Particle Physics

Dr Victoria Martin, Spring Semester 2012
Lecture 5: Dirac Spinors
Describing Fermions and Bosons

★Schrödinger Equation
★Klein-Gordon Equation
★Dirac Equation
★Spinors
★Helicity and handedness
★Bosons: Polarisation Vectors

Notation Review

- A μ sub- or super- script represents a four vector, e.g. x^μ, p^μ, p_μ
  - μ runs from 0 to 3

$$p^\mu = (p^0, p^1, p^2, p^3) = (E, p_x, p_y, p_z)$$

- This lecture also introduce other quantities with μ index, μ=0,1,2,3

- The scalar product of two four vectors

$$a^\mu b_\mu = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$$

- The three dimension differential operator

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

- Four dimension differential operator

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Schrödinger Equation

- Classical energy-momentum relationship:

$$E = \frac{p^2}{2m} + V$$

- Substitute QM operators:

$$\hat{p} = -i\hbar\vec{\nabla} \qquad \hat{E} = i\hbar\frac{\partial}{\partial t}$$

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\hbar^2\frac{\nabla^2}{2m} + V\right)\psi = \hat{H}\psi$$

Schrödinger equation!

- 1st order in $\partial/\partial t$; 2nd order in $\partial/\partial x$. Space and time not treated equally.

Klein-Gordon Equation

- Relativistic energy-momentum relationship is:

$$E^2 = \vec{p}^{\,2}c^2 + m^2c^4$$

in covariant notation with $c=1$: $p_\mu p^\mu = E^2 - \vec{p}^{\,2} = m^2$

- Again substitute the operators: $\hat{p} = -i\hbar\vec{\nabla} \quad \hat{E} = i\hbar\frac{\partial}{\partial t}$
- To give the relativistic Klein-Gordon equation:

$$-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \nabla^2\psi = \left(\frac{mc}{\hbar}\right)^2\psi$$

- This is the four-dimensional wave equation. Solutions are plane wave solutions introduced in lecture 4:

$$\psi = e^{-ip\cdot x} \qquad p\cdot x = p^\mu x_\mu = \hbar(\vec{k}\cdot\vec{x} - \omega t)$$

- **The Klein-Gordon equation describes spin-0 bosons.**
- KG equation is 2nd order in $\partial/\partial t$ and $\partial/\partial x$

Negative Energy & the Dirac Equation

- The relativistic energy-momentum equation is quadratic, negative energy solutions are possible:

$$E^2 = \vec{p}^{\,2} + m^2 \qquad \Rightarrow \qquad E = \pm\sqrt{\vec{p}^{\,2} + m^2}$$

- (From last lecture) probability density for plane wave = ***2E.***
- Negative energy solutions imply negative probability density!

- In 1931 Dirac predicted that the negative energy solutions could describe antiparticles - before its discovery! However negative probability density was still a problem

- Dirac searched for 1st order relationship between energy and momentum, using coefficients α^1 α^2 α^3 and β

$$\hat{E}\,\psi = (\vec{\alpha}\cdot\vec{p} + \beta m)\psi = i\frac{\partial\psi}{\partial t}$$

- Need to find solutions for α and β

Dirac Equation

- Solution is more elegant defining $\gamma^0 \equiv \beta$, $\gamma^1 \equiv \beta\alpha^1$, $\gamma^2 \equiv \beta\alpha^2$, $\gamma^3 \equiv \beta\alpha^3$
- The Dirac equation can be written (with $c = \hbar = 1$) as:

$$i\left(\gamma^0\frac{\partial\psi}{\partial t} + \vec{\gamma}\cdot\vec{\nabla}\right)\psi = m\psi$$

in covariant notation: $i\gamma^\mu\partial_\mu\psi = m\psi$

- Multiplying the Dirac equation by its complex conjugate must give KG:

$$\left(-i\gamma^0\frac{\partial}{\partial t} - i\vec{\gamma}\cdot\vec{\nabla} - m\right)\left(i\gamma^0\frac{\partial}{\partial t} + i\vec{\gamma}\cdot\vec{\nabla} - m\right) = 0$$

- This leads to a set of conditions on the four coefficients γ^μ:

$$(\gamma^0)^2 = 1 \qquad (\gamma^1)^2 = -1 \qquad (\gamma^2)^2 = -1 \qquad (\gamma^1)^3 = -1$$
$$\{\gamma^i, \gamma^j\} = \gamma^i\gamma^j + \gamma^j\gamma^i = 0$$

γ^μ are unitary and anticommute

The Gamma Matrices

- To satisfy unitarity and anticommutation the γ^μ must be at least 4 × 4 matrices (exercise to check this!)
- There is a choice of representation, but the conventional one is:

$$\gamma^0 = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix}$$

- where $\mathbf{I}$ and $\mathbf{0}$ are the 2 × 2 identity and null matrices:

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

- and the $\boldsymbol{\sigma}^i$ are the 2 × 2 Pauli spin matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- γ^μ are not four vectors! They do however remain constant under Lorentz transformations

Solutions of the Dirac Equation

- The Dirac equation: $(i\gamma^\mu\partial_\mu - m)\psi = 0$
- The wavefunctions are written as a combination of a plane wave and a **Dirac spinor** u, a function of the four momentum, p^μ :

$$\psi = u(p^\mu)e^{-ip\cdot x}$$

- For a particle at rest $p^\mu = (m, 0)$
- The Dirac equation becomes:

$$\left(-i\gamma^0\frac{\partial\psi}{\partial t} - i\vec{\gamma}\cdot\vec{\nabla} - m\right)\psi = (-i\gamma^0(-iE) - m)\psi = 0$$

$$Eu = \begin{pmatrix} m\mathbf{I} & 0 \\ 0 & -m\mathbf{I} \end{pmatrix} u$$

- Defines four energy eigenstates, u^μ
  - u^1 and u^2 have $E = m$ (fermions)
  - u^3 and u^4 have $E = -m$ (antifermions)
  - $u^3(p)$ and $u^4(p)$ are often written as $v^1(p)=u^4(-p)$ and $v^2(p)=u^3(-p)$

Spinors at rest

- The spinors of a particle are written as 1 × 4 matrices. (They are not four-vectors.)
- Making the equation first order in all derivatives introduces new degrees of freedom!
- For a particle at rest they take the trivial form:

$$u^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad u^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad u^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad u^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- The wavefunctions are:

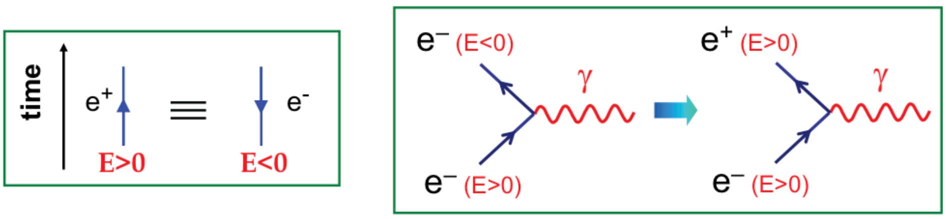
$$\psi^1 = u^1\, e^{-imt} \qquad \psi^2 = u^2\, e^{-imt} \qquad \psi^3 = u^3\, e^{+imt} \qquad \psi^4 = u^4\, e^{+imt}$$

- The $E = -m$ solution still exists!
- However we will see later that the probability density is strictly positive for both particles and antiparticles.

Negative Energy Solutions

- We can’t escape negative energy solutions. How should we interpret them?
- **Modern Feynman-Stückelberg Interpretation:**

  A negative energy solution is a negative energy particle which propagates backwards in time or equivalently a positive energy anti-particle which propagates forwards in time.



$$e^{-i(-E)(-t)} \longrightarrow e^{-iEt}$$

- This is why in Feynman diagrams the backwards pointing lines represent anti-particles.

Discovery of Positron

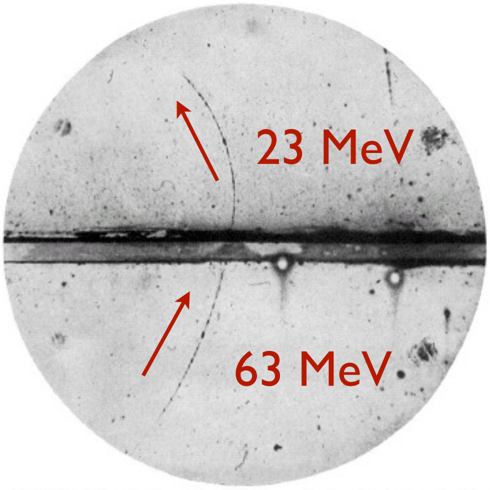
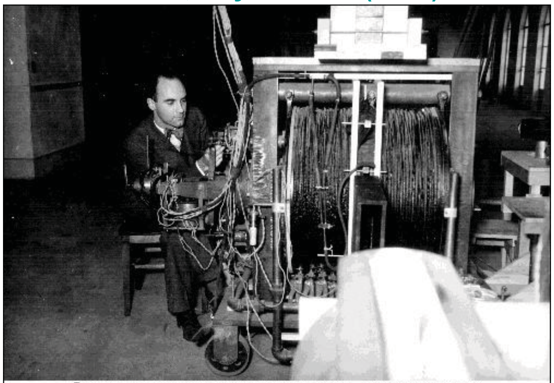


FIG. 1. A 63 million volt positron ($H\rho = 2.1 \times 10^5$ gauss-cm) passing through a 6 mm lead plate and emerging as a 23 million volt positron ($H\rho = 7.5 \times 10^4$ gauss-cm). The length of this latter path is at least ten times greater than the possible length of a proton path of this curvature.

C.D.Anderson, Phys Rev 43 (1933) 491



- e^+ enters at bottom, slows down in the lead plate - know direction
- Curvature in B-field shows that it is a positive particle
- Can't be a proton as would have stopped in the lead

Spinors for moving particles

- For the derivation of these see Griffiths Pp. 231-234
- Using v for antiparticles

Fermions:

$$u^1 = \begin{pmatrix} 1 \\ 0 \\ p_z/(E+m) \\ (p_x+ip_y)/(E+m) \end{pmatrix} \qquad u^2 = \begin{pmatrix} 0 \\ 1 \\ (p_x-ip_y)/(E+m) \\ -p_z/(E+m) \end{pmatrix}$$

Antifermions:

$$v^2 = \begin{pmatrix} p_z/(E+m) \\ (p_x+ip_y)/(E+m) \\ 1 \\ 0 \end{pmatrix} \qquad v^1 = \begin{pmatrix} (p_x-ip_y)/(E+m) \\ -p_z/(E+m) \\ 0 \\ 1 \end{pmatrix}$$

- The u and v are the solutions of

$$(\gamma^\mu p_\mu - m)u = 0 \qquad (\gamma^\mu p_\mu + m)v = 0$$

Fermion Wavefunctions

- The two different solutions for each of fermions and antifermions corresponds to the two possible spin states.
- A fermion with 4-momentum p is either:

$\psi = u^1(p)\, e^{-ip\cdot x}$ spin ↑ parallel to p_z

$\psi = u^2(p)\, e^{-ip\cdot x}$ spin ↓ antiparallel to p_z

- An antifermion with 4-momentum p is either:

$\psi = v^1(p)\, e^{-ip\cdot x} = u^4(-p)\, e^{-i(-p)\cdot x}$ spin ↑ parallel to p_z

$\psi = v^2(p)\, e^{-ip\cdot x} = u^3(-p)\, e^{-i(-p)\cdot x}$ spin ↓ antiparallel to p_z

Spinors, u^1, u^2, v^1, v^2 are only eigenstates of $\hat{S}_z$ for $p_z = \pm|\vec{p}|$

However u^1, u^2, v^1, v^2 do describe a complete set of solutions for the Dirac equation

Helicity

- Define helicity, $\hat{h}$, the component of the spin along a particle's direction of flight.

$$\hat{h} = \frac{\vec{S}\cdot\vec{p}}{|\vec{S}||\vec{p}|} = \frac{2\vec{S}\cdot\vec{p}}{|\vec{p}|}$$

- For a S=½ fermion, the project of spin along any axis can only be ±½.
- For a S=½ fermion, eigenvalues of $\hat{h}$ are ±1.
- We call h=+1, "right-handed", h=−1 "left handed".

- Massless fermions with (p=E) are purely left-handed (only u^2)
- Massless antifermions are purely right-handed (only v^1)
- Non-massless particles need a superposition of u^1 and u^2 to fully describe the state
- Non-massless antiparticles need a superposition of v^1 and v^2.

Handedness and Projection Operators

- The concept of handedness is very useful and plays a **key role** in describing the interactions of the forces.
- Helicity not Lorentz invariant instead use Lorentz invariant **chirality**.

- **LH projection** operator $P_L = (1 - \gamma^5)/2$ projects out **left-handed chiral** state
- **RH projection** operator $P_R = (1 + \gamma^5)/2$ projects out **right-handed chiral** state

where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is 4×4 matrix: $\gamma^5 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$

It has the properties: $(\gamma^5)^2 = \mathbf{1}$, $\{\gamma^5, \gamma^i\} = \gamma^5\gamma^i + \gamma^i\gamma^5 = \mathbf{0}$

- For massless particles or high energies $(E>>m)$ chirality & helicity are the same.
- $P_L + P_R = 1 \Rightarrow \psi = P_L\psi + P_R\psi$, a state can always be written as the sum of LH and RH components
- Also, $P_L^2 = P_L$ $\quad P_R^2 = P_R$ $\quad P_L P_R = 0$

Summary and Reading List

- The Dirac Equation describes spin-½ particles.

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

- Solutions include four component **spinors**, u and v.

$$(\gamma^\mu p_\mu - m)u = 0 \qquad (\gamma^\mu p_\mu + m)v = 0$$

$$\psi = u(p)e^{-ip\cdot x} \qquad \psi = v(p)e^{-ip\cdot x}$$

- γ^μ, $\mu=0,1,2,3$ are 4 × 4 Gamma matrices
- Four components describe e.g. two spin states of the electron and two spin states of the positron.
- Negative energy solutions, $E=-\sqrt{(p^2+m^2)}$ are predicted by the Dirac equation.
- Modern interpretation is to is reverse the sign of x^μ and p^μ: giving a positive energy anti-particle travelling forwards in time.
- Any particle can be written in terms of left handed and right handed components: $\psi = (1 - \gamma^5)\psi + (1 + \gamma^5)\psi = \psi_L + \psi_R$
- Spin-1 bosons are described by the **polarisation vector,** ε^μ: $A^\mu = \varepsilon^\mu(p\,;s)\,e^{-ip\cdot x}$
- Next Lecture: The Electromagnetic Force. Griffiths 7.5 & 7.6