Lecture 9: Quantum Chromodynamics (QCD)

- Colour charge and symmetry
- Gluons
- QCD Feynman Rules
- $q \bar{q} \rightarrow q \bar{q}$ scattering
- QCD potential
Symmetries in Particle Physics

• The EM, Weak and Strong forces all display a property known as Gauge Symmetry.

• In QM, a symmetry is present if physical observables (e.g. cross section, decay widths) are invariant under the following change in the wavefunction:

\[ \psi \rightarrow \psi' = \hat{U} \psi \]

• e.g. in electromagnetism, the physical observable fields \(E\) and \(B\) are independent of the value of the EM potential, \(A_\mu\):

\[ A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu \chi \]
\[ A_\mu = (V, \vec{A}) \text{ with } \vec{B} = \nabla \times \vec{A} \]

• The conditions on \(U\) are that \(U\) is unitary, and commutes with the Hamiltonian:

\[ \hat{U}^\dagger \hat{U} = 1 \quad [\hat{U}, \hat{H}] = 0 \]

• e.g. for EM, \(\hat{U} = e^{i\phi}\) where \(\phi\) is an arbitrary phase: \(\psi \rightarrow \psi' = e^{i\phi} \psi\)
Symmetries in QED

• Instead of a global phase transformation $e^{i\phi}$ imagine a local phase transformation, where the phase $\phi \sim q \chi$ is a function of $x^\mu$: $\chi(x^\mu)$.
  
  • $q$ is a constant (will be electric charge)
  
  $$\psi \rightarrow \psi' = \hat{U} \psi = e^{iq\chi(x^\mu)} \psi$$

• Substitute into Dirac Equation $(i\gamma^\mu \partial_\mu - m)\psi = 0$

  $$(i\gamma^\mu \partial_\mu - m)\psi' = 0$$

  $$(i\gamma^\mu \partial_\mu - m)e^{iq\chi(x)}\psi = 0$$

  $$i\gamma^\mu(e^{iq\chi(x)}\partial_\mu \psi + iq\partial_\mu \chi \psi) - me^{iq\chi(x)}\psi = 0$$

  $$(i\gamma^\mu \partial_\mu - m)e^{iq\chi(x)}\psi - q\gamma^\mu \partial_\mu \chi \psi = 0$$

• An interaction term $-q\gamma^\mu \partial_\mu \chi \psi$ term appears in the Dirac Equation.

• To cancel this, modify the Dirac Equation for interacting fermions:

  $$(i\gamma^\mu \partial_\mu + iqA_\mu - m)\psi = 0$$

• With $A^\mu$ transforming as:

  $$A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu \chi$$

  to cancel interaction term
Gauge Symmetry in QED & QCD

• Demanding that QED is invariant by a local phase shift:

\[ \psi \rightarrow \psi' = \hat{U} \psi = e^{iq\chi(x^\mu)} \psi \]

• Tells us that fermions interact with the photon field as:

\[ q \gamma^\mu A_\mu \psi \]

• This invariance of QED under the local phase shift status is know as a local U(1) gauge symmetry.

• Today we will see the consequences of a symmetry in QCD, but with a different symmetry, known as SU(3).

⇒ QCD exhibits a local SU(3) gauge symmetry.
Colour Charge

- Each quark carries a colour charge: red, blue or green.
- The coupling strength is the same for all three colours.
- To describe a quark, use a spinor plus a colour column vector:

\[ r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad g = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]

- Mathematically, this is described by an SU(3) symmetry. QCD interactions are invariant under SU(3) rotations in colour space.

- Gluons responsible for exchanging momentum and colour between quarks.

- Each gluon contains colour and anti-colour.
- Naively expect nine gluons: \( r \bar{r} \quad r \bar{b} \quad r \bar{g} \quad b \bar{r} \quad b \bar{b} \quad b \bar{g} \quad g \bar{r} \quad g \bar{b} \quad g \bar{g} \)
- However gluons are described by the generators of the SU(3) group, giving eight linear colour-anti-colour combinations of these
Eight Gluons

- The Gell-Mann matrices describe the allowed colour configurations of gluons. (The Gell-Mann matrices are the *generators* of the SU(3) symmetry.)

\[
\begin{align*}
\lambda^1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda^2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda^3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda^4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \lambda^5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \lambda^6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda^7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} & \lambda^8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{align*}
\]

Each gluon is described by:

\[
g^i = \left( \begin{array}{c} \text{r} \\ \text{b} \\ \text{g} \end{array} \right) \lambda^i \left( \begin{array}{c} \text{r} \\ \text{b} \\ \text{g} \end{array} \right)
\]

\[
\begin{align*}
g^1 &= \frac{1}{\sqrt{2}} (r\bar{b} + b\bar{r}) & g^2 &= \frac{i}{\sqrt{2}} (r\bar{b} - b\bar{r}) & g^3 &= \frac{1}{\sqrt{2}} (r\bar{r} - b\bar{b}) \\
g^4 &= \frac{1}{\sqrt{2}} (r\bar{g} + g\bar{r}) & g^5 &= \frac{i}{\sqrt{2}} (r\bar{g} - g\bar{r}) & g^6 &= \frac{1}{\sqrt{2}} (b\bar{g} + g\bar{b}) \\
g^7 &= \frac{i}{\sqrt{2}} (b\bar{g} - g\bar{b}) & g^8 &= \frac{1}{\sqrt{6}} (r\bar{r} + b\bar{b} - 2g\bar{g})
\end{align*}
\]
Feynman Rules for QCD

External Lines
- incoming quark: $u(p)$
- outgoing quark: $\bar{u}(p)$
- incoming anti-quark: $\bar{v}(p)$
- outgoing anti-quark: $v(p)$
- incoming gluon: $\varepsilon_{\mu}(p)$
- outgoing gluon: $\varepsilon_{\mu}(p)^*$

Internal Lines (propagators)
- spin 1 gluon

Vertex Factors
- spin 1/2 quark

\[ \alpha_S = \frac{g_S^2}{4\pi} \]

\[ g_{\mu\nu} \delta^{ab} \]

\[ \lambda^a_{ji} \gamma^\mu \]

\[ a, b = 1, 2, ..., 8 \] are gluon colour indices

\[ i, j = 1, 2, 3 \] are quark colours,

\[ \lambda^a \quad a = 1, 2, ..., 8 \] are the Gell-Mann SU(3) matrices

The $\lambda^a_{ij}$ terms account for the quark colour
Gluon-Gluon Interactions

- Gluons also carry colour charge and can therefore self-interact.

- Two allowed possibilities:

  - Gluon interactions are believed to give rise to **colour confinement**
    - Try to separate an electron-positron pair
    
    \[ V_{\text{QED}}(r) = -\frac{q_2 q_1}{4\pi\varepsilon_0 r} = -\frac{\alpha}{r} \]

  - Try to separate an quark anti-quark pair
    
    \[ V_{\text{QCD}}(r) \sim \lambda r \]

  - A **gluon flux tube** of interacting gluons is formed. Energy \( \sim 1 \text{ GeV/fm} \).

- Gluon-gluon interactions are responsible for holding quarks in mesons and baryons.
\( \overline{q}q \rightarrow q\bar{q} \) scattering

• To write down the matrix element, follow the fermion arrows backwards.
  ➡ For the quark line \( j \rightarrow i \): \( \lambda_{ji} \) term at vertex
  ➡ For the antiquark line \( k \rightarrow l \): \( \lambda_{kl} \) term at vertex

\[
\mathcal{M} = \left[ \bar{u}_j \frac{g_S}{2} \lambda^a_{ji} \gamma^\mu u_i \right] \left[ \frac{g^{\mu\nu}}{q^2} \delta^{ab} \left[ \bar{v}_k \frac{g_S}{2} \lambda^b_{kl} \gamma^\nu v_l \right] \right]
\]

\[
\mathcal{M} = \frac{g_S^2}{q^2} \frac{1}{4} \left[ \bar{u}_j \gamma^\mu u_i \right] \left[ \bar{v}_k \gamma^\mu v_l \right] \lambda^a_{ji} \lambda^a_{kl}
\]

• The matrix element looks very similar to electromagnetic scattering except \( e \rightarrow gs \), and the addition of the terms \( \lambda^a_{ji} \lambda^a_{kl} / 4 \)

• In the lowest order approximation, the dynamics of the \( \overline{q}q \rightarrow q\overline{q} \) scattering is the same as electromagnetic \( e^+e^- \rightarrow e^+e^- \) scattering.

• Describe in terms of Coloumb-like potential \( V_{q\bar{q}} = -\frac{f\alpha_S}{r} \)

• The colour factor \( f = \frac{1}{4} \lambda^a_{ji} \lambda^a_{kl} = \frac{1}{4} \sum_a \lambda^a_{ji} \lambda^a_{kl} \) is a sum over elements in the \( \lambda \) matrices.
Colour Factor for $q \bar{q} \rightarrow q \bar{q}$

- Need to calculate the **colour factor**
  \[ f = \frac{1}{4} \lambda^a_{ji} \lambda^a_{kl} = \frac{1}{4} \sum_a \lambda^a_{ji} \lambda^a_{kl} \]

- For the calculation we choose colours for $q$ and $\bar{q}$. As the theory is invariant under rotations in colour space any choice of colours will give the same answer.

- Three colour options:
  1. $i=1 \ k=\bar{1} \rightarrow j=1 \ l=\bar{1}$ e.g. $r \bar{r} \rightarrow r \bar{r}$
     \[ f_1 = \frac{1}{4} \sum_a \lambda^a_{11} \lambda^a_{11} \]
  2. $i=1 \ k=\bar{2} \rightarrow j=1 \ l=\bar{2}$ e.g. $r \bar{b} \rightarrow r \bar{b}$
     \[ f_2 = \frac{1}{4} \sum_a \lambda^a_{11} \lambda^a_{22} \]
  3. $i=1 \ k=\bar{1} \rightarrow j=2 \ l=\bar{2}$ e.g. $r \bar{r} \rightarrow b \bar{b}$
     \[ f_3 = \frac{1}{4} \sum_a \lambda^a_{21} \lambda^a_{12} \]

- Calculate option 2. The only matrices with non-zero element in the red-red (11) and blue-blue (22) elements are $\lambda^3$ and $\lambda^8$.
  \[ f_2 = \frac{1}{4} \sum_a \lambda^a_{11} \lambda^a_{22} = \frac{1}{4} \left[ \lambda^3_{11} \lambda^3_{22} + \lambda^8_{11} \lambda^8_{22} \right] = \frac{1}{4} \left[ (1)(-1) + \left( \frac{1}{\sqrt{3}} \right) \left( \frac{1}{\sqrt{3}} \right) \right] = -\frac{1}{6} \]

- Similarly,
  \[ f_1 = \frac{1}{4} \sum_a \lambda^a_{11} \lambda^a_{11} = \frac{1}{4} \left[ \lambda^3_{11} \lambda^3_{11} + \lambda^8_{11} \lambda^8_{11} \right] = \frac{1}{4} \left[ (1)(1) + \left( \frac{1}{\sqrt{3}} \right) \left( \frac{1}{\sqrt{3}} \right) \right] = \frac{1}{3} \]
  \[ f_3 = \frac{1}{4} \sum_a \lambda^a_{21} \lambda^a_{12} = \frac{1}{4} \left( \lambda^1_{21} \lambda^1_{12} + \lambda^2_{12} \lambda^2_{21} \right) = \frac{1}{4} \left[ (-i)(i) + (1)(1) \right] = \frac{1}{2} \]
Colour Factor for Mesons

- Mesons are colourless $qq$ states in a “colour singlet”: $rr + gg + bb$
- Calculate colour factor for $qq \rightarrow qq$ scattering in a meson.

- Two possibilities for colour combinations:
  - Quarks stay the same colour e.g. $rr \rightarrow rr$ $f_r = \frac{1}{3}$
  - Quarks change colour e.g. $rr \rightarrow bb$ and $rr \rightarrow gg$ each contributes $f_3 = \frac{1}{2}$

- Sum over all possible final states for $rr \rightarrow qq$ gives $f_r = \frac{1}{3} + \frac{1}{2} + \frac{1}{2} = \frac{4}{3}$
- Average over all possible initial states, $rr, gg, bb$:
  \[
  f = \frac{1}{3} \left( rr \rightarrow qq + gg \rightarrow qq + bb \rightarrow qq \right) = \frac{1}{3} \left( \frac{4}{3} + \frac{4}{3} + \frac{4}{3} \right) = \frac{4}{3}
  \]

- The colour factor for the $qq$ interactions within a meson is $\frac{4}{3}$
  - The potential within a meson (to lowest order) is:
    \[
    V_{qq} = -\frac{4}{3} \frac{\alpha_s}{r}
    \]
QCD Potential

- At large distances: gluon-gluon interactions
  \[ V_{QCD}(r) \sim \lambda r \]
- At short distances: \( \bar{q}q \rightarrow \bar{q}q \) scattering
  \[ V_{q\bar{q}} = -\frac{4}{3} \frac{\alpha_S}{r} \]
- Overall potential is:
  \[ V_{QCD}(r) = -\frac{4}{3} \frac{\alpha_S}{r} + \lambda r \]

This model provides a good description of the bound states of heavy quarks:
- charmonium (c \( \bar{c} \))
- bottomonium (b \( \bar{b} \))

\[ \text{Diagram showing charmonium and bottomonium mass spectra} \]

Agreement of data with prediction provides strong evidence that \( V_{QCD} \) has the expected form.