Outline Solutions to Particle Physics Problem Sheet 1

1. **List all fundamental fermions in the Standard Model**

   There are six leptons and six quarks. Leptons: $e^-, \nu_e, \mu^-, \nu_\mu, \tau^-, \nu_\tau$. Quarks: u, d, c, s, t, b. Plus their anti-particle counterparts, *i.e.* 24 fermions in total.

   In fact, we should really list the quarks by their individual colour charges. There's red-up-quarks, $\mathrm{u^R}$, blue-up-quarks, $\mathrm{u^B}$ and green-up-quarks, $\mathrm{u^G}$, and similarly for all of the other quarks flavours.

Generation	Quarks		Leptons	
I	$\mathrm{u^R, u^B, u^G}$	$\mathrm{d^R, d^B, d^G}$	e^-	ν_e
II	$\mathrm{c^R, c^B, c^G}$	$\mathrm{s^R, s^B, s^G}$	μ^-	ν_μ
III	$\mathrm{t^R, t^B, t^G}$	$\mathrm{b^R, b^B, b^G}$	τ^-	ν_τ

2. **What quantum numbers are associated with leptons? Are they conserved in strong, weak and electromagnetic interactions?**

   L_e, L_μ, L_τ, total lepton number $L = L_e + L_\mu + L_\tau$ (and electric charge!) They are conserved in all interactions.

3. **What quantum numbers are associated with quarks, and how are they defined? Are they conserved in strong, weak and electromagnetic interactions?**

   - Total quark number, $N_q = N(\mathrm{q}) - N(\bar{\mathrm{q}})$.
   - Up quark number $N_\mathrm{u} \equiv N(\mathrm{u}) - N(\bar{\mathrm{u}})$
   - Down quark number, $N_\mathrm{d} \equiv N(\mathrm{d}) - N(\bar{\mathrm{d}})$
   - Strange quark number $N_\mathrm{s} \equiv N(\mathrm{s}) - N(\bar{\mathrm{s}})$
   - Charm quark number, $N_\mathrm{c} \equiv N(\mathrm{c}) - N(\bar{\mathrm{c}})$
   - Bottom quark number, $N_\mathrm{b} \equiv N(\mathrm{b}) - N(\bar{\mathrm{b}})$
   - Top quark number, $N_\mathrm{t} \equiv N(\mathrm{t}) - N(\bar{\mathrm{t}})$

   Total quark number and charge are conserved in all interactions. $N_\mathrm{u}, N_\mathrm{d}, N_\mathrm{s}, N_\mathrm{c}, N_\mathrm{b}, N_\mathrm{t}$ are conserved in strong and electromagnetic interactions. They are not (necessarily) conserved in weak interactions.

4. **What are the charge and quark flavour quantum numbers for the $\bar{\mathrm{u}}$, $\bar{\mathrm{d}}$ and $\bar{\mathrm{s}}$ quarks? What are the quantum numbers of the lambda anti-baryon, $\bar{\Lambda}^0$, and of the antiproton, $\bar{p}$? Make sure you understand these in terms of quark content!**

   Antiproton $\bar{p} = (\bar{\mathrm{u}}\bar{\mathrm{u}}\bar{\mathrm{d}})$ and $\bar{\Lambda}^0 = (\bar{\mathrm{u}}\bar{\mathrm{d}}\bar{\mathrm{s}})$.

particle	charge $Q(e)$	N_d	N_u	N_s	N_c	total quark number, N_q
$\bar{\mathrm{u}}$	$-2/3$	0	-1	0	0	-1
$\bar{\mathrm{d}}$	$+1/3$	-1	0	0	0	-1
$\bar{\mathrm{s}}$	$+1/3$	0	0	-1	0	-1
$\bar{p}$	-1	-2	-1	0	0	-3
$\bar{\Lambda}^0$	0	-1	-1	-1	0	-3

5. **Use the Pauli exclusion principle to argue why in the Δ^{++} baryon (which has total spin, $S = 3/2\hbar$ and consists of three up quarks) all the quarks have a different colour charge.**

   The Pauli exclusion principle states that no two fermions in a multi-particle state can have identical quantum numbers.

   The three quarks are fermions, so we have to apply the Pauli exclusion principle to them. They all have identical up-quark number ($N_{\rm u} = +1$), they all have identical spin (e.g. $S = 1/2\hbar, S_z = 1/2\hbar$, to make overall total spin of $3/2\hbar$). Therefore there needs to be another quantum number which is different for the different quarks, and that's the colour charge quantum number. Therefore all the up quarks must carry a different colour charge quantum number: one red, one blue and one green.

6. **Explain why the decays $\mu^- \rightarrow e^-\bar{\nu}_e\nu_\mu$ and $\mu^+ \rightarrow e^+\nu_e\bar{\nu}_\mu$ are allowed and why $\mu^+ \rightarrow e^+\gamma$ and $\mu^+ \rightarrow e^+e^-e^+$ are forbidden.**

   Lepton family number conservation — For all processes, L_e, L_μ & L_τ are conserved quantum numbers, *e.g.*

	μ^-	$\rightarrow$	$e^-\bar{\nu}_e\nu_\mu$			μ^+	$\rightarrow$	$e^+e^-e^+$	
L_e :	0		$1+(-1)+0=0$	OK!	L_e :	0		$1+(-1)+1=+1$	forbidden
L_μ :	1		$0+0+1=1$	OK!	L_μ :	1		$0+0+0=0$	forbidden

   What about $\mu^+ \rightarrow e^-\bar{\nu}_e\nu_\mu$? We don't even need to think about lepton number here, this decay violates conservation of charge!

7. **What is 1 fm in inverse GeV? How many seconds is 1 inverse GeV?**

   I've written this answer in quite a lot of detail, as this concept can be quite confusing.

   - We need to write length in terms of inverse energy, and some (as yet unknown) factors of c and $\hbar$. We can do this by balancing dimensions:

     $$[L] = [E]^{-1}[\hbar]^m[c]^n = [E]^{-1}[E]^m[T]^m[L]^n[T]^{-n} = [E]^{m-1}[T]^{m-n}[L]^n$$

     Where we have used the fact that $\hbar$ is measured in $[E][T]$ and c is measured in $[L][T]^{-1}$. The dimensions on both side of the equation have to balance. So we have to set $n = 1$ and $m = 1$. Therefore length is measured in $\text{GeV}^{-1}\hbar c$. What we want to know is how many $\text{GeV}^{-1}\hbar c$ corresponds to 1 fm:

     $$1\ \text{fm} = l\ \text{GeV}^{-1}\hbar c \quad \Rightarrow \quad l = 1\ \text{fm}/(\hbar c)$$

     We have to use measurements of $\hbar$ and c in appropriate units: GeV s and fm/s. In lecture 2, page 5 there is an expression in the appropriate units:

     $$\hbar c = 197\ \text{MeV fm} = 0.197\ \text{GeV fm}$$

     Therefore $l = 1/0.197 \approx 5$. Or $1\ \text{fm} \approx 5\ \text{GeV}^{-1}$.

   - To write time in units of inverse energy:

     $$[T] = [E]^{-1}[\hbar]^m[c]^n = [E]^{-1}[E]^m[T]^m[L]^n[T]^{-n} = [E]^{m-1}[T]^{m-n}[L]^n$$

     This time to make the dimensions balance $m = 1, n = 0$. Then:

     $$1\ \text{s} = t\ \text{GeV}^{-1}\hbar \quad \Rightarrow \quad t = 1\ \text{s}/(\hbar)$$

     $\hbar$ in appropriate units is $= 6.58\times10^{-22}$ MeV s $= 6.58\times10^{-25}$ GeV s. Therefore, $t = 1/6.58 \times 10^{-25} = 1.5 \times 10^{24}$.

     We can invert this to get the answer: $1\ \text{GeV}^{-1} = 6.58 \times 10^{-25}$s.

8. **Write down the typical lifetimes for particles that decay by:**
   (a) **The strong force**: 10^{-23} to 10^{-20}s
   (b) **The electromagnetic force**: 10^{-20} to 10^{-16}s
   (c) **The weak force**: 10^{-13} to 10^{3}s

   By looking at the liftetimes on the Particle Properties sheet, which force is responsible for the decay of π^0, B^+, ω^0?
   - π^0 lifetime is 0.83×10^{-16} s, electromagnetic decay.
   - B^+ lifetime is 1.5×10^{-12} s, weak decay.
   - ω^0 lifetime is 0.8×10^{-22} s, strong decay.

9. **The lifetime of the η'^0 has not been measured directly. The total width of the η'^0 has been measured to be $\Gamma(\eta'^0) = 0.203 \pm 0.016$ MeV. What is the lifetime of the η'^0? What force is responsible for its decay?**
   - The total width of any particle is given by $\Gamma = \hbar/\tau \Rightarrow \tau(\eta'^0) = \hbar/\Gamma(\eta'^0)$.
   - Using $\hbar = 6.58 \times 10^{-22}$ MeV s, gives $\tau(\eta'^0) = 3.24 \times 10^{-21}$ s.
   - By looking at the lifetime we see that the strong force is responsible for the decay of η'^0.
   - You could also do this problem in natural units ($\hbar = c = 1$). Then $\tau = 1/\Gamma = 4.93$ MeV^{-1}. We can convert this back to seconds using the answer to question 3: 1 s $= 1.5 \times 10^{24}$ GeV^{-1} $= 1.5 \times 10^{21}$ MeV^{-1}, therefore: $\tau(\eta'^0) = 4.93/1.5 \times 10^{21} = 3.24 \times 10^{-21}$ s.

10. **What are the Centre-of-Momentum (CM) energies of the following machines:**
    - **LEP1: e^+e^- collider, both beams 45.6 GeV**
    - **LHC: pp collider, both beams 7 TeV**
    - **HERA: ep collider, $E_\mathrm{e} = 30$ GeV and $E_\mathrm{p} = 820$ GeV.**

    If HERA were a fixed target machine what energy would the electron require to give an equivalent CM energy?

    We write the four-momenta of the two beams as: $p_a = (E_a/c, \vec{p}_a)$ and $p_b = (E_b/c, \vec{p}_b)$. The four momenta is the thing you wrote in Dynamics and Relativity as $\underline{\underline{p_a}}$.

    The first thing to note is that if we square any four momenta, we get the mass of the particle:

$$p_a^2 = (E_a/c)^2 - \vec{p}_a \cdot \vec{p}_a = m_a^2 c^2$$

    This is much easier if we use natural units, so set $c = 1$. Then we get $p_a = (E_a, \vec{p}_a)$ and $p_b = (E_b, \vec{p}_b)$, and $p_a^2 = m_a^2$.

    The centre-of-mass energy, $\sqrt{s}$, is calculated from:

$$\begin{aligned} s = (E_{\mathrm{CoM}})^2 &\equiv (p_a + p_b)^2 \\ &= p_a^2 + p_b^2 + 2p_a p_b \\ &= m_a^2 + m_b^2 + 2(E_a\, E_b - \vec{p}_a \cdot \vec{p}_b) \\ &= m_a^2 + m_b^2 + 2(E_a\, E_b - |p_a||p_b|\cos\theta) \approx 4E_a E_b \end{aligned}$$

    Where we have assumed the beams collide head on: $\theta = 180°$, and the energies of the beam is much larger than the mass of the colliding particles: $E_a, E_b \gg m_a, m_b$.

- Here $E_e = 45.6$ GeV, much larger than the mass of the electron, $m_e = 511$ keV. Then $\sqrt{s} = 2E_e = 91.2$ GeV. (*The mass of the Z-boson, which was the whole point of LEP!*)
- At LHC, again we can ignore the proton mass: $\sqrt{s} = 2E_p = 14$ TeV.
- At HERA $s = 4 \times 30 \times 820 = 98400$ GeV^2 or $\sqrt{s} = 314$ GeV/c.

Note that at proton colliders not all this energy is in practise available, since only a fraction of the proton momenta is carried by the quarks and gluons, which are the particles actually involved in the scattering.

At a fixed target machine $s = 2m_\mathrm{p}E_\mathrm{e}$ (electron and proton mass can be neglected) which implies that $E_\mathrm{e} = 52.5$ TeV would be necessary to achieve the same CoM-energy. As a consequence colliding beam machines are the only viable means of reaching the highest energies.

11. **The $\boldsymbol{\Delta^{++}}$ baryon can be produced as a resonance by aiming a pion beam onto a hydrogen target to produce the reaction $\boldsymbol{\pi^+ p \rightarrow \Delta^{++} \rightarrow \pi^+ p}$. Calculate the energy and momentum of the pions in the $\boldsymbol{\Delta^{++}}$ centre-of-mass frame.**

We need the masses of the particles, from the particle properties sheet: $m_\pi = 139.6$ MeV/c^2, $m_p = 938.3$ MeV/c^2 and $m_\Delta = 1232$ MeV/c^2.

In the centre-of-mass frame of the Δ^{++}, it is at rest. Therefore its three momentum is 0, and therefore the energy of the Δ is $E_\Delta = m_\Delta c^2$.

We can write down the four-momenta for the Δ^{++}, π^+ and p as:

$$\underline{\underline{p}}_\Delta = (m_\Delta c, \vec{0}\,) \qquad \underline{\underline{p}}_\pi = (E_\pi/c, \vec{p}_\pi) \qquad \underline{\underline{p}}_p = (E_p/c, \vec{p}_p)$$

Using four momentum conservation:

$$\underline{\underline{p}}_\Delta = \underline{\underline{p}}_\pi + \underline{\underline{p}}_p \qquad \Rightarrow \qquad \underline{\underline{p}}_p = \underline{\underline{p}}_\Delta - \underline{\underline{p}}_\pi$$

Squaring this we get:

$$\begin{aligned} \left(\underline{\underline{p}}_p\right)^2 &= \left(\underline{\underline{p}}_\Delta\right)^2 + \left(\underline{\underline{p}}_\pi\right)^2 - 2\underline{\underline{p}}_\Delta \cdot \underline{\underline{p}}_\pi \\ m_p^2 c^2 &= m_\Delta^2 c^2 + m_\pi^2 c^2 - 2m_\Delta c \cdot E_\pi/c \end{aligned}$$

Where we have used the key fact: $\left(\underline{\underline{p}}\right)^2 = m^2c^2$. Rearranging we get:

$$E_\pi/c^2 = \frac{1}{2m_\Delta}(m_\Delta^2 + m_\pi^2 - m_p^2) \qquad \Rightarrow \qquad E_\pi = 266.6 \text{ MeV}$$

We can get the three momentum of the pion using $E^2 = \vec{p}^{\,2}c^2 + m^2c^4$:

$$|\vec{p}_\pi| = \sqrt{(E_\pi/c)^2 - (m_\pi c)^2} = 227.1 \text{ MeV/c}$$

From the measured total width $\Gamma(\Delta)$=120 MeV calculate the lifetime of the Δ^{++}.

Using $\hbar = 6.58 \times 10^{-22}$ MeV s we obtain the Δ^{++} lifetime $\tau_\Delta = \frac{\hbar}{\Gamma} = 5.5 \times 10^{-24}$ s.

12. **The B_d meson has a mass of 5.28 GeV/c^2 and mean lifetime of 1.54 ps. At the LEP B_d mesons are produced with an average energy of 32 GeV. Calculate the mean decay length of a B_d meson.**

    The average decay length in the lab frame is $L = \gamma\beta c\tau_B$.

    We can calculate the Lorentz factors $\gamma\beta$ from the average B_d momentum:

    $$p_B = \sqrt{(E_B/c)^2 - (m_B c)^2} = 31.56 \text{ GeV/c}$$

    This gives $\gamma\beta = pc/mc^2 = p/mc = 31.56/5.28 = 5.98$

    Also, $c\tau_B = 3 \times 10^8 \times 1.54 \times 10^{-12} = 4.62 \times 10^{-4}$m

    Then the mean decay length is: $L = \gamma\beta c\tau_B = 5.98 \times 4.62 \times 10^{-4} = 0.00276$ m = 2.67mm, which is a measurable distance.

13. **The cross section to make b-quarks at LEP with $E_{\text{CM}} = 91.2$ GeV was $\sigma(e^+e^- \to \text{b}\bar{\text{b}}) = 4.5$ nb. How many $e^+e^- \to \text{b}\bar{\text{b}}$ events were produced with a integrated luminosity of $\int \mathcal{L}dt = 100$ pb^{-1}?**

    $N = \sigma \int \mathcal{L}dt = 4.5 \text{ nb} \times 100 \text{ pb}^{-1} = 4.5 \text{ nb} \times 100,000 \text{nb}^{-1} = 450,000.$

    450,000 $e^+e^- \to \text{b}\bar{\text{b}}$ events were produced with 100 pb^{-1}.