8 Probing the Proton: Electron - proton scattering

Scattering of charged leptons by protons is an electromagnetic interaction. Electron beams have been used to probe the structure of the proton (and neutron) since the 1960s, with the most recent results coming from a high energy electron-proton collider called HERA at DESY in Hamburg.

These experiments provide direct evidence for the composite nature of protons and neutrons, and measure the distributions of the quarks and gluons inside the nucleon. The results of $e^- p \rightarrow e^- p$ scattering depend strongly on the wavelength $\lambda = E/\hbar c$.

- At very low electron energies $\lambda >> r_p$, where $r_p$ is the radius of the proton, the scattering is equivalent to that from a point-like spin-less object
- At low electron energies $\lambda \sim r_p$, the scattering is equivalent to that from a extended charged object
- At high electron energies $\lambda < r_p$: the wavelength is sufficiently short to resolve sub-structure. Scattering from constituent quarks
- At very high electron energies $\lambda << r_p$: the proton appears to be a sea of quarks and gluons.

8.1 Form Factors

Extended object - like the proton - have a matter density $\rho(r)$, normalised to unit volume: $\int d^3\vec{r} \rho(\vec{r}) = 1$. The Fourier Transform of $\rho(r)$ is the form factor, $F(q)$:

$$F(q) = \int d^3\vec{r} \exp\{i\vec{q} \cdot \vec{r}\} \rho(\vec{r}) \Rightarrow F(0) = 1$$

(8.1)

Cross section from extended objects are modified by the form factor:

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{extended}} \approx \left.\frac{d\sigma}{d\Omega}\right|_{\text{point-like}} |F(q)|^2$$

(8.2)

For $e^- p \rightarrow e^- p$ scattering we need form factors are required: $F_1$ to describe the distribution of the electric charge $F_2$ to describing the recoil of the proton.

8.2 Elastic Scattering

The elastic scattering of a pointlike spin-1/2 electron by a pointlike spin 1/2 target is described in the relativistic limit $p_e = E_e$ by the Mott formula:

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{point}} = \frac{\alpha^2}{4p_e^2 \sin^4 \frac{\theta}{2}} \left(\cos^2 \frac{\theta}{2} - \frac{q^2}{2m_p^2 \sin^2 \theta} \frac{\theta}{2}\right)$$

(8.3)
In this formula $\theta$ and $p_e$ are in the Lab frame. The no recoil limit corresponds to a very massive target $m_p^2 \gg q^2$.

In the non-relativistic limit $p_e \ll m_e$ this reduces to Rutherford scattering:

$$\frac{d\sigma}{d\Omega}_{\text{NR}} = \frac{\alpha^2}{4 m_e^2 v^4 \sin^4 \frac{\theta}{2}}$$ (8.4)

$$e^-(p_1) \xrightarrow{i e \gamma^\mu} e^-(p_3)$$

$$p(p_2) \xrightarrow{-i e K^\nu} p(p_4)$$

$$2m_p \nu = Q^2 = -q^2$$ (8.5)

Note that $\nu > 0$ by energy conservation, so $Q^2 > 0$ and the mass squared of the virtual photon is negative, $q^2 < 0$!

### 8.3 Form Factors

Deviations from the point-like Mott scattering are described in the no recoil limit by a form factor $F(q^2)$, related to the finite size of a charge distribution inside the proton:

$$\frac{d\sigma}{d\Omega} = \left. \frac{d\sigma}{d\Omega} \right|_{\text{point}} |F(q^2)|^2$$ (8.6)

At low $q^2$ the distances probed are large compared to the size of the proton, so the scattering still appears point-like with $F(0) = 1$. As $q^2$ gets larger the electron probes deeper into the proton, and $F(q^2)$ is found to decrease. Mathematically, the form factor is the Fourier transform of the charge distribution inside the proton:

$$F(q^2) = \int \rho(\vec{x}) \exp(i \vec{q} \cdot \vec{x}) d^3x$$ (8.7)

The charge distribution $\rho(\vec{x})$ is assumed to be spherically symmetric, and normalised to one. It has a mean square radius $< r^2 >$:

$$\int \rho(r) d^3r = 1 \quad < r^2 >= \int r^2 \rho(r) d^3r$$ (8.8)

The matrix element for elastic scattering is written in terms of electron and proton currents:

$$\mathcal{M}(e^-p \to e^-p) = \frac{e^2}{(p_1 - p_3)^2} (\bar{u}_3 \gamma^\mu u_1) (\bar{u}_4 K^\mu u_2)$$ (8.9)
where the proton is treated as an extended structure, with a current operator $K^\mu$ that
is more complex than the point-like $\gamma^\mu$:

$$K^\mu = \gamma^\mu F_1(q^2) + \frac{i\kappa_p}{2m_p} F_2(q^2)\sigma^{\mu\nu}q_\nu + q^\nu F_3(q^2)$$  \hspace{1cm} (8.10)

In this general form there are three form factors $F_1$, $F_2$ and $F_3$ which are functions of $q^2$. However, electromagnetic current conservation $\delta_\mu(\bar{u}_3 K^\mu u_2) = 0$ implies that $F_3 = 0$. $F_1$ is the electrostatic form factor, while $F_2$ is associated with the recoil of the proton.

The **anomalous magnetic moment** of the proton is defined by:

$$\mu_p = \frac{e(1 + \kappa_p)}{2m_p} \kappa_p = 1.79$$  \hspace{1cm} (8.11)

The differential cross section for elastic electron-proton scattering is

$$\frac{d\sigma}{d\Omega}_{\text{lab}} = \frac{\alpha^2}{4E_1^2 \sin^4 \frac{\theta}{2} E_1} \left\{ \left( F_1^2 - \frac{\kappa^2 q^2}{4m_p^2} F_2^2 \right) \cos^2 \frac{\theta}{2} - \frac{q^2}{2m_p^2} (F_1 + \kappa F_2)^2 \sin^2 \frac{\theta}{2} \right\}$$  \hspace{1cm} (8.12)

where the two form factors $F_1$ and $F_2$ are functions of $q^2$ which parameterise the structure of the proton. They have to be determined by experiment. For a point-like spin 1/2 particle, $F_1 = 1$, $\kappa = 0$, and the above equation reduces to the Mott scattering result.

It is common to use linear combinations of the form factors:

$$G_E = F_1 + \frac{\kappa q^2}{4m_p^2} F_2 \quad G_M = F_1 + \kappa F_2$$  \hspace{1cm} (8.13)

which are referred to as the **electric** and **magnetic** form factors, respectively.

The differential cross section can be rewritten as:

$$\frac{d\sigma}{d\Omega}_{\text{lab}} = \frac{\alpha^2}{4E_1^2 \sin^4 \frac{\theta}{2} E_1} \left( \frac{G_E^2 + \tau G_M^2}{1 + \tau} - \frac{2\tau G_M^2 \sin^2 \frac{\theta}{2}}{1 + \tau} \right)$$  \hspace{1cm} (8.14)

where we have used the abbreviation $\tau = Q^2/4m_p^2$. The experimental data on the form factors as a function of $q^2$ are in good agreement with a “dipole” fit:

$$G_E = \frac{G_M}{\mu_p} = \left( \frac{\beta^2}{\beta^2 + Q^2} \right)^2 \quad \beta = 0.84\text{GeV}$$  \hspace{1cm} (8.15)

This corresponds to an exponential charge distribution:

$$\rho(r) = \rho_0 \exp(-r/r_0) \quad 1/r_0^2 = 0.71\text{GeV}^2 \quad <r^2> = 0.81fm^2$$  \hspace{1cm} (8.16)

The whole of the above discussion can be repeated for electron-neutron scattering, with similar form factors for the neutron.
Note that even though the neutron has zero total charge, it has an anomalous magnetic moment:

\[ \mu_n = \frac{e\kappa_n}{2m_n} \quad \kappa_n = -1.91 \]  \hspace{1cm} (8.17)

The anomalous magnetic moments are themselves evidence that the proton and neutron are not pointlike Dirac fermions.
8.4 Deep Inelastic Scattering

During inelastic scattering the proton can break up into its constituent quarks which then form a hadronic jet. At high $q^2$ this is known as deep inelastic scattering (DIS).

\[
\begin{array}{c}
\text{e}^{-}(p_1) \\
\text{i}e\gamma^{\mu} \\
\text{q} \\
\text{e}^{-}(p_3) \\
p(p_2) \\
X
\end{array}
\]

The invariant mass, $W$, of the final state hadronic jet is:

\[
W^2 = m_p^2 + 2m_p\nu + q^2 \tag{8.18}
\]

Since $W \neq m_p$ the four-momentum and energy transfer, $q^2$ and $\nu$, are two independent variables in DIS, and it is necessary to measure $E_1$, $E_3$ and $\theta$ in the Lab frame to determine the full kinematics.

It is useful to introduce two dimensionless variables, a parton energy $x$, and a rapidity $y$, which replace $\nu$ and $q^2$

\[
x = \frac{Q^2}{2m_p\nu} = \frac{-q^2}{2m_p\nu} \quad y = \frac{p_2 \cdot q}{p_2 \cdot p_1} = \frac{\nu}{E_1} \tag{8.19}
\]

It is an exercise to show that the allowed kinematic ranges of these variables are $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

8.5 Structure Functions

The matrix element squared for DIS can be factorised into lepton and hadron currents:

\[
|M|^2 = \frac{e^4}{q^2} L_{\nu}^{\mu\nu}(W_{\text{hadron}})_{\mu\nu} \tag{8.20}
\]

where the hadronic part $W_{\text{hadron}}^{\mu\nu}$ describes the inelastic breakup of the proton. As in elastic scattering, there are two independent form factors in $W_{\text{hadron}}$, which are known as structure functions $W_1(\nu, Q^2)$ and $W_2(\nu, Q^2)$.

In the Lab frame, the doubly differential cross section for deep inelastic scattering is:

\[
\left. \frac{d\sigma}{dE_3 d\Omega_{\text{lab}}} \right|_{\text{lab}} = \frac{\alpha^2}{4E_1^2 \sin^4 \frac{\theta}{2}} \left( W_2(\nu, Q^2) \cos^2 \frac{\theta}{2} + 2W_1(\nu, Q^2) \sin^2 \frac{\theta}{2} \right) \tag{8.21}
\]
Bjorken predicted that at high energy the structure functions should exhibit a property called **scaling**:

\[
m_\nu W_1(\nu, Q^2) \to F_1(x) \quad \nu W_2(\nu, Q^2) \to F_2(x)
\]

(8.22)

where \(F_1(x)\) and \(F_2(x)\) are now functions of \(x\) alone.

*Note that the structure functions \(F_1(x)\) and \(F_2(x)\) in DIS are different from the elastic form factors \(F_1(q^2)\) and \(F_2(q^2)\).*

The experimental data for \(F_2\) are shown in figure 8.1. For intermediate regions of \(x\) and \(Q^2\) scaling holds, but at high \(Q^2\) and low \(x\) there is a significant amount of **scaling violation** which we will discuss later.

![HERA F2](image)

Figure 8.1: Structure function \(F_2\) for large \(Q^2\) and small \(x\), as measured at HERA using collisions between 30 GeV electrons and 830 GeV protons.
The parton model was proposed by Feynman in 1969, to describe deep inelastic scattering in terms of point-like constituents inside the nucleon known as **partons** with an effective mass \( m < m_p \). Nowadays partons are identified as being quarks or gluons.

\[
e^-(p_1) \rightarrow q + m
\]

\[
e^-(p_3)
\]

The parton model restores the elastic scattering relationship between \( q^2 \) and \( \nu \), with \( m \) replacing \( m_p \):

\[
\nu + \frac{q^2}{2m} = 0
\] (8.23)

and the cross section for elastic electron-parton scattering is:

\[
\frac{d\sigma}{d\Omega}_{\text{lab}} = \frac{Z^2 \alpha^2}{4E^2 \sin^4 \frac{\theta}{2}} \frac{E_3}{E_1} \left( \cos^2 \frac{\theta}{2} - \frac{q^2}{2m^2} \sin^2 \frac{\theta}{2} \right)
\] (8.24)

where \( Z \) is the charge of the parton (+2/3 or −1/3 for quarks).

Effectively the DIS structure functions for scattering off a single parton, have become delta functions:

\[
2W_1 = \frac{Q^2}{2m^2} \delta(\nu - \frac{Q^2}{2m}) \quad W_2 = \delta(\nu - \frac{Q^2}{2m})
\] (8.25)

It can be seen that the parton energy variable:

\[
x = \frac{Q^2}{2m_p \nu} = \frac{m}{m_p}
\] (8.26)

is the fraction of the proton rest mass carried by the parton.

### 8.7 Parton Distribution Functions

We introduce the parton distribution functions, \( f_i(x) \), defined as the probability that a parton of type \( i \) carries a fraction \( x \) of the proton mass. The structure functions can then be written:

\[
W_1(x) = \frac{F_1(x)}{m_p} \quad F_1(x) = \frac{1}{2} \sum_i Z_i^2 f_i(x)
\] (8.27)

\[
W_2(x) = \frac{F_2(x)}{\nu} \quad F_2(x) = \sum_i xZ_i^2 f_i(x)
\] (8.28)
The structure functions $F_1$ and $F_2$ satisfy the Callan-Gross relation:

$$2xF_1(x) = F_2(x) = \sum_i xZ_i^2 f_i(x) \quad (8.29)$$

The valence quark distributions are written $u(x)$ and $d(x)$. In a proton they have normalisations:

$$\int_0^1 u(x)dx = 2 \quad \int_0^1 d(x)dx = 1 \quad (8.30)$$

and the contribution of the valence quarks to $F_2$ are:

$$F_2^p(x) = \frac{4}{9} xu(x) + \frac{1}{9} xd(x) \quad (8.31)$$

For a neutron $u$ and $d$ are interchanged:

$$F_2^n(x) = \frac{4}{9} xd(x) + \frac{1}{9} xu(x) \quad (8.32)$$
There is an additional sea of quarks and antiquarks at low $x$, which also contribute to $F_2$, shown as $\bar{u}(x) = \bar{d}(x) = \bar{s}(x)$.

8.8 Quark, Antiquark & Gluon Fractions

The total number of valence quarks can be obtained from:

$$F_3^{CC}(\nu N) = \left[ u(x) - \bar{u}(x) + d(x) - \bar{d}(x) \right]$$

$$\int_0^1 F_3^{CC}(\nu N)dx = 3$$

The fraction of sea quarks is obtained from:

$$\frac{\int \bar{q}(x)dx}{\int q(x)dx} = \frac{3R - 1}{3 - R} \approx 0.1$$

where $R$ is the ratio of the antineutrino to neutrino total cross-sections. For a pointlike spin 1/2 fermion, $f$:

$$R = \frac{\sigma(\bar{\nu}f)}{\sigma(\nu f)} = \frac{1}{3}$$

whereas for an antifermion $R = 3$.

If we integrate the area under the electron DIS structure function $F_2(x)$ we measure the total fraction carried by the valence and sea quarks.

$$\int_0^1 F_2^p(x)dx = \frac{4}{9}f_u + \frac{1}{9}f_d$$

$$\int_0^1 F_2^n(x)dx = \frac{4}{9}f_d + \frac{1}{9}f_u$$

We obtain the surprising results:

$$\int_0^1 F_2^p(x)dx = 0.18$$

$$\int_0^1 F_2^n(x)dx = 0.12$$

$$f_u = 0.36$$

$$f_d = 0.18$$

The quarks constitute only 54% of the proton rest mass!

The remainder is carried by gluons which must also be considered as partons.

Understanding the gluon component of the proton presents a problem, since the lowest order scattering processes of electrons and neutrinos only couple to the quarks.

There are additional scattering processes that involve gluons either in the initial state (“hard scattering”) or in the final state (“gluon emission”), where there is an additional hadronic jet from the gluon. These have different kinematics from the lowest order scattering.
Figure 8.2: Gluon emission from a final state quark $\gamma^* q \rightarrow qg$

Figure 8.3: Hard scattering off an initial state gluon $\gamma^* g \rightarrow q\bar{q}$

9 Quantum Chromodynamics

Quantum Chromodynamics or QCD is the theory of strong interactions between quarks and gluons. It is a quantum field theory similar to QED but with some crucial differences.

9.1 Feynman rules for QCD amplitudes

The calculation of Feynman diagrams containing quarks and gluons has the following changes compared to QED:

- The coupling constant $\alpha$ becomes $\alpha_s$ (where $\sqrt{\alpha_s} = g_s$).
- $\alpha_s(q^2)$ decreases rapidly as a function of $q^2$.
  At small $q^2$ it is large, and QCD is a non-perturbative theory.
  At large $q^2$ QCD becomes perturbative like QED.
- A quark has one of three colour states (replacing electric charge).
  Antiquarks have anticolour states.
- A gluon propagator has one of eight colour-anticolour states.
- A quark-gluon vertex has a factor $-ig_s\lambda^{\alpha}\gamma^\nu$.
- As a consequence of the gluon colour states, gluons can self-interact in three or four gluon vertices.
Figure 8.4: Evidence for gluon emission from transverse momentum squared of jets in deep inelastic scattering of muons. The difference between the solid and dashed lines is due to gluon emission.

- There is a complicated factor for a three-gluon vertex:

\[ -g_s f^{abc} [g_{\mu\nu}(q_1 - q_2)_{\lambda} + g_{\nu\lambda}(q_2 - q_3)_{\mu} + g_{\lambda\mu}(q_3 - q_1)_{\nu}] \]  

(9.1)

where \( f^{abc} \) are known as colour structure constants.

### 9.2 SU(3) Colour

Quarks carry one of three colour states, red \( r \), green \( g \), and blue \( b \). Antiquarks carry the anticolour states, \( \bar{r} \), \( \bar{g} \), and \( \bar{b} \). We define three quark eigenstates \( r, g, \) and \( b \):

\[
c_r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad c_g = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad c_b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

(9.2)

The colour states are related by an SU(3) symmetry group which defines rotations in colour. **Strong interactions are invariant under SU(3) colour.**

The coupling is independent of the colour states of the quarks and gluons.

#### 9.2.1 Gluon States

At a quark-gluon vertex a quark can either change colour or remain the same. A change of colour requires that gluons also have colour states.
This differs from QED where the photon has no charge.

A gluon carries a colour and an anti-colour state $c_i \bar{c}_j$. Naively you would think that there are nine possible gluon states:

$$r \bar{b} \quad b \bar{r} \quad r \bar{g} \quad g \bar{r} \quad b \bar{g} \quad g \bar{b} \quad r \bar{r} \quad b \bar{b} \quad g \bar{g}$$

of which the last three are apparently colour neutral. The symmetry properties of the SU(3) group actually classify the states as $3 \otimes \bar{3} = 8 \oplus 1$. There is a colour-\textit{octet} of allowed gluon states which are colour antisymmetric:

$$G_1 = \frac{1}{\sqrt{3}} (r \bar{b} + b \bar{r}) \quad G_2 = \frac{1}{\sqrt{3}} (r \bar{b} - b \bar{r})$$
$$G_4 = \frac{1}{\sqrt{3}} (r \bar{g} + g \bar{r}) \quad G_5 = \frac{1}{\sqrt{3}} (r \bar{g} - g \bar{r})$$
$$G_6 = \frac{1}{\sqrt{3}} (b \bar{g} + g \bar{b}) \quad G_7 = \frac{1}{\sqrt{3}} (b \bar{g} - g \bar{b})$$
$$G_3 = \frac{1}{\sqrt{2}} (r \bar{r} - b \bar{b}) \quad G_8 = \frac{1}{\sqrt{6}} (r \bar{r} + b \bar{b} - 2g \bar{g})$$

and a colour-\textit{singlet}, which is the only colour symmetric state:

$$G_0 = \frac{1}{\sqrt{3}} (r \bar{g} + g \bar{r} + b \bar{b})$$

The singlet state $G_0$ is forbidden for gluons, because it would give rise to long-range strong interactions.

\subsection*{9.2.2 The $\lambda$ matrices}

There are eight $\lambda$ matrices which are the \textbf{generators} of SU(3):

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The $\lambda$ matrices can be identified with the eight gluon states. The operators $\lambda^{1,2,4,5,6,7}$ are the ones that change quark colour states. The diagonal operators $\lambda^{3,8}$ are the ones that do not change the colour states.

The SU(3) \textit{structure constants} describe the commutators of the $\lambda$ matrices:

$$[\lambda^a, \lambda^b] = 2i \sum_c f_{abc} \lambda^c$$
Non-zero values are:

\[ f_{123} = 1 \quad f_{458} = f_{678} = \frac{\sqrt{3}}{2} \]  
\[ f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2} \]

(9.6)

(9.7)

The \( f_{abc} \) are related to each other by the property that they are \textit{antisymmetric} under the interchange of any pair of indices, e.g. \( f_{132} = f_{213} = f_{321} = -1 \) and \( f_{231} = f_{312} = 1 \) from the value \( f_{123} = 1 \). The \( f_{abc} \) not covered by permutations of the values given above are zero.

More familiar are the three Pauli matrices \( \sigma \) which are the generators of SU(2). They can be seen as subsets of \( \lambda^{1,4,6} (\sigma^1) \), \( \lambda^{2,5,7} (\sigma^2) \), and \( \lambda^3 (\sigma^3) \).

Note that the Pauli matrices have only one structure constant \( \epsilon_{ijk} = f_{123} \).

### 9.2.3 Non-Abelian Gauge Symmetry

A rotation in colour space is written as:

\[ U = e^{-i\alpha_a \lambda^a} \]

(9.8)

where \( \alpha_a \) are the equivalent of “angles” in colour space. QCD amplitudes can be shown to be invariant under this \textit{non-Abelian} gauge transformation.

The transformations of the quark and gluon states are:

\[ q \rightarrow (1 + i\alpha_a \lambda^a)q \quad G^a_{\mu} \rightarrow G^a_{\mu} - \frac{1}{g_s} \partial_\mu \alpha_a - f_{abc} \alpha_b G^c_{\mu} \]

(9.9)

### 9.3 Quark-antiquark scattering

The matrix element for quark-antiquark scattering is written:

\[ M = \left[ \bar{u}_3 c_3^\dagger \right] \left[ -\frac{ig_s}{2} \lambda^a \gamma^\mu \right] [u_1 c_1] \left[ g_{\mu\nu} \delta^{ab} \right] \left[ \bar{v}_2 c_2^\dagger \right] \left[ -\frac{ig_s}{2} \lambda^b \gamma^\nu \right] [v_4 c_4] \]

(9.10)

\[ M = \frac{\alpha_s}{4q^2} \left[ \bar{u}_3 \gamma^\mu u_1 \right] [\bar{v}_2 \gamma_\mu v_4] (c_3^\dagger \lambda^a c_1)(c_2^\dagger \lambda^a c_4) \]

(9.11)
This looks very similar to the matrix element for electron-positron scattering, except that \( \alpha \) is replaced by \( \alpha_s \) and there is a colour factor \( c_f \):

\[
c_f = \frac{1}{2}(c_3^\dagger \lambda^a c_1)(c_4^\dagger \lambda^a c_4)
\]  

The calculation of the \( c_f \) can be found in Halzen & Martin P. 67-69:

<table>
<thead>
<tr>
<th>quark states</th>
<th>gluon states</th>
<th>( c_f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( rr \leftrightarrow rr )</td>
<td>( G_7, G_8 )</td>
<td>+2/3</td>
</tr>
<tr>
<td>( r\bar{r} \leftrightarrow r\bar{r} )</td>
<td>( G_7, G_8 )</td>
<td>-2/3</td>
</tr>
<tr>
<td>( rb \leftrightarrow rb )</td>
<td>( G_8 )</td>
<td>-1/3</td>
</tr>
<tr>
<td>( rb \leftrightarrow br )</td>
<td>( G_1 )</td>
<td>1</td>
</tr>
<tr>
<td>( r\bar{r} \leftrightarrow b\bar{b} )</td>
<td>( G_1 )</td>
<td>1</td>
</tr>
<tr>
<td>( rb \leftrightarrow r\bar{b} )</td>
<td>( G_8 )</td>
<td>+1/3</td>
</tr>
</tbody>
</table>

### 9.4 Strong Coupling Constant \( \alpha_s \)

The coupling strength \( \alpha_s \) is large, which means that higher order diagrams are important. At low \( q^2 \) higher order amplitudes are larger than the lowest order diagrams, so the sum of all diagrams does not converge, and QCD is non-perturbative. At high \( q^2 \) the sum does converge, and QCD becomes perturbative. The coupling constant \( \alpha_s \) can be renormalised at a scale \( \mu \) in a similar way to \( \alpha \):

\[
\alpha(q^2) = \frac{\alpha(\mu^2)}{1 - \frac{\alpha(\mu^2)}{3\pi} \log \left( \frac{q^2}{\mu^2} \right)} \quad \text{(QED)}
\]

In QCD the renormalization is attributed both to the colour screening effect of virtual \( q\bar{q} \) pairs and to anti-screening effects from gluons since they have colour-anticolour states. This leads to a running of the strong coupling constant:

\[
\alpha_s(q^2) = \frac{\alpha_s(\mu^2)}{1 + \beta \alpha_s(\mu^2) \ln \left( \frac{q^2}{\mu^2} \right)} \quad \beta = \frac{(11N_c - 2N_f)}{12\pi} \quad \text{(QCD)}
\]

where \( N_c = 3 \) is the number of colours, and \( N_f \leq 6 \) is the number of active quark flavours which is a function of \( q^2 \).

From the positive sign of \( \beta \) it can be seen that the anti-screening effect of the gluons dominates, and the strong coupling constant decreases rapidly as a function of \( q^2 \). The running of the coupling constant is usually written:

\[
\alpha_s(q^2) = \frac{12\pi}{(33 - 2N_f) \ln \left( \frac{q^2}{\Lambda_{QCD}^2} \right)} \quad \text{(9.13)}
\]

where \( \Lambda_{QCD} = 217 \pm 25 \) MeV is a reference scale which defines the onset of a strongly coupled theory where \( \alpha_s \approx 1 \).

There are many independent determinations of \( \alpha_s \) at different scales \( \mu \), which are summarized in the figures on the next page. The results are compared with each other by adjusting them to a common scale \( \mu = M_Z \) where \( \alpha_s = 0.1184(7) \).
Figure 9.1: Top: Running of the strong coupling constant $\alpha_s(q^2)$. Bottom: Measurements of $\alpha_s(M_Z^2)$, the equivalent strong coupling at $q^2 = M_Z^2$.

9.5 Confinement and Asymptotic Freedom

In deep inelastic scattering experiments at large $q^2$, the quarks and gluons inside the proton can be observed as partons. This property is known as **asymptotic freedom**. It is associated with the small value of $\alpha_s(q^2)$, which allows the strong interaction corrections from gluon emission and hard scattering to be calculated using a perturbative expansion of QCD. The perturbative QCD treatment of high $q^2$ strong interactions has been well established over the past 20 years by experiments at high energy colliders.

In contrast, at low $q^2$ the quarks and gluons are tightly bound into hadrons. This is known as **confinement**. For large $\alpha_s$, QCD is not a perturbative theory and different mathematical methods have to be used to calculate the properties of hadronic systems. A rigorous numerical approach is provided by **Lattice gauge theories**.

The breakdown of the perturbation series is due to the colour states of the gluons and the contributions from three-gluon and four-gluon vertices which do not have an analogue in QED. A pictorial way of thinking of this is as a **colour flux tube**, connecting the quarks in a hadron. Starting from the familiar dipole field between two charges, imagine the colour field lines as being squeezed down into a tight line between the two quarks:

The additional energy density stored in the flux tube as the $q\bar{q}$ pair are pulled apart can be parametrized by a string tension $k$, and the QCD potential can be written as the sum of a Coulomb-like potential and the string potential energy:

$$ V_{q\bar{q}}(r) = -\frac{4}{3} \frac{\alpha_s}{r} + kr $$  \hspace{1cm} (9.14)

As a consequence of the positive term in the strong interaction potential, a $q\bar{q}$ pair cannot be separated since infinite energy would be required. Instead the flux tube can
break creating an additional $q\bar{q}$ pair in the middle which combines with the original $q\bar{q}$ pair to form two separate hadrons.

There are no free quarks or gluons!
10 QCD at Colliders

10.1 $e^+e^- \rightarrow$ Hadrons

In the electromagnetic process $e^+e^- \rightarrow q\bar{q}$, the flavour of the $q$ and $\bar{q}$ must be the same. This is known as associated production. The final state quark and antiquark each form a jet by a process known as fragmentation. The two jets follow the directions of the quarks, and are back-to-back in the center-of-mass system.

There is an initial colour flux between the quark and antiquark, but this is broken by the energy of the collision. The quarks then radiate gluons, which couple to quark-antiquark pairs, eventually forming a large number of bound states known as hadrons. The fragmentation process is characterised by the transverse momentum, $p_T$, of the hadrons relative to the jet axis.

The cross section for $e^+e^- \rightarrow q\bar{q}$ can be calculated using QED. The matrix element is similar to $e^+e^- \rightarrow \mu^+\mu^-$ apart from the final state charges and a colour factor, $N_c = 3$

$$\sigma(e^+e^- \rightarrow q\bar{q}) = N_c Z_q^2 \sigma(e^+e^- \rightarrow \mu^+\mu^-) \quad (10.1)$$

Summing over all the active quark flavours gives the ratio:

$$R \equiv \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} = 3 \sum_q Z_q^2 \quad (10.2)$$
Note the importance of the factor 3. The measurement of the ratio $R$ shows that there are three colour states of quarks!

$$\sigma$$ and $R$ in $e^+e^-$ Collisions

![Graph showing $\sigma$ and $R$ versus $\sqrt{s}$ in GeV]

10.2 Gluon Jets

Sometimes one of the quarks radiates a hard gluon which carries a large fraction of the quark energy, $e^+e^- \rightarrow q\bar{q}g$. In this process there are three jets, with one coming from the gluon. This provided the first direct evidence for the gluon in 1979. The jets are no longer back-to-back, and the gluon jet has a different $p_T$ distribution from the quark jets. The production rate of three-jet versus two-jet events is proportional to $\alpha_s$.

Including higher order processes, the ratio $R$ is:

$$R = 3 \sum_q Z_q^2 \left[ 1 + \frac{\alpha_s(q^2)}{\pi} \right]$$  \hspace{1cm} (10.3)

This is used to measure the strong coupling constant as a function of $q^2$.

10.3 Hadron Colliders

Proton-antiproton colliders were used to discover the $W$ and $Z$ bosons at CERN in the 1980s, and the top quark at the Tevatron (Fermilab) in 1995. In 2010 the Large
Hadron Collider (LHC) began operation at CERN. This collides two proton beams with an initial CM energy of 7 TeV. This is half the eventual design energy, but is already the world’s highest energy collider.

10.3.1 Parton Level Scattering

![Figure 10.1: Stylized representation of a hadron-hadron collision from “Jet Physics at the Tevatron” by A.Bhatti & D.Lincoln, arXiv:1002.1708.](image)

- Each initial hadron provides one parton, which can be either a quark, antiquark or gluon. At lower energy colliders the valence quarks dominate, whereas at the LHC most scattering is between low $x$ gluons.

- The remnants of the two protons form backward and forward jets similar to the jet in electron-proton DIS. These remnants are known as the “underlying event”, and are usually regarded as background.

- The partons undergo “hard scattering”, a strong interaction with a large $q^2$ transfer. This leads to two or more quarks or gluons in the final state. The amplitude can be calculated using perturbative QCD.

- The final state quarks and gluons form hadronic jets in a similar way to the quarks (and gluons) in $e^+e^- \rightarrow q\bar{q}(g)$.

- Additional gluons can be emitted from either the initial or final state quarks (and gluons). A distinction is made between “hard” gluons, and “soft collinear” gluons.

- Most collisions have no missing energy and no high mass final state particles $W, Z, H, b, t$. These events are said to be “minimum biased”.

- A large transverse momentum (high $p_t$) jet or isolated charged lepton is a signature for the production of a high mass final state particle.
• Large missing transverse energy \((\text{missing } E_t)\) is a signature for neutrinos or other weakly interacting neutral particles.

10.3.2 Description of Jets

The hadrons that constitute a jet are summed to provide the kinematic information about the jet:

- Jet energy \(E = \sum_i E_i\) and momentum \(\vec{p} = \sum_i \vec{p}_i\)
  The direction of \(\vec{p}\) defines the jet axis.
- Jet invariant mass \(W^2 = E^2 - |\vec{p}|^2\)
- Transverse energy flow within the jet \(|p_T|^2 = \sum_i |\vec{p}_i - \vec{p}|^2\)
- Transverse jet momentum \(p_t = \sqrt{p_z^2 + p_y^2}\), relative to beam axis.
- “Pseudorapidity” \(y = \ln [(E + p_z)/(E - p_z)]/2\), related to \(\cos \theta\).
  This goes to \(\infty\) along the beam axis, and is zero at \(90^\circ\).
- Azimuthal angle \(\phi = \tan^{-1}(p_y/p_x)\)

In the case of \(e^+e^- \rightarrow q\bar{q}(g)\) it is straightforward to decide which hadrons belong to which jets. At a hadron collider a minimum bias event has at least two jets from the hard scatter, as well as two jets from the underlying event. With large numbers of jets it is hard to decide which hadrons belong to which jets. What is done is to define a radial distance from the jet axis for each hadron:

\[
R_i^2 = (y_i - y_{jet})^2 + (\phi_i - \phi_{jet})^2
\]

(10.4)

An initial estimate is made of \(y_{jet}\) and \(\phi_{jet}\) using a subset of the highest momentum hadrons. Then a cone radius \(R\) is defined containing all the \(i\) hadrons that make up the jet. These are then used to recalculate \(y_{jet}\) and \(\phi_{jet}\). The process can be iterated until it converges. The choice of radius for the cone is critical. A typical value is \(R = 0.7\).

The cross-section for jet production as a function of \(p_t\) and \(y\) can be measured and compared to perturbative QCD calculations. These are usually carried out to next-to-leading order (NLO), with one additional hard gluon, and sometimes to next-to-next-to-leading order (NNNLO). The measurements are used to constrain gluon and quark parton density functions, and to determine the strong coupling constant \(\alpha_S(q^2)\).

10.3.3 Jet Fragmentation

The initial stages of jet fragmentation involve the emission of soft collinear gluons from the scattered partons. These can still be understood using perturbative QCD. Eventually the energy scales of the fragmentation become too low, and hadronisation into bound states begins. This part is modelled in several different ways:
• PYTHIA breaks colour flux tubes between quarks and gluons.
• HERWIG clusters the quarks and gluons according to colour matching.
• SHERPA uses a cutoff in the transverse energy of a quark or gluon within a jet to stop the production of lower energy jet fragments.

The fragmentation models are compared to data and tuned appropriately. The most relevant jet parameter for this is the transverse energy flow within the jet $|p_T|^2$, although the hadron multiplicity and individual momenta can also be looked at.

10.4 Production of Heavy Quarks

10.4.1 Discovery of the $J/\psi$

In November 1974, two groups simultaneously announced the discovery of a narrow $J/\psi$ resonance with a mass of 3100 MeV. At SLAC the resonance was observed in $e^+e^- \rightarrow q\bar{q}$, while at Brookhaven the inverse process was studied, $e^+e^-$ pair production by a proton beam on a Beryllium target.

Figure 10.2: Discovery of $J/\psi$ in 1974 at Brookhaven (left) and SLAC (right).

The $J/\psi$ is identified as the lightest of the $c\bar{c}$ states, known as charmonium. Its width $\Gamma = 0.087$ MeV, is much smaller than the experimental resolution, which is a surprise.

10.4.2 $b\bar{b}$ Production

The narrow $b\bar{b}$ bound state $\Upsilon(1S)$ with mass 9.5 GeV, was first identified in hadron collisions at Fermilab in 1977. It is also produced in $e^+e^-$ collisions in a similar way to
The production of \( b \) quarks has a significant role at hadron colliders, because it is possible to “tag” jets that come from \( b \) quarks. This is done by measuring detached decay vertices due to the finite lifetime of the \( b \) quark, \( \tau_b = 1.5 \text{ps} \). Measurements of the production cross-section for \( b \) jets can then be compared with perturbative QCD calculations. These are particularly useful for constraining the antiquark and gluon parton density functions, since \( b \bar{b} \) pairs are produced either by quark-antiquark annihilation or by gluon fusion.

An important use of \( b \) tagging is to identify jets that have been produced by the decay of a heavy particle into a \( b \) quark. Examples of these are \( t \to bW \), \( Z \to b\bar{b} \), and eventually for a light Higgs boson, \( H \to b\bar{b} \).

### 10.4.3 The Top Quark

The top quark was not discovered until 1995, because of its unexpectedly large mass \( m_t = 172.0 \pm 1.6 \text{ GeV} \). It was observed at the Tevatron (Fermilab) in proton-antiproton collisions at a centre-of-mass energy of 1.8 TeV. The main production mechanism is quark-antiquark annihilation giving \( t\bar{t} \), followed by the decays \( t \to bW \):

\[
t \to W^+ b \quad \bar{t} \to W^- \bar{b} \quad W^+ \to \ell^+ \nu_t \quad W^- \to q\bar{q}
\]  

There are no narrow \( t\bar{t} \) resonances because of the short lifetime of the top quark. The total cross-section for \( t\bar{t} \) production at the Tevatron is \( 7.5 \pm 0.5 \text{ pb} \). There is also single top production via a \( W \) coupling to the valence quarks, \( W \to t\bar{b} \), with a cross-section \( 2.3 \pm 0.6 \text{ pb} \).

Recently the CDF experiment at the Tevatron has measured a large forward-backward asymmetry in \( t\bar{t} \) production from proton-antiproton collisions. This is not expected in the Standard Model! (see preprint arXiv:1101.0034)
11 Mesons and Baryons

11.1 Formation of Hadrons

As a result of the colour confinement mechanism of QCD only bound states of quarks are observed as free particles, known as hadrons. They are colour singlets, with gluon exchange between the quarks inside the hadrons, but no colour field outside.

Mesons are formed from a quark and an antiquark with colour and anticolour states with a symmetric colour wavefunction:

\[ \chi_c = \frac{1}{\sqrt{3}} (r\bar{r} + g\bar{g} + b\bar{b}) \]  \hspace{1cm} (11.1)

Baryons are formed from three quarks, all with different colour states, with an anti-symmetric colour wavefunction:

\[ \chi_c = \frac{1}{\sqrt{6}} (rgb - rbg + grb - grb + brg - bgr) \]  \hspace{1cm} (11.2)

There may be other types of hadrons which are colour singlets: four quark states \((q\bar{q}q\bar{q})\), pentaquarks \((qqqq\bar{q})\), hybrid mesons \((q\bar{q}g)\), or glueballs \((gg, ggg)\). There is some experimental evidence for these, but it is not yet convincing.

11.2 Isospin

Strong interactions are the same for \(u\) and \(d\) quarks. This is an SU(2) flavour symmetry, known rather confusingly as isospin. The \(u\) and \(d\) quarks are assigned to an isospin doublet:

\[ u : \quad I = \frac{1}{2}, \quad I_3 = +\frac{1}{2} \quad d : \quad I = \frac{1}{2}, \quad I_3 = -\frac{1}{2} \]  \hspace{1cm} (11.3)

The lowest lying meson states are the pions, which are pseudoscalars with spin \(J = 0\) (\(\uparrow \downarrow\)). They form an \(I=1\) triplet:

\[ \pi^+ [1,1] = u\bar{d} \quad \pi^0 [1,0] = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \quad \pi^- [1,-1] = d\bar{u} \]  \hspace{1cm} (11.4)

There is also an \(I=0\) singlet, the eta meson:

\[ \eta [0,0] = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) \]  \hspace{1cm} (11.5)

For baryons the lowest lying states are the \(J = 1/2\) proton and neutron:

\[ p \ [1/2, +1/2] = uud \quad n \ [1/2, -1/2] = ddu \]  \hspace{1cm} (11.6)

Isospin symmetry means that protons, neutrons and pions all have the same strong interactions.
11.3 SU(3) Flavour Symmetry

In 1961, before the discovery of quarks, Gell-Mann proposed a structure to classify the hadrons, which was known as the **eightfold way**. This classification is now understood as an **approximate SU(3) flavour symmetry** between the three lightest quarks, $u$, $d$, and $s$. This symmetry is broken by the mass of the strange quark. The $s$ quark is assigned a **strangeness**, $S=1$ and $I=0$, and the $u$ and $d$ quarks have $S=0$ and $I=1/2$.

The mathematics of SU(3) flavour is identical to that of SU(3) colour which was discussed in lecture 8. The eight generators of the group are the $\lambda^a$ matrices. The matrices $\lambda^1$, $\lambda^2$, and $\lambda^3$ form the $SU(2)$ isospin part of SU(3) flavour which deals with the $u$ and $d$ quarks. The two diagonal matrices, $\lambda^3$ and $\lambda^8$ are related to eigenstates of isospin, $I_3$, and **hypercharge**, $Y$, respectively:

$$ I_3 = \frac{1}{2} \lambda^3 \quad Y = S + B = \frac{1}{\sqrt{3}} \lambda^8 $$

where $B=1/3$ is the baryon number of the quarks. The quarks charges can be written as:

$$ Q = I_3 + \frac{Y}{2} $$

The quark and antiquark flavours can be represented as 2-dimensional $SU(3)$ multiplets of isospin and hypercharge:

![SU(3) Flavour Diagram](image)

11.4 Hadron Multiplets

The SU(3) multiplet structure for the $q\bar{q}$ meson states is $3 \otimes 3 = 8 \oplus 1$, i.e. it contains a flavour octet and a flavour singlet. The lowest lying $J = 0$ ($\uparrow\downarrow$) pseudoscalar and $J = 1$ ($\uparrow\uparrow$) vector mesons are shown on the next pages.

Three of the nine $J = 0$ mesons are neutral with $I_3 = Y = 0$. These states are $\pi^0$, $\eta_1$ and $\eta_8$, where $\eta_1$ is the SU(3) singlet. The physically observed states, $\eta$ and $\eta'$, are mixtures of $\eta_1$ and $\eta_8$. For the $J = 1$ mesons, the three neutral states that are observed are $\rho^0$, $\omega$ and $\phi$, where the $\phi$ is purely an $s\bar{s}$ state.

According to SU(3) symmetry, the baryon states are classified as $3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$, i.e. the three light quarks form a decuplet, two octets and a singlet. The lowest lying
baryon octet with J=1/2 (↑↓↓) contains p, n, Λ, Σ⁺, Σ⁰, Σ⁻, Ξ⁰ and Ξ⁻ states.

The decuplet has J = 3/2 (↑↑↓). It consists of Δ, Σ*, Ξ* states and the Ω⁻ which is an sss state.

11.4.1 Δ++ and Proton Wavefunctions

The flavour wavefunctions for decuplet baryons are symmetric, e.g.:

\[
\frac{1}{\sqrt{6}} (dus + uds + sud + sdu + dsu + usd)
\] (11.9)

The Δ++(uuu) has symmetric (S) flavour, spin and orbital wavefunctions, and an antisymmetric (A) colour wavefunction. It is overall antisymmetric, as it must be for a system of identical fermions.

The wavefunctions for the lowest baryon octet have a combined symmetry of the flavour and spin parts:

\[
uud(\uparrow\uparrow\uparrow + \downarrow\uparrow\downarrow - 2 \uparrow\downarrow\downarrow) + uud(\downarrow\uparrow\uparrow + \uparrow\downarrow\downarrow - 2 \uparrow\downarrow\downarrow) + duu(\uparrow\downarrow\downarrow + \uparrow\downarrow\downarrow - 2 \uparrow\downarrow\downarrow)
\] (11.10)

Hence the proton also has an overall antisymmetric wavefunction, \(\psi\):

<table>
<thead>
<tr>
<th>Hadron</th>
<th>(\chi_c)</th>
<th>(\chi_L)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Delta^{++})</td>
<td>A</td>
<td>S</td>
</tr>
<tr>
<td>p</td>
<td>A</td>
<td>S</td>
</tr>
</tbody>
</table>

Note that there are no J=1/2 states uuu, ddd, sss because the flavour symmetric part would have to be combined with an antisymmetric spin part.

11.4.2 Hadron Masses*

In the \(\overline{MS}\) renormalization scheme of QCD the masses of the u and d quarks, \(m_u\) and \(m_d\) are only a few MeV, and the mass of the s quark, \(m_s\) ≈ 100 MeV. In discussing hadron masses and magnetic moments we use constituent quark masses \(m_u = m_d \approx 300\) MeV and \(m_s \approx 500\) MeV.

Gell-mann and Okubo introduced a semi-empirical mass relation for baryons:

\[
M = M_0 + YM_1 + M_2 \left[ I(I + 1) - Y^2/4 \right]
\] (11.11)

For the \(J = 3/2\) baryon decuplet \(Y = B + S = 2 \left( I - 1 \right)\), and the formula reduces to:

\[
M = M_\Delta + (m_d - m_s)S
\] (11.12)

For the \(J = 1/2\) baryon octet the masses are related by:

\[
3M_\Delta + M_\Sigma = 2M_N + 2M_\Xi
\] (11.13)

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Pseudoscalar mesons
$J^{PC} = 0^{-+}$

$\pi^0 = (d\bar{d} - u\bar{u})/\sqrt{2}$
$\eta_8 = (d\bar{d} + u\bar{u} - 2s\bar{s})/\sqrt{6}$
$\eta_1 = (d\bar{d} + u\bar{u} + s\bar{s})/\sqrt{3}$

Vector mesons
$J^{PC} = 1^{--}$

$\rho^0 = (d\bar{d} - u\bar{u})/\sqrt{2}$
$\omega = (d\bar{d} + u\bar{u})/\sqrt{2}$
$\phi = s\bar{s}$
These formulae are accurate to $\approx 1\%$.

The semi-empirical mass relation fails for mesons, but the mass differences between $J=0$ and $J=1$ mesons and between $J=1/2$ and $J=3/2$ baryons can be understood as hyperfine splitting due to spin-spin coupling between quarks.

11.5 Resonances

The $J=1$ vector mesons decay to the $J=0$ pseudoscalar mesons through strong interactions in which a second quark-antiquark pair is produced, e.g. $\rho^0 \rightarrow \pi^+\pi^-$. Similarly the $J=3/2$ baryons (except for the $\Omega^-$), decay to the $J=1/2$ baryons through strong interactions, e.g. $\Delta^{++} \rightarrow \pi^+ p$. The decay widths for these processes are $\approx 100$ MeV, corresponding to lifetimes of $\approx 10^{-23}$s. Hadronic states that can decay through strong interactions are known as resonances. They are observed as broad mass peaks in the combinations of their daughter particles.

![Figure 11.1: The $\Delta^{++}$ resonance observed in $\pi^+p$ scattering.](image1)

![Figure 11.2: Quark line diagram of a $\rho^+$ meson coupling to $\pi^+\pi^0$.](image2)

The Feynman diagram for resonance production can be drawn in reduced form showing just the quark lines, which are continuous and do not change flavour. Since the process is a low energy strong interaction, there are lots of gluons coupling to the quark lines which form a colour flux between them.
11.6 Heavy Quark States

The $c$ and $b$ quarks can replace the lighter quarks to form heavy hadrons. However, the $t$ quark is too short-lived to form observable hadrons.

The equivalent of the $K$ and $K^*$ mesons are known as charm, $D^{(*)}$, and beauty, $B^{(*)}$ mesons. There are three possible $D$ states (and antipartners):

$$D^+ (c\bar{d}) \quad D^0 (c\bar{u}) \quad D_{s}^+ (c\bar{s})$$ (11.14)

and four possible $B$ states:

$$B^+ (\bar{b}u) \quad B^0 (\bar{b}d) \quad B_{s}^0 (\bar{b}s) \quad B_{c}^+ (\bar{b}c)$$ (11.15)

Similarly there are heavy baryons such as $\Lambda_c$ and $\Lambda_b$, where a heavy quark replaces one of the light quarks.

The masses of heavy quark states are dominated by the heavy quark constituent masses $m_c \approx 1.5$ GeV and $m_b \approx 4.6$ GeV.

11.6.1 Charmonium and Bottomonium

The $J/\psi$ meson is identified as the lowest $^3S_1$ bound state of $c\bar{c}$. The width of this state is narrow because $M_{J/\psi} < 2M_D$, so it cannot decay into a $D\bar{D}$ meson pair. It decays to light hadrons by quark-antiquark annihilation into gluons. The quark line diagram is drawn as being disconnected:

There is a complete spectroscopy of $c\bar{c}$ charmonium states with angular momentum states equivalent to atomic spectroscopy. Only the states with $M(c\bar{c}) < 2M_D$ have narrow widths.

A similar spectroscopy is observed for $b\bar{b}$ bottomonium states.
Figure 11.3: Spectroscopy of lightest charmonium states with $M(c\bar{c}) < 2M_D$. Note that out of these states only the $J/\psi$ and $\psi(2S)$ are produced in $e^+e^-$ collisions. The singlet $\eta_c$ states, and the P-wave $\chi_c$ and $h_c$ states, are observed via electromagnetic transitions.
12 Weak Decays of Hadrons

12.1 Selection Rules for Decays

There are selection rules to decide if a hadronic decay is weak, electromagnetic or strong, based on which quantities are conserved:

- For strong interactions total isospin $I$, and its projection $I_3$ are conserved. Strong interactions always lead to hadronic final states.

- For electromagnetic decays $I$ is not conserved, but $I_3$ is. There are often photons or charged lepton-antilepton pairs in the final state.

- For weak decays $I$ and $I_3$ are not conserved. They are the only ones that can change quark flavour, e.g. $\Delta S = 1$ when $s \to u$.

- Weak decays can lead to hadronic, semileptonic, or leptonic final states. A neutrino in the final state is a clear signature of a weak interaction.

The lightest hadrons cannot decay to other hadrons via strong interactions. The $\pi^0$, $\eta$ and $\Sigma^0$ decay electromagnetically. The $\pi^\pm$, $K^\pm$, $K^0$, $n$, $\Lambda$, $\Sigma^\pm$, $\Xi$ and $\Omega^-$ decay via flavour-changing weak interactions, with long lifetimes.

12.2 Pion and Kaon Decays

12.2.1 Neutral Pion Decay $\pi^0 \to \gamma \gamma$

The $\pi^0$ meson is a $(u\bar{u} - d\bar{d})$ state which decays through the annihilation of its $u\bar{u}$ or $d\bar{d}$ quarks into a pair of photons. This is an electromagnetic decays that conserves $I_3$ and charge conjugation symmetry $C$. The quantum numbers are $[I, I_3]_{\pi^0} = [1, 0]$, $C_{\pi^0} = +1$ and $I_\gamma = 0$, $C_\gamma = -1$, so the allowed decay is to two photons. The decay to three photons has not been observed, $\mathcal{B}(\pi^0 \to 3\gamma) < 3 \times 10^{-8}$.

The lifetime $\tau_{\pi^0} = (8.4 \pm 0.6) \times 10^{-17}$ s, is the shortest lifetime ever measured experimentally. The decay width is:

$$\Gamma(\pi^0 \to 2\gamma) = \alpha^2 N_c^2 g_{\pi^0}^2 m_{\pi^0}^3$$

where $N_c = 3$ is the colour factor, and $g_{\pi^0} = 92$ MeV is the $\pi^0$ decay constant, which represents the probability that the quark-antiquark pair will meet inside the meson and annihilate.
12.2.2 Charged Pion Decay $\pi^+ \rightarrow \mu^+ \nu_\mu$

Charged pions decay mainly to a muon and a neutrino:

$$\pi^+ \rightarrow \mu^+ \nu_\mu \quad \pi^- \rightarrow \mu^- \bar{\nu}_\mu$$ (12.2)

This occurs through the annihilation of the $u\bar{d}$ quark-antiquark into a charged $W$ boson, which is described in terms of a pion decay constant, $f_\pi$. The matrix element for the decay is:

$$\mathcal{M} = \frac{V_{ud} G_F}{\sqrt{2}} f_\pi q^\mu \left( \bar{u}_\mu \gamma^\mu \frac{1}{2} (1 - \gamma^5) u_\nu \right)$$ (12.3)

The matrix element squared in the rest frame of the pion is:

$$|\mathcal{M}|^2 = 4|V_{ud}|^2 G_F^2 f_\pi^2 m_\mu^2 [p_3 \cdot p_4]$$ (12.4)

and the total decay rate is:

$$\Gamma = \frac{1}{\tau_\pi} = \frac{|V_{ud}|^2 G_F^2 f_\pi^2 m_\pi m_\mu^2}{8\pi} \left( 1 - \frac{m_\mu^2}{m_\pi^2} \right)^2$$ (12.5)

From the charged pion lifetime, $\tau_{\pi^+} = 26\text{ns}$, the pion decay constant can be deduced, $f_\pi = 131 \text{ MeV}$. Note that this is very similar to the charged pion mass, $m_{\pi^+} = 139.6 \text{ MeV}$.

The decay of a charged pion to an electron and a neutrino $\pi^+ \rightarrow e^+ \nu_e$, is helicity suppressed. This can be seen from the factor $m_\mu^2/m_e^2$ in the decay rate. Replacing $m_\mu$ with $m_e$ reduces the decay rate by a factor $\approx 10^{-4}$. The suppression is associated with the helicity states of neutrinos, which force the spin-zero $\pi^+$ to decay to a left-handed neutrino and a left-handed $\mu^+$, or the $\pi^-$ to decay to a right-handed antineutrino and a right-handed $\mu^-$. The precise experimental measurement $\mathcal{B}(\pi^+ \rightarrow e^+ \nu_e) = 1.230(4) \times 10^{-4}$, is a test of the lepton universality of weak couplings.

12.2.3 Charged Kaon Decays

The charged Kaon has a mass of 494MeV and a lifetime $\tau_{K^+} = 12\text{ns}$.

Its main decay modes are:

- Purely leptonic decays, where the $\bar{s}u$ annihilate, similar to $\pi^+$ decay:

$$\mathcal{B}(K^+ \rightarrow \mu^+ \nu_\mu) = 63.4\%$$ (12.6)

The Kaon decay constant $f_K = 160 \text{ MeV}$ is 20% larger than $f_\pi$ due to SU(3) flavour symmetry breaking.
• Semileptonic decays, where $\bar{s} \to \bar{u}$ and the $W^+$ boson couples to either $e^+\nu_e$ or $\mu^+\nu_\mu$:

$$\mathcal{B}(K^+ \to \pi^0\ell^+\nu_\ell) = 8.1\%$$ (12.7)

Semileptonic decays satisfy the $\Delta Q = \Delta S$ rule, i.e. the change in strangeness is equal to the lepton charge, $K^+ \to \ell^+$ and $K^- \to \ell^-$.  

• Hadronic decays, where $\bar{s} \to \bar{u}$ and the $W^+$ boson couples to $ud$:

$$\mathcal{B}(K^+ \to \pi^+\pi^0) = 21.1\%$$ (12.8)

$$\mathcal{B}(K^+ \to \pi^+\pi^+\pi^-) = 5.6\%$$ (12.9)

Hadronic weak decays prefer an isospin change $\Delta I = 1/2$ to $\Delta I = 3/2$. The origins of this rule are not well understood.

12.2.4 The Cabibbo Angle

The relationship between the couplings of the $W$ boson to $ud$ and $us$ quarks is described by the Cabibbo angle, $\theta_C = 12.7^\circ$:

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos \theta_C & \sin \theta_C \\ -\sin \theta_C & \cos \theta_C \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}$$ (12.10)

and the vertex couplings which modify the usual weak coupling $G_F$ are:

$$V_{ud} = \cos \theta_C = 0.974(1) \quad V_{us} = \sin \theta_C = 0.220(3)$$ (12.11)

A factor $|V_{us}|^2$ appears in all the decay rates for charged Kaons.

12.3 Decays of Heavy Quarks

Hadrons with charm decay mainly by weak $c \to s$ transitions, while beauty hadrons decay via weak $b \to c$ transitions. There are also suppressed $c \to d$ and $b \to u$ couplings to the $W$. The lifetimes are quite short but measurable: $\tau_{D^0} = 410(2)\,fs$ and $\tau_{B^0} = 1.53(1)\,ps$.

12.3.1 Semileptonic decays

Semileptonic decays of heavy quarks are used to determine the $W$ couplings. $D \to K\ell\nu$ and $D \to \pi\ell\nu$ give $|V_{cs}| = \cos \theta_C$ and $|V_{cd}| = \sin \theta_C$. More interestingly $B \to D^{(*)}\ell\nu$ is used to measure $|V_{cb}| \approx 10^{-2}$, and $B \to \pi\ell\nu$ is used to measure $|V_{ub}| \approx 10^{-3}$. The extraction of these couplings requires a knowledge of hadronic form factors, which describe the $B \to D^{(*)}$ or $\pi$ transitions in terms of the overlap of initial and final state hadronic wavefunctions. These are usually calculated non-perturbatively using Lattice QCD.
12.3.2 $B \to \tau \nu$ and $D_s \to \mu \nu$

The purely leptonic decays of heavy mesons have small branching fractions, $\mathcal{B}(D_s \to \mu \nu) = 6 \times 10^{-3}$ and $\mathcal{B}(B^{+} \to \tau \nu) = 1.8 \times 10^{-4}$. The decay $B \to \tau \nu$ was first observed by the Belle experiment in 2006. It measures the combination $f_B^2|V_{ub}|^2$, where $f_B \approx 190$MeV is the $B$ decay constant, also obtained from Lattice QCD. The measured branching fraction is larger than expected from other determinations of $|V_{ub}|$.

12.3.3 $b \to s \gamma$

The flavour-changing neutral current (FCNC) transitions $b \to s$, $b \to d$ and $c \to u$ are not allowed at first order in the Standard Model. They do occur as a second order weak interaction through a “penguin” diagram containing a loop with a $W$ boson and a $t$ quark (or a $b$ quark in the case of $c \to u$). The decay $b \to s \gamma$ was first observed by the CLEO experiment in 1992. It has a branching fraction $\mathcal{B}(b \to s \gamma) = 3.5 \times 10^{-4}$. This is a strong constraint on new physics beyond the Standard Model, because new heavy particles could replace the $W$ and $t$ in the loop, giving significant changes to the decay rate. The decay $b \to d \gamma$ has recently been observed by the Babar and Belle experiments, but $c \to u$ transitions have not yet been seen.

12.4 The Cabibbo-Kobayashi-Maskawa (CKM) Matrix

The $3 \times 3$ CKM matrix is an extension of the $2 \times 2$ Cabibbo matrix to include the heavy quarks:

$$
\begin{pmatrix}
  d' \\
  s' \\
  b'
\end{pmatrix} =
\begin{pmatrix}
  V_{ud} & V_{us} & V_{ub} \\
  V_{cd} & V_{cs} & V_{cb} \\
  V_{td} & V_{ts} & V_{tb}
\end{pmatrix}
\begin{pmatrix}
  d \\
  s \\
  b
\end{pmatrix}
$$

(12.12)

where the mass eigenstates of the $Q = -1/3$ quarks are $d, s, b$ and the weak eigenstates which couple to the $W$ are $d', s', b'$. The matrix is unitary, and its elements satisfy:

$$
\sum_i V_{ij}^2 = 1 \quad \sum_j V_{ij}^2 = 1 
$$

(12.13)

$$
\sum_i V_{ij}V_{ik} = 0 \quad \sum_j V_{ij}V_{kj} = 0 
$$

(12.14)

By considering the above unitarity constraints it can be deduced that the CKM matrix can be written in terms of just four parameters. Starting from the definition of the Cabibbo angle, the four CKM parameters can be written as three angles $s_i = \sin \theta_i$, $c_i = \cos \theta_i$, and a complex phase $\delta$:

$$
\begin{pmatrix}
  c_1 & s_1 c_3 & s_1 s_3 \\
  -s_1 c_3 & c_1 c_2 c_3 - s_2 s_3 e^{i \delta} & c_1 c_2 s_3 + s_2 c_3 e^{i \delta} \\
  s_1 s_2 & -c_1 s_2 c_3 - c_2 s_3 e^{i \delta} & -c_1 s_2 s_3 + c_2 c_3 e^{i \delta}
\end{pmatrix}
$$

(12.15)

It is more common to see the CKM matrix written as an expansion in powers of the Cabibbo angle $\lambda = \sin \theta_C$. This is known as the Wolfenstein parametrization. It satisfies
unitarity to $O(\lambda^3)$:

$$
\begin{pmatrix}
1 - \lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\
-\lambda & 1 - \lambda^2/2 & A\lambda^2 \\
A\lambda^3(1 - \rho - i\eta) & A\lambda^2 & 1
\end{pmatrix}
$$

(12.16)

The relative sizes of the CKM elements in powers of $\lambda$ vary from 1 on the diagonal, to $\lambda^3$ in the off-diagonal corners. Note that the complex phase $\eta$ is associated with the smallest elements of the matrix $V_{ub}$ and $V_{td}$.

Figure 12.1: Constraints on the $\rho$ and $\eta$ parameters of the CKM matrix. Plot comes from the CKM fitter group at http://ckmfitter.in2p3.fr. More details on the measurements can be found at the Heavy Flavour Averaging Group http://www.slac.stanford.edu/xorg/hfag/.

The measured values of the CKM elements are:

- $|V_{cs}| = 0.97(12)$ from $D \to K\ell\nu$.
- $|V_{cd}| = 0.224(12)$ from $D \to \pi\ell\nu$ and neutrino production of charm $\nu d \to \ell c$.
- $|V_{cb}| = 0.042(1)$ from $b \to c\ell\nu$. This determines the parameter $A$.
- $|V_{ub}| = 0.0044(4)$ from $b \to u\ell\nu$.
- $|V_{td}| \approx 0.008$ and $|V_{ts}| \approx 0.04$ from $\Delta m_d$ and $\Delta m_s$ in $B_d$ and $B_s$ mixing (see Lecture 13).
- $|V_{tb}| \approx 1$ from top decays at the Tevatron.

The constraints on the $\rho$ and $\eta$ parameters of the CKM matrix are shown in Figure 12. We will discuss $\epsilon_K$ and the angles $\alpha$, $\beta$ and $\gamma$ of the “unitarity triangle” in Lecture 13.

From a full fit of all the constraints on the CKM matrix the Wolfenstein parameters are:

$$
\lambda = 0.225(1) \quad A = 0.81(2) \quad \rho = 0.14(3) \quad \eta = 0.34(2)
$$

These are fundamental parameters of the Standard Model.
13 Symmetries in Particle Physics

Symmetries play an important role in particle physics. The mathematical description of symmetries uses group theory, examples of which are SU(2) and SU(3):

A serious student of elementary particle physics should plan eventually to study this subject in far greater detail. (Griffiths P.115)

There is a relation between symmetries and conservation laws which is known as Noether’s theorem. Examples of this in classical physics are:

- invariance under change of time → conservation of energy
- invariance under translation in space → conservation of momentum
- invariance under rotation → conservation of angular momentum

In particle physics there are many examples of symmetries and their associated conservation laws. There are also cases where a symmetry is broken, and the mechanism has to be understood. The breaking of electroweak symmetry and the associated Higgs field will be discussed in lectures 15 and 16.

13.1 Gauge Symmetries

The Lagrangian is \( L = T - V \), where \( T \) and \( V \) are the kinetic and potential energies of a system. It can be used to obtain the equations of motion. The Dirac equation follows from a Lagrangian of the form:

\[
L = i\bar{\psi}\gamma_{\mu}\delta^{\mu}\psi - m\bar{\psi}\psi
\]  

(13.1)

It can be seen that this Lagrangian is invariant under a phase transformation:

\[
\psi \rightarrow e^{i\alpha}\psi \quad \bar{\psi} \rightarrow e^{-i\alpha}\bar{\psi}
\]  

(13.2)

This is an example of a gauge invariance.

13.1.1 U(1) Symmetry of QED

The Lagrangian for QED is written:

\[
L = \bar{\psi}(i\gamma_{\mu}\delta^{\mu} - m)\psi + e\bar{\psi}\gamma_{\mu}A_{\mu}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}
\]  

(13.3)

where \( A^{\mu} \) represents the photon, and the second term can be thought of as \( J_{\mu}A^{\mu} \) where \( J_{\mu} = e\bar{\psi}\gamma^{\mu}\psi \) is an electromagnetic current. \( F_{\mu\nu} \) is the electromagnetic field tensor:

\[
F_{\mu\nu} = \delta_{\mu}A_{\nu} - \delta_{\nu}A_{\mu}
\]  

(13.4)
The gauge transformation is:

\[ \psi \rightarrow e^{i\alpha} \psi \quad A_\mu \rightarrow A_\mu + \frac{1}{e} \delta_\mu \alpha \quad (13.5) \]

This has the property that it conserves the current, and hence conserves electric charge. A mass term \( m A_\mu A^\mu \) is forbidden in the Lagrangian by gauge invariance. This explains why the photon must be massless.

The gauge invariance of QED is described mathematically by a U(1) group.

### 13.1.2 SU(3) Symmetry of QCD

This gauge symmetry was already introduced in Lecture 8. A rotation in colour space is written as:

\[ U = e^{-i\alpha_a \lambda^a} \quad (13.6) \]

where \( \alpha_a \) are the equivalent of “angles”, and \( \lambda^a \) are the generators of SU(3). QCD amplitudes can be shown to be invariant under this non-Abelian gauge transformation.

The transformations of the quark and gluon states are:

\[ q \rightarrow (1 + i\alpha_a \lambda^a)q \quad G_\mu^a \rightarrow G_\mu^a - \frac{1}{g_s} \partial_\mu \alpha_a - f_{abc} \alpha_b G_\mu^c \quad (13.7) \]

The Lagrangian for QCD is written:

\[ L = \bar{q} (i\gamma_\mu \delta^\mu - m) q + g_s \bar{q} \gamma_\mu \lambda^a G_\mu^a q - \frac{1}{4} G_\mu^a G_\nu^a G^{\mu\nu} \quad (13.8) \]

where the gluon states \( G_\mu^a \) replace the photon, and \( g_s \) replaces \( e \). The gluon field energy contains a term for the self-interactions of the gluons:

\[ G_\mu^a G^\mu_a = \delta_\mu^a G^\mu_a - \delta^\mu G_\mu^a - g_s f_{abc} G_\mu^b G_\mu^c \quad (13.9) \]

The absence of a mass term \( m G_\mu^a G^\mu_a \) makes the gluon massless.

### 13.2 Flavour Symmetries

In Lecture 10 we met isospin symmetry, which is a flavour symmetry of the strong interactions between \( u \) and \( d \) quarks. It is described by SU(2), and can be extended to SU(3) with the addition of the \( s \) quark. The SU(3) symmetry is partially broken by the \( s \) quark mass. In principle this could be extended further to an SU(6) symmetry between all the quark flavours, but at this point the level of symmetry breaking becomes rather large. The interesting question, to which we do not yet have an answer, is what causes the breaking of quark flavour symmetry.

Similar questions can be asked about lepton flavour symmetry. There are precise tests of lepton flavour conservation, and of the universality of lepton couplings in electromagnetic and weak interactions. However, flavour symmetry is broken by the lepton
masses for reasons that are not understood.

There is overall conservation of baryon number (B) and lepton number (L). However, to generate the matter-antimatter asymmetry of the universe B has be violated. In many models for this B−L is then conserved.

13.3 Discrete Symmetries

13.3.1 Parity

The parity operation $P$ performs a spatial inversion through the origin:

$$P\psi(\vec{r}) = \psi(-\vec{r})$$

(13.10)

This is NOT a mirror reflection through an axis, e.g. $\psi(x) \to \psi(-x)$.

Many books get this wrong!

Applying parity twice restores the original state, $P^2 = 1$. From this the parity of a wavefunction $\psi(\vec{r})$ has to be either even, $P = +1$, or odd, $P = -1$. For example $\psi(x) = \cos kx$ is even, and $\psi(x) = \sin kx$ is odd.

The hydrogen atom wavefunctions are a product of a radial function $f(r)$ and the spherical harmonics $Y^m_L(\theta, \phi)$, where $L$ and $m$ are the orbital angular momentum of the state and its projection along an axis. In spherical polar coordinates the parity operation changes $\{r, \theta, \phi\} \to \{r, \pi - \theta, \pi + \phi\}$. From the properties of $Y^m_L$ the wavefunctions have parity $P = (-1)^L$. It is observed that single photon transitions between atomic states obey the selection rule $\Delta L = \pm 1$. From this it can be deduced that the intrinsic parity of the photon is:

$$(-1)^L = (-1)^{L\pm 1} \times P_\gamma \quad P_\gamma = -1$$

(13.11)

The parity of the photon can also be obtained from the gauge symmetry of QED discussed in the previous section.

13.3.2 Intrinsic Parity of Fermions

Applying a spatial inversion to the Dirac equation gives

$$\left( i\gamma^0 \frac{\delta}{\delta t} - i\vec{\gamma} \cdot \nabla - m \right) \psi(-\vec{r}, t) = 0$$

(13.12)

This is not the same as the Dirac equation because there is a change of sign of the first derivative in the spatial coordinates. If we multiply from the left by $\gamma^0$ and use the relations $(\gamma^0)^2 = 1$ and $\gamma^0\gamma^i + \gamma^i\gamma^0 = 0$ we get back a valid Dirac equation:

$$\left( i\gamma^0 \frac{\delta}{\delta t} + i\vec{\gamma} \cdot \nabla - m \right) \gamma^0 \psi(-\vec{r}, t) = 0$$

(13.13)
We identify the parity operator with $\gamma^0$:
\[
\psi(\vec{r}, t) = P\psi(-\vec{r}, t) = \gamma^0\psi(-\vec{r}, t)
\]

(13.14)

Applying this to the Dirac spinors:
\[
P u_1 = P u_2 = +1 \quad P v_1 = P v_2 = -1
\]

(13.15)

The intrinsic parity of fermions is $P = +1$ (even)
The intrinsic parity of antifermions is $P = -1$ (odd)

Parity is a multiplicative quantum number, so the parity of a many particle system is equal to the product of the intrinsic parities of the particles, and the parity of the spatial wavefunction which is $(-1)^L$. As an example, positronium is an $e^+e^-$ atom with:
\[
P(e^+e^-) = P e^-P e^+(-1)^L = (-1)^{L+1}
\]

(13.16)

where $L$ is the relative orbital angular momentum between the $e^+$ and $e^-$. 

### 13.3.3 Charge Conjugation

Charge Conjugation is a discrete symmetry that reverses the sign of the charge and magnetic moment of a particle. Like the parity operator it satisfies $C^2 = 1$, and has possible eigenvalues $C = \pm 1$. Electromagnetism is $C$ invariant, since Maxwell’s equations apply equally to $+$ and $-$ charges. However the electromagnetic fields change sign under $C$, which means the photon has:

\[
C_\gamma = -1
\]

(13.17)

For fermions charge conjugation changes a particle into an antiparticle, so fermions themselves are not eigenstates of $C$, but combinations of fermions are. Positronium has:

\[
C(e^+e^-) = (-1)^{L+S}
\]

(13.18)

where $S$ is the sum of the spins which can be either 0 or 1. Positronium states with even $L+S$ decay to two photons, and those with odd $L+S$ decay to three photons. This shows that electromagnetic interactions are invariant under charge conjugation and parity, and conserve $C$ and $P$.

We can also determine the $P$ and $C$ states of mesons. The lowest pseudoscalar mesons have $J^{PC} = 0^{--}$ and the vector mesons have $J^{PC} = 1^{--}$.

### 13.3.4 Time Reversal

Time reversal $T\psi(t) = \psi(-t)$, is another discrete symmetry operator with $T^2 = 1$, and possible eigenvalues $T = \pm 1$. The solutions of the Dirac equation describe antifermion states as equivalent to fermion states with the time and space coordinates reversed.
13.3.5 Summary of Discrete Symmetry Transformations

- A polar vector such as momentum, $\vec{p}$, transforms under parity $P = -1$.
- An axial vector such as angular momentum, $\vec{L} = \vec{r} \times \vec{p}$ transforms as $P\vec{L} = \vec{L}$. This implies that parity does not affect the spin of a particle.
- Charge conjugation reverses the charge, but does not change the direction of the spin vector or the momentum of a particle.
- Time reversal changes the sign of both the spin and momentum.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Notation</th>
<th>P</th>
<th>C</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Position</td>
<td>$\vec{r}$</td>
<td>-1</td>
<td>+1</td>
<td>+1</td>
</tr>
<tr>
<td>Momentum (Vector)</td>
<td>$\vec{p}$</td>
<td>-1</td>
<td>+1</td>
<td>-1</td>
</tr>
<tr>
<td>Spin (Axial Vector)</td>
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<td>+1</td>
<td>-1</td>
</tr>
<tr>
<td>Helicity</td>
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<td>+1</td>
<td>+1</td>
</tr>
<tr>
<td>Electric Field</td>
<td>$\vec{E}$</td>
<td>-1</td>
<td>-1</td>
<td>+1</td>
</tr>
<tr>
<td>Magnetic Field</td>
<td>$\vec{B}$</td>
<td>+1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>Magnetic Dipole Moment</td>
<td>$\vec{\sigma} \cdot \vec{B}$</td>
<td>+1</td>
<td>-1</td>
<td>+1</td>
</tr>
<tr>
<td>Electric Dipole Moment</td>
<td>$\vec{\sigma} \cdot \vec{E}$</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>Transverse Polarization</td>
<td>$\vec{\sigma} \cdot (\vec{p}_1 \times \vec{p}_2)$</td>
<td>+1</td>
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<td>-1</td>
</tr>
</tbody>
</table>

13.4 Parity Violation in Weak Interactions

In contrast to electromagnetic interactions it is found that weak interactions maximally violate both parity and charge conjugation symmetries. The original evidence for parity violation came from the study of the $\beta$ decay of polarized $^{60}\text{Co}$, where it was observed that the electron was emitted preferentially in the direction opposite to the spin of the nucleus. The distribution of the decay electrons can be described by:

$$\frac{dN}{d\Omega} = 1 - \frac{\vec{\sigma} \cdot \vec{p}}{E}$$  \hspace{1cm} (13.19)

The parity operator reverses the direction of the electron but not the spin of the nucleus, so the $\vec{\sigma} \cdot \vec{p}$ term is parity-violating. A similar parity violation is observed in muon decay. A $\mu^+$ emits an $e^+$ preferentially along the direction of the $\mu^+$ spin, whereas a $\mu^-$ emits an $e^-$ preferentially in the direction opposite to the $\mu^-$ spin. The difference between $\mu^+$ and $\mu^-$ shows that charge conjugation is also violated. However a comparison of the two decay distributions shows that the combined operation $CP$ is conserved.
14 \( CP \) and \( CP \) Violation

14.1 \( CPT \) Theorem

The \( CPT \) theorem requires that all interactions that are described by localized Lorentz invariant gauge theories must be invariant under the combined operation of \( C, P \) and \( T \) in any order. The proof of the \( CPT \) theorem is based on very general field theoretic assumptions. It can be thought of as a statement about the invariance of Feynman diagrams under particle/antiparticle interchange, and interchange of the initial and final states.

The \( CPT \) theorem predicts that particles and antiparticles must have the same mass and lifetime, but opposite electric charge and magnetic moment. Experimental tests of the \( CPT \) theorem have shown very precise agreement.

The \( CPT \) theorem also means that the transformation properties of gauge theories under the discrete symmetries \( C, P \) and \( T \) are related to each other:

\[
CP \leftrightarrow T \quad CT \leftrightarrow P \quad PT \leftrightarrow C \quad (14.1)
\]

The first of these establishes that time reversal invariance is equivalent to \( CP \) invariance.

In the Big Bang model of the universe there is an \textbf{arrow of time} so \( T \) may not be a valid symmetry of the universe. It is believed that matter and antimatter were originally created in equal amounts, but we observe that we live in a matter dominated universe, with a baryon density compared to photons of \( N_b/N_\gamma = 10^{-9} \), and no evidence for primordial antibaryons. In 1966 Sakharov postulated three conditions that are necessary for our matter-dominated universe to exist:

- An epoch with no thermal equilibrium
- Baryon number violation
- \( CP \) Violation (or equivalently \( T \) violation)
15 Measuring CP Violation

15.1 Neutral Meson Mixing

The couplings $V_{td}$ and $V_{ts}$ have been inferred from second order weak processes in which top quarks appear inside loops with $W$ boson. These involve mixing diagrams which describe the transitions between neutral mesons $K^0 \leftrightarrow \bar{K}^0$, $B^0 \leftrightarrow \bar{B}^0$ and $B_s \leftrightarrow \bar{B}_s$.

Figure 15.1: Second order weak diagram for neutral meson mixing.

15.1.1 Mixing of Neutral Kaons

A state that is initially $K^0$ or $\bar{K}^0$ will evolve as a function of time due to the mixing diagram:

$$\psi(t) = a(t)|K^0 > + b(t)|\bar{K}^0 > \quad i\frac{d\psi}{dt} = H\psi(t)$$  \hspace{1cm} (15.1)

where $H$ is the effective Hamiltonian which can be written in terms of $2 \times 2$ mass and decay matrices $M$ and $\Gamma$:

$$H = M - \frac{i}{2} \Gamma$$  \hspace{1cm} (15.2)

The diagonal elements of these matrices are associated with flavour-conserving transitions, while the off-diagonal elements are associated with the mixing transitions $K^0 \leftrightarrow \bar{K}^0$. The matrix $H$ has two eigenvectors corresponding to the mass and weak decay eigenstates $K_L$ and $K_S$. The real parts of the eigenvectors are the masses, $m_L$ and $m_S$, and the imaginary parts are the decay widths. The eigenstates can be expressed as a linear superposition of $K^0$ and $\bar{K}^0$:

$$|K_S > = p|K^0 > + q|\bar{K}^0 > \quad |K_L > = p|K^0 > - q|\bar{K}^0 >$$  \hspace{1cm} (15.3)

where $|q|^2 + |p|^2 = 1$ and:

$$\frac{q}{p} = \frac{2M_{12}^* - i/2\Gamma_{12}^*}{\Delta m_K - i/2\Delta \Gamma_K}$$  \hspace{1cm} (15.4)

The differences in the masses and decay widths of the weak eigenstates are:

$$\Delta m_K = m_L - m_S = (3.52 \pm 0.01) \times 10^{-12}\text{MeV} = 0.529 \times 10^{10}\text{s}^{-1}$$  \hspace{1cm} (15.5)
\[ \Delta \Gamma_K = \frac{1}{\tau_L} - \frac{1}{\tau_S} = 1.1 \times 10^{10} \text{s}^{-1} \]  

(15.6)

The mass difference \( \Delta m_K \) is very small compared to the neutral Kaon mass!

---

Figure 15.2: Time evolution of an initial \( K^0 \) state

The time evolution of a state that is initially pure \( K^0 \) is given by:

\[ |\psi_{K^0}(t)|^2 = \frac{1}{4} \left( e^{-\Gamma_L t} + e^{-\Gamma_S t} + 2e^{-\frac{(\Gamma_L + \Gamma_S)}{2} t} \cos \Delta m_K t \right) \]  

(15.7)

\[ |\psi_{\bar{K}^0}(t)|^2 = \frac{1}{4} \left( e^{-\Gamma_L t} + e^{-\Gamma_S t} - 2e^{-\frac{(\Gamma_L + \Gamma_S)}{2} t} \cos \Delta m_K t \right) \]  

(15.8)

Neutral meson mixing leads to flavour oscillations, with a frequency given by the mass difference between the weak eigenstates.

### 15.1.2 Mixing of \( B \) mesons

There are two systems of neutral \( B \) mesons, the \( B_d \) states \( B^0 \) and \( \bar{B}^0 \), and the equivalent \( B_s \) states where an \( s \) quark replaces the \( d \) quark. They are expected to mix in a similar way to the \( K^0 \) states, but in this case the mixing diagram is dominated by the top quark, and the off-diagonal elements of the mixing matrix are given by

\[ M_{12} \propto (V_{tb}V_{td}^*)^2 \quad \frac{q}{p} = \frac{V_{tb}V_{td}}{V_{tb}V_{td}^*} \]  

(15.9)

Oscillations of \( B_d \) mesons have been observed with:

\[ \Delta m_d = 0.508(4) \text{ps}^{-1} \quad \tau_{B_d} = 1.53(1) \text{ps} \]  

(15.10)
$B_s$ oscillations were observed at the Tevatron in 2006 using an amplitude scan to Fourier analyse their $B_s$ decays.

$$\Delta m_s = 17.8(1) \text{ps}^{-1} \quad \tau_{B_d} = 1.47(6) \text{ps}$$

(15.11)

Note the much larger oscillation frequency which makes the direct observation of the oscillations difficult, although it should be possible at the LHC.

From the ratio of the two oscillation frequencies it is possible to determine:

$$\left| \frac{V_{td}}{V_{ts}} \right| = 0.206(1)$$

(15.12)

The main uncertainty in this ratio is now coming from the theoretical calculation of the hadronic properties of $B$ mesons, the decay constants $f_B$ and the "bag" constants $B_B$. It should be noted that most of the uncertainties cancel in the ratio, and the individual determinations of $|V_{td}|$ and $|V_{ts}|$ have theoretical errors which are $\times 10$ larger.

### 15.2 CP Violation in Neutral Kaon Decays

The $CP$ eigenstates of neutral Kaons are:

$$K_1 = \frac{1}{\sqrt{2}} [K^0 + \bar{K}^0] \quad K_2 = \frac{1}{\sqrt{2}} [K^0 - \bar{K}^0]$$

(15.13)

$$CP|K_1\rangle = +1 \quad CP|K_2\rangle = -1$$

(15.14)

If $CP$ is violated the weak decay eigenstates are not the same as the $CP$ eigenstates:

$$K_L = \frac{1}{\sqrt{1 + \epsilon^2}} [\epsilon K_1 + K_2] \quad K_S = \frac{1}{\sqrt{1 + \epsilon^2}} [K_1 - \epsilon K_2]$$

(15.15)

where $\epsilon$ is a complex number.

The notation $K_L$ and $K_S$ refers to the long and short lifetimes:

$$\tau_L = 5.2 \times 10^{-8} \text{s} \quad \tau_S = 0.9 \times 10^{-10} \text{s}$$

(15.16)
15.2.1 \( CP \) Violation in \( K \to 2\pi \) Decays

There are neutral Kaon decays to two and three pion final states with:

\[
CP|\pi^+\pi^-\rangle = CP|\pi^0\pi^0\rangle = +1 \\
CP|\pi^+\pi^-\pi^0\rangle = CP|\pi^0\pi^0\pi^0\rangle = -1
\]  

(15.17) (15.18)

If \( CP \) is conserved it is expected that \( K_1 \to 2\pi \) and \( K_2 \to 3\pi \), with \( K_1 = K_S \) and \( K_2 = K_L \).

In 1964 the decay \( K_L \to \pi^+\pi^- \) was observed which violates \( CP \). The ratios of decays are written as:

\[
\eta_{+-} = \frac{K_L \to \pi^+\pi^-}{K_S \to \pi^+\pi^-} = \epsilon + \epsilon' \\
\eta_{00} = \frac{K_L \to \pi^0\pi^0}{K_S \to \pi^0\pi^0} = \epsilon - 2\epsilon'
\]

(15.19) (15.20)

where the parameter \( \epsilon' \) represents a “direct” \( CP \) violation between the \( \Delta I = 1/2 \) and \( \Delta I = 3/2 \) amplitudes in \( K \to 2\pi \) decays.

The \( \eta \) have measured magnitudes and phases:

\[
|\eta_{+-}| = 2.286(14) \times 10^{-3} \quad \phi_{+-} = 43.4(7)^\circ \\
|\eta_{00}| = 2.276(14) \times 10^{-3} \quad \phi_{00} = 43.6(8)^\circ
\]

(15.21) (15.22)

From a comparison of the charged and neutral pion decays:

\[
\text{Re}(\epsilon'/\epsilon) = 1.67(26) \times 10^{-3}
\]

(15.23)

15.2.2 \( CP \) and \( T \) Violation in Semileptonic Decays

In semileptonic decays the charge of the lepton is given by the charge of the \( W \) boson. Thus a \( K^0 \) decay by an \( s \to \bar{u} \) transition gives an \( \ell^+ \), and a \( \bar{K}^0 \) decay by an \( s \to u \) transition gives an \( \ell^- \). This is known as the \( \Delta Q = \Delta S \) rule. The charge of the lepton gives a \textit{flavour tag} to the neutral Kaon decay.

If there is no \( CP \) violation, the \( K_L \) is an equal superposition of \( K^0 \) and \( \bar{K}^0 \), so it should decay equally to \( \ell^+ \) and \( \ell^- \) with no charge asymmetry. If we add in the small amount of \( CP \) violation \( \epsilon \), then a charge asymmetry is predicted:

\[
\delta_{SL} = \frac{\Gamma(K_L \to \pi^+\ell^+\nu) - \Gamma(K_L \to \pi^+\ell^-\bar{\nu})}{\Gamma(K_L \to \pi^+\ell^+\nu) + \Gamma(K_L \to \pi^+\ell^-\bar{\nu})}
\]

(15.24)

\[
\delta_{SL} = \frac{(1 + \epsilon)^2 - (1 - \epsilon)^2}{(1 + \epsilon)^2 + (1 - \epsilon)^2} = 2\text{Re}(\epsilon)
\]

(15.25)
The experimental measurement of this asymmetry is:
\[ \delta_{SL} = 3.27(12) \times 10^{-3} \quad (15.26) \]

There is another elegant measurement that can be made with semileptonic decays that explicitly demonstrates time-reversal violation. We start with a pure \( K^0 \) or \( \bar{K}^0 \) state, and let it oscillate and then decay semileptonically. The T violation is observable as a rate asymmetry:
\[ \Gamma(K^0 \rightarrow \bar{K}^0 \rightarrow \pi^+ \ell^{-} \bar{\nu}) \neq \Gamma(\bar{K}^0 \rightarrow K^0 \rightarrow \pi^- \ell^+ \nu) \quad (15.27) \]

The amount of T violation corresponds to the amount of CP violation, so CPT symmetry is preserved. A direct test of CPT violation in semileptonic decays would be:
\[ \Gamma(\bar{K}^0 \rightarrow \pi^+ \ell^{-} \bar{\nu}) \neq \Gamma(K^0 \rightarrow \pi^- \ell^+ \nu) \quad (15.28) \]

The bounds on a CPT violating parameter \( \delta_{CPT} \) in neutral kaon decays are actually only one order of magnitude below \( \epsilon \):
\[ \delta_{CPT} = (2.9 \pm 2.7) \times 10^{-4} \quad (15.29) \]

### 15.3 General Formalism for CP violation

There is an excellent review of CP violation in the Particle Data Group compilation at http://pdg.lbl.gov/2009/reviews.

In a more general notation the weak eigenstates are labelled \( M_L \) and \( M_H \) (for light and heavy mass), and are not assumed to be the same as the \( CP \) eigenstates:
\[ M_L = pM^0 + q\bar{M}^0 \quad M_H = pM^0 - q\bar{M}^0 \quad (15.30) \]
with the normalisation \(|p|^2 + |q|^2 = 1\).

We use the following notation for the mass and decay width differences:
\[ \Delta m = M_H - M_L \quad \Delta \Gamma = \Gamma_H - \Gamma_L \quad (15.31) \]
\[ \Gamma = \frac{\Gamma_H + \Gamma_L}{2} \quad x = \frac{\Delta m}{\Gamma} \quad y = \frac{\Delta \Gamma}{\Gamma} \quad (15.32) \]

The amplitudes for the decays of the flavour eigenstates \( M^0 \) and \( \bar{M}^0 \) to a final state \( f \) or \( \bar{f} \), are written as \( A \) and \( \bar{A} \). If the final state is a \( CP \) eigenstate \( f = \bar{f} \), but \( A \) and \( \bar{A} \) are not necessarily equal.

The time dependent decay rates of the flavour eigenstates to a \( CP \) eigenstate \( M^0 \to f \) and \( \bar{M}^0 \to f \), are given in the most general form by:
\[ \frac{d\Gamma}{dt} = e^{-\Gamma t} [\alpha \cosh(\Delta \Gamma t) + \beta \cos(\Delta m t) + 2 \text{Re}[\gamma] \sinh(\Delta \Gamma t) - 2 \text{Im}[\gamma] \sin(\Delta m t)] \quad (15.33) \]
\[ \alpha = |A|^2 + \left| \frac{q}{p} \bar{A} \right|^2 \quad \beta = |A|^2 - \left| \frac{q}{p} \bar{A} \right|^2 \quad \gamma = \frac{q}{p} A^* \bar{A} \]  

(15.34)

\[ \frac{d\Gamma}{dt} = e^{-\Gamma t} \left[ \tilde{\alpha} \cosh(\Delta \Gamma t) - \tilde{\beta} \cos(\Delta m t) + 2\text{Re}[\tilde{\gamma}] \sinh(\Delta \Gamma t) + 2\text{Im}[\tilde{\gamma}] \sin(\Delta m t) \right] \]  

(15.35)

\[ \tilde{\alpha} = \left| \frac{p}{q} A \right|^2 + |\bar{A}|^2 \quad \tilde{\beta} = \left| \frac{p}{q} A \right|^2 - |\bar{A}|^2 \quad \tilde{\gamma} = \frac{p}{q} A \bar{A}^* \]  

(15.36)

Note the changes in sign of the second and fourth terms in the decay rates.

The sin and cos terms give the mixing oscillations with frequency \( \Delta m \). The amplitudes of these oscillations depend on \( \gamma \), and include a possible \( CP \) violation through mixing.

### 15.3.1 Types of \( CP \) violation

There are three types of \( CP \) violation that can be observed:

- **\( CP \) violation in the mixing amplitude**, due to the mass eigenstates being different from the \( CP \) eigenstates, \( |q/p| \neq 1 \).
  
  In the neutral kaon system this is represented by the semileptonic charge asymmetry \( \delta_{SL} \).

- **\( CP \) violation in the amplitudes** \( A \) and \( \bar{A} \) for decays to a particular final state, \( |A/\bar{A}| \neq 1 \) and phase differences between them. This is commonly known as **direct \( CP \) violation**. It does not require mixing, and can be found in both charged and neutral meson decays.
  
  In the decays \( K_{L,S} \to 2\pi \) it is represented by \( \epsilon' \).

- **\( CP \) violation in the interference between mixing and decay amplitudes**, which requires an overall weak phase \( \text{Im}[\lambda] \neq 0 \), where \( \lambda = q\bar{A}/pA \).
  
  In the decays \( K_{L,S} \to 2\pi \) it is represented by \( \epsilon \).

### 15.4 \( CP \) Violation in \( B \) Meson Decays

All three types are \( CP \) violation are expected to occur in the decays of neutral \( B \) mesons. Due to the dominance of the \( t \) quark contribution inside mixing and penguin diagrams, and the presence of the suppressed CKM couplings \( V_{ub} \) and \( V_{td} \), measurements of \( CP \) violation in \( B \) decays provide important additional information compared to the neutral kaon system.

#### 15.4.1 \( B_d \to J/\psi K_S \) and \( \sin 2\beta \)

\( CP \) violation through interference between mixing and decay amplitudes was first observed in the decay \( B_d \to J/\psi K_S \) in 2001 by the BaBar and Belle experiments. For
this decay:

\[
\lambda = \frac{q\bar{A}_f}{pA_f} = \left( \frac{V_{tb}^* V_{td}}{V_{tb} V_{td}^*} \right) \left( \frac{V_{cs}^* V_{cb}}{V_{cs} V_{cb}^*} \right) \left( \frac{V_{cs}^* V_{cd}}{V_{cs} V_{cd}^*} \right)
\]

where the three set of CKM factors respectively account for \( B_d \) mixing, the \( B \to J/\psi K_S \) decay amplitude and final state \( K^0 \) mixing.

The measurement is done by producing a \( B^0 \bar{B}^0 \) pair in \( e^+ e^- \to \Upsilon(4S) \) collisions. One of the \( B \) mesons decays into a flavour-specific final state providing a tag of the flavour of the other \( \bar{B} \) at the time of the first \( B \) decay. Then the \( CP \) asymmetry is measured as a function of the time difference, \( \Delta t \), between the \( B \) and \( \bar{B} \) decays. The amplitude of the asymmetry as a function of the mixing oscillations \( \sin \Delta m_d \Delta t \) is proportional to \( \text{Im}(\lambda) \). The measured \( CP \) violation is large, \( \text{Im}(\lambda) = \sin 2\beta = 0.67(2) \), with the angle \( \beta = 21^\circ \) being the complex phase of \( V_{td} \) in the Standard Model.

### 15.4.2 \( B_d \to \pi\pi, B_d \to \rho\rho \) and the angle \( \alpha \)

A similar measurement of time-dependent \( CP \) asymmetries can be made with the rare hadronic final states \( B^0 \to \pi^+\pi^- \) and \( B^0 \to \rho^+\rho^- \). In this case the decay amplitude is proportional to \( V_{ub} \), and the angle \( \alpha \) is measured which is the complex phase between \( V_{td} \) and \( V_{ub} \) in the Standard Model.

There is a complication due to an additional contribution to the amplitude from a \( b \to d \) penguin diagram. This is rather important for \( B \to \pi\pi \), and its effect has to be determined from an isospin analysis including the modes \( B^+ \to \pi^+\pi^0 \) and \( B^0 \to \pi^0\pi^0 \).

The BaBar experiment obtains \( \alpha = (96 \pm 6)^\circ \) from a fit to \( B \to \rho\rho \) decays where the penguin diagram has a much smaller effect.

### 15.4.3 Direct \( CP \) violation in \( B \to K\pi \)

In 2004 the BaBar and Belle experiments made the first observation of a direct \( CP \) violation in the decay amplitudes for \( B^0 \to K\pi \) decays:

\[
A_{CP} = \frac{\Gamma(\bar{B}^0 \to K^-\pi^+) - \Gamma(B^0 \to K^+\pi^-)}{\Gamma(B^0 \to K^-\pi^+) - \Gamma(B^0 \to K^+\pi^-)}
\]

(15.38)

The latest world average for this is \( A_{CP} = -0.10 \pm 0.01 \). However, significant direct \( CP \) asymmetries have not been observed in the decays \( B^\pm \to K^{\pm}\pi^0 \) and \( B^\pm \to K_S\pi^\pm \), for reasons that are not well understood.
15.5 *CP* violation in *D* mesons

*CP* violation in *D* mesons was observed for the first time in 2011 by the LHCb collaboration at CERN.
16 Neutrino Oscillations

This topic is well covered by Chapter 11 of Griffiths, and there is also a good web site at http://neutrinooscillation.org/.

In the Standard Model neutrinos are described as massless neutral fermions which come in three different flavours, $\nu_e$, $\nu_\mu$ and $\nu_\tau$. It is necessary to invoke separate conservation of each lepton flavour number $L_e$, $L_\mu$ and $L_\tau$, to avoid flavour-changing weak couplings of leptons to $W$ and $Z$ bosons.

If lepton flavours are not separately conserved, and neutrinos have finite masses, then it might be possible for the different neutrino flavours to mix in a similar way to the neutral mesons in the last lecture. This was first suggested by Pontecorvo in 1957. The weak interaction eigenstates remain the flavour eigenstates, $\nu_e$, $\nu_\mu$ and $\nu_\tau$, but the mass eigenstates $\nu_1$, $\nu_2$ and $\nu_3$ are linear superpositions of the weak eigenstates.

*Note that neutrino mixing requires a new lepton-flavour violating interaction which is not present in the Standard Model. We do not yet know what this interaction is!*

16.0.1 Neutrino States

A massless neutrino is purely left-handed, and a massless antineutrino is purely right-handed. The $P$ operator changes the direction of $\vec{p}$ but not $\vec{\sigma}$, so it reverses the helicity state. The $C$ operator changes a neutrino into an antineutrino. Each of these operators by itself changes a physical state into a forbidden state, again showing that $P$ and $C$ must be maximally violated in weak interactions with neutrinos. The combined operator $CP$ changes a left-handed neutrino into a right-handed antineutrino which is allowed.

16.1 Description of Oscillations

16.1.1 Two Neutrino Flavours

Neutrinos are produced in weak decays, so they start off as weak eigenstates. However, they propagate through space-time as plane waves corresponding to their mass eigenstates:

$$\psi_1(t) = \psi_1(0)e^{-iE_1t} \quad \psi_2(t) = \psi_2(0)e^{-iE_2t}$$

(16.1)

We have the equivalent of the Cabibbo angle to describe the mixing of two neutrino states:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos \theta_{12} & \sin \theta_{12} \\ -\sin \theta_{12} & \cos \theta_{12} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

(16.2)

where $\theta_{12}$ is a new parameter not present in the Standard Model.

If the neutrino masses $m_1$ and $m_2$ are different, the energies $E_1$ and $E_2$ are different.
Assuming highly relativistic neutrinos with \( m \ll E, \ p \approx E \):

\[
E_i = p + \frac{m_i}{2p^2} \quad \Delta E = \frac{\Delta m_{12}^2}{E}
\]  
(16.3)

If we start off with a pure \( \nu_e \) beam, the amplitude for \( \nu_e \) at a later time \( t \) is:

\[
\nu_e(t) = \nu_e(0) \left[ 1 - \sin \theta_{12} \cos \theta_{12} (-e^{-iE_1t} + e^{-iE_2t}) \right]
\]  
(16.4)

and the probability of observing an oscillation to \( \nu_\mu \) is:

\[
P(\nu_e \rightarrow \nu_\mu) = |\nu_\mu(t)|^2 = 1 - |\nu_e(t)|^2 = \sin^2 2\theta_{12} \sin^2 \left( \frac{E_2 - E_1}{2} \right) t
\]  
(16.5)

For experimental convenience this is usually expressed as:

\[
P(\nu_e \rightarrow \nu_\mu) = \sin^2 2\theta_{12} \sin^2 \left( \frac{1.27\Delta m_{12}^2 L}{E} \right)
\]  
(16.6)

where the numerical factor 1.27 applies if we express \( \Delta m^2 \) in eV\(^2\), the distance from the source \( L \) in metres, and the neutrino energy \( E \) in MeV.

To observe these oscillations experimentally a “near” detector measures the initial \( \nu_e \) flux, and a “far” detector measures either disappearance of \( \nu_e \), or appearance of \( \nu_\mu \). The choice of the “baseline”, \( L \), has to be matched to the oscillation frequency \( 1.27\Delta m_{12}^2 / E \), and the amplitude of the oscillations is related to the mixing angle by \( \sin^2 2\theta_{12} \).

*Note that the maximum possible mixing is for \( \theta_{12} = 45^\circ \).*

### 16.1.2 The PMNS Mixing Matrix

For the full case of three neutrinos we have the equivalent of the CKM matrix which is known as the PMNS (Pontecorvo, Maki, Nakagawa, Sakata) matrix. It is usually written out as the product of three matrices representing the three different types of two neutrino mixings:

\[
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix} = V_{PMNS} \begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix}
\]

(16.7)

\[
V_{PMNS} = \begin{pmatrix}
1 & 0 & 0 \\
0 & c_{23} & s_{23} \\
0 & -s_{23} & c_{23}
\end{pmatrix} \begin{pmatrix}
c_{13} & 0 & s_{13}e^{i\delta} \\
0 & 1 & 0 \\
-s_{13}e^{-i\delta} & 0 & c_{13}
\end{pmatrix} \begin{pmatrix}
c_{12} & s_{12} & 0 \\
-s_{12} & c_{12} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

(16.8)

It is parameterised by three angles, where \( s_{ij} = \sin \theta_{ij}, \ c_{ij} = \cos \theta_{ij} \), and one complex phase \( \delta \). The observations of neutrino oscillations, described in the next sections, can be accounted for by small mass differences \( \Delta m_{12} \) and \( \Delta m_{23} \), and large mixing angles \( \theta_{12} \) and \( \theta_{23} \). At present there is only an upper limit on the third angle \( \theta_{13} \). As in the CKM case, the phase \( \delta \) can give rise to CP violation in neutrino oscillations, but only if \( \theta_{13} \neq 0 \).
16.2 Solar Neutrinos

16.2.1 The Standard Solar Model

The sun creates energy by fusion of light nuclei. During this process a large flux of low energy electron neutrinos are released from $\beta^+$ decays of the fusion products. Most of the flux comes from the p-p fusion process, in which neutrinos are emitted up to a maximum energy of 400 keV. There is a small component of higher energy neutrinos, up to a maximum of 15 MeV, associated with $^8$B.

A large amount of work, mostly by Bahcall, has gone into calculating the flux of solar neutrinos using a Standard Solar Model (SSM).

16.2.2 The Davis Experiment

From 1970-1995 Ray Davis looked for solar neutrinos using a large tank containing 100,000 gallons of cleaning fluid placed in a mine in South Dakota. The neutrinos from $^8$B and $^7$Be are detected by the interaction:

$$\nu_e + ^{37}\text{Cl} \rightarrow ^{37}\text{Ar} + e^-$$

(16.9)

Only 0.5 Argon atoms are produced per day!(1). The whole cleaning tank is analysed radiochemically every few months to count these atoms. The observed rate is $2.56 \pm 0.23$ SNU, where 1 SNU(solar neutrino unit) is $10^{-36}$ captures per atom per second. The predicted rate from the SSM is $7.7 \pm 1.2$ SNU.

This is the famous solar neutrino deficit factor of $0.33 \pm 0.06$. There was a long discussion about whether the radiochemical extraction of the Argon atoms was reliable, and an
equally long discussion about whether the predictions of the Standard Solar Model were reliable. Now it is accepted that the deficit is real, and is attributable to $\nu_e \rightarrow \nu_\mu$ (or $\nu_\tau$) oscillations.

*Note that due to the low neutrino energy it is impossible to detect $\nu_\mu$ or $\nu_\tau$ by charged current interactions.*

### 16.2.3 Recent Solar Neutrino Experiments

The Kamiokande experiment used 50,000 tons of water as a Cherenkov detector for the $^8$B neutrinos. The electron scattering process $\nu_e + e^- \rightarrow \nu_e + e^-$ gives a recoil electron that produces light which is detected by 11,000 photomultiplier tubes. This method allows the direction of the neutrinos to be determined, proving that they come from the sun. Like the Davis experiment, Kamiokande measures a solar neutrino deficit of $0.45 \pm 0.02$.

Two experiments, Gallex and SAGE, used large quantities of metallic Gallium to measure the flux of the lower energy p-p neutrinos. In this case $\nu_e + ^{71}\text{Ga} \rightarrow ^{71}\text{Ge} + e^-$, and the Germanium atoms are again counted by radiochemical means. Both experiments measure $71 \pm 5$ SNU, compared to the SSM prediction of 129 SNU. The importance of these measurements is that the rate of p-p neutrinos is determined precisely by the thermal output of the sun, so the SSM prediction is rather reliable. At this point most people believed that the solar neutrino deficit was due to about half of the electron neutrinos oscillating into another neutrino flavour.

The SNO experiment proved this between 2000 and 2006 using 1,000 tons of heavy water ($\text{D}_2\text{O}$) to detect neutrinos in three different ways:

- Scattering on electrons $\nu_e + e^- \rightarrow \nu_e + e^-$
- Charged current scattering on deuterium $\nu_e + d \rightarrow p + p + e^-$
- Neutral current scattering on deuterium $\nu + d \rightarrow n + p + \nu$
As indicated by the lack of a subscript, the last process does not distinguish between \( \nu_e, \nu_\mu \) and \( \nu_\tau \). The difference between the neutral and charged current scattering on deuterium shows that the \( \nu_\mu \) (or \( \nu_\tau \)) flux is exactly what is required to account for the solar neutrino deficit.

### 16.2.4 The MSW Effect

In 1978 Wolfenstein noted that the effect of flavour-specific neutrino interactions must be taken into account when considering neutrino propagation in the presence of matter. Since matter contains electrons but not muons, electron neutrinos experience a potential energy due to interactions, \( U_e \propto G_F N_e \), where \( N_e \) is the electron density of the matter. This potential has an equivalent effect to a mass difference, i.e. it changes the energy with which the electron neutrinos propagate. This leads to matter-induced electron neutrino oscillations, with an effective mixing angle in matter \( \theta_m \), which differs from \( \theta \) in vacuum:

\[
\sin^2 2\theta_m = \frac{\sin^2 2\theta}{\left(\cos 2\theta - a\right)^2 + \sin^2 2\theta} \quad \text{a} \propto G_F E_\nu N_e / \Delta m^2
\]

In the sun, the electron density \( N_e \) varies with radius, and there can be a radius where \( a = \cos 2\theta \) and \( \sin^2 2\theta_m = 1 \) leads to **resonance-enhanced** oscillations of electron neutrinos. This is known as the MSW effect.

Combining all solar neutrino results, and including the MSW effect, the parameters of the solar neutrino oscillations have been determined to be:

\[
\Delta m^2_{12} = (7.6 \pm 0.2) \times 10^{-5} \text{eV}^2 \quad \sin^2 2\theta_{12} = 0.87 \pm 0.03
\]

There is an alternative solution with vacuum oscillations and no MSW effect. This has a much smaller \( \Delta m^2 \), but it is ruled out by reactor experiments.
16.3 Atmospheric Neutrinos

Neutrinos are produced in the upper atmosphere by the interactions of cosmic rays. The initial strong interaction of protons with nuclei produces charged (and neutral) pions. The charged pions decay via $\pi^+ \rightarrow \mu^+ \nu_\mu$, $\mu^+ \rightarrow e^+ \nu_e \bar{\nu}_\mu$, and the charge conjugate $\pi^-$ decays. This gives ratios of two muon (anti)neutrinos to one electron (anti)neutrino. Note that atmospheric neutrinos have much higher energies than solar neutrinos, in the GeV range.

SuperKamiokande detected atmospheric neutrinos via the charged current interactions $\nu_e + p \rightarrow p + e^-$, $\nu_\mu + p \rightarrow p + \mu^-$. The muon and electron can be identified and used to tag the flavour of the incoming neutrino. What is observed is a deficit of upward going muons, produced by muon neutrinos coming from the atmosphere on the other side of the earth:

Note that there is no up-down asymmetry for electrons from the electron neutrinos, and that the expected $\mu : e$ ratio of a factor of two is observed for the downward neutrinos.

This observation is interpreted as the oscillation of muon neutrinos into unobserved tau neutrinos over the earth’s diameter, with parameters:

$$\Delta m^2_{23} = (2.4 \pm 0.1) \times 10^{-3} \text{eV}^2 \quad \sin^2 2\theta_{23} = 1.00 \pm 0.05 \quad (16.12)$$

Note that this mass difference squared is 30 times larger than the solar neutrino mass difference, and that the mixing is consistent with being maximal.

16.4 Accelerator Neutrino Experiments

A typical accelerator neutrino beam is either $\nu_\mu$ or $\bar{\nu}_\mu$, produced from the decays of $\pi^\pm$ and $K^\pm$ mesons. There is $\approx 1\%$ contamination of $\nu_e$ from semileptonic decays. The beam energies are in the range 100 MeV to 10 GeV, and the corresponding baselines
range from 1 to 1000 km.

Accelerator beams have been used to confirm the oscillations of $\nu_\mu \rightarrow \nu_\tau$. The K2K experiment fired a 1 GeV beam across Japan from KEK to Kamiokande (L=250 km). They measured the disappearance of $\nu_\mu$, and obtained results consistent with the atmospheric neutrinos. More recently the MINOS experiment fired a 10 GeV beam from Fermilab to Soudan (L=735 km), to obtain the world’s most accurate values for $\Delta m^2_{23}$ and $\sin^2 2\theta_{23}$.

Both MINOS and a Japanese experiment, T2K, are now looking for $\nu_\mu \rightarrow \nu_e$ appearance to try and measure the small mixing angle $\theta_{13}$, for which the current limit is $\sin^2 2\theta_{13} < 0.15$.

16.5 Open Questions on Neutrinos

In the past decade we have confirmed the existence of neutrino oscillations and explained the solar neutrino deficit, but there remain several open questions:

- We do not know the absolute neutrino mass scale. It could be $m \approx \Delta m$, or the masses could be degenerate $m \gg \Delta m$.

- We do not know the mass hierarchy, because we determine the magnitudes but not the signs of the mass differences. It could be normal $m_1, m_2 < m_3$, or inverted $m_3 < m_2, m_1$.

- We have measured two mass differences and two mixing angles. The third mass difference must be $\Delta m_{13} \approx 10^{-3}$ eV$^2$, but the third mixing angle only has an upper limit $\sin^2 2\theta_{13} < 0.15$.

- If $\theta_{13}$ is large enough, it may eventually be possible to measure the CP-violating phase $\delta$. This will require an accelerator known as a “neutrino factory” which is currently being designed.

- Why is the PMNS matrix very close to tri-bimaximal mixing: $\sin^2 \theta_{23} = 1/2, \sin^2 \theta_{12} = 1/3, \theta_{13} = 0$?

16.6 Neutrinos in Astrophysics*

Finally some comments on the role of neutrinos in astrophysics:

- The Big Bang model predicts a large relic density of very low energy neutrinos, similar to the microwave background of photons. However, the mass of neutrinos is too small to account for dark matter.

- In 1987 a few electron neutrinos with $E=10-40$ MeV were observed coming from the Supernova SN1987A in the Large Magellanic Cloud (L=175k light years).
The energies and spread of arrival times constrain the neutrino mass, and may eventually provide information on the initial stages of a supernova explosion. We just have to wait for the next one...

- Detectors such as AMANDA at the South Pole, and ANTARES in the Mediterranean detect very high energy neutrinos from outer space. The advantage of neutrinos is that they are unaffected on their path from a point source to the earth.