, p_3\}$ or $\{n, l, m\}$. The spherical symmetry guarantees that m does not enter and hence reduces it to two quantum numbers. So a rank two symmetry (or two Casimir operators) can be expected to be sufficient to obtain the solutions with group theory alone.

[14] The Hamiltonian of a bound state problem usually has negative eigenvalues and is therefore chosen $-H$ under the square root.

Epilogue

I hope you have enjoyed this (brief) journey into group theory and some of its applications in quantum mechanics. Group theory is an indispensable tool, as outlined at the beginning, as it often partly solves complex dynamical equations. Even if you have not understood all the details of this course I would hope that you take this on into your professional activities. There are various points where one could have gone deeper in this course. For example a generic study of the simple and compact Lie groups in Fig. 5.1. E.g. the classification of Dynkin and Cartan. Or the study of the representations of the Poincaré group; in particular the so-called spinor calculus. The course as such certainly has a lot of material already and hence it is time to close. Good luck with the exam!