Groups and Symmetries: Lectures 6 and 7

SO(3) and Conserved Quantities

William Barter

email me: w.barter@imperial.ac.uk

Recap – Last Time

Lecture 4: Operators and Vector spaces

- Operators within Vector Spaces
- Linking Operators and Matrices, and Groups
- Types of Operators – Hermitian, Unitary

Lecture 5:

- Conservation in a classical physics context
- Conservation in a Quantum Mechanical context, and Ehrenfest's theorem.
- Key Implications : the link between symmetries (unitary transformations that commute with the Hamiltonian) and conserved quantities.

Today

Lecture 6:

- Rotations in 3D – SO(3)
- Interpretations in terms of conserved quantities
- Why spin is a form of angular momentum

Lecture 7:

- Spherical Harmonics and SO(3)
- Structure Constants
- Regular Representation
- Rank of a group, and Casimir Operators

Rotations in Three Dimensions

- This is governed by the group of three dimensional rotational matrices – which are orthogonal matrices with determinant 1.
- This is the group SO(3).
- We will now look at this group in some detail.

Rotations in Three Dimensions

- Let's start by looking at the defining representation – the rotation matrices for three dimensional space.
- And we can start that by looking at rotations about the z-axis – which rotates the x and y axes.
  - This is just the two dimensional rotation we've already looked at – SO(2) is a subgroup of SO(3).

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \mathbf{R}_3(\alpha_3) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\alpha_3 & -\sin\alpha_3 & 0 \\ \sin\alpha_3 & \cos\alpha_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Rotations in Three Dimensions

- We can also find the generator by considering an infinitesimal rotation:

$$g(\delta\alpha) \approx e - i\delta\alpha X$$

$$\mathbf{R}_3(\delta\alpha_3) \approx \begin{pmatrix} 1 & -\delta\alpha_3 & 0 \\ \delta\alpha_3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \delta\alpha_3 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$-i\mathbf{X}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Longrightarrow \quad \mathbf{X}_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Rotations in Three Dimensions

- Similarly, rotations around the x and y axes give:

$$\mathbf{X}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \mathbf{X}_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

- These matrices are all Hermitian, and we can write them as $(X_i)_{jk} = -i\varepsilon_{ijk}$, where ε_{ijk} is completely anti-symmetric.

- The three generators can be thought of as forming a $3 \times 3 \times 3$ tensor.

Rotations in Three Dimensions

- In three dimensions the order we apply rotations matters:
  - A rotation about the x axis, followed by a rotation about the y axis, is different to the same rotation about the y axis followed by the same rotation about the x axis.
  - Mathematically, this is captured by the rotation matrices in three dimensions not commuting. This is a general property, and means the group is non-Abelian.
- This property is captured by the generators as well:

$$[\mathbf{X_i}, \mathbf{X_j}] = i\epsilon_{ijk}\mathbf{X_k}$$

Rotations in Three Dimensions

- A general rotation can be written as:

$$\mathbf{R}(\alpha_1, \alpha_2, \alpha_3) = e^{-i(\alpha_1 \mathbf{X}_1 + \alpha_2 \mathbf{X}_2 + \alpha_3 \mathbf{X}_3)} = e^{-i\alpha_i \mathbf{X}_i}$$

  though this is not the same as the three rotations applied sequentially.

- Let's look at some other representations of SO(3), and learn more about this symmetry, and its implications for physics.

Rotations in Three Dimensions - Scalars

- We considered transformations of different objects in SO(2): a vector, a tensor, a scalar field...
- Let’s take a step back as we look at SO(3).
- How does a scalar transform? What is the generator? $S' = \left[I - i\delta\alpha X_i^{(S)}\right] S$
- The scalar’s value doesn’t change: $S' = S$

$$\Longrightarrow \; X_i^{(S)} = 0$$

- Therefore in this representation, all generators are 0, and all group elements are mapped onto the identity element.
- The generators trivially obey the same commutation relation we saw in the defining representation: $[\mathbf{X_i}, \mathbf{X_j}] = i\epsilon_{ijk}\mathbf{X_k}$

Rotations in Three Dimensions – Scalar Function

- What about a scalar function? This is a less trivial object, as the value of the scalar does change through space.
- Therefore, as with SO(2), the value of the function changes after rotation because the rotation takes us to a new point in space.
- However, we can write – as with SO(2) – that the value of the function after the rotation, at the rotated coordinates, is the same as the value before the rotation:

$$f'(\vec{r'}) = f(\vec{r})$$

Or:

$$f'(\vec{r'}) = f(\mathbf{R}^{-1}(\vec{\alpha})\vec{r'}) = \mathbf{U}(\vec{\alpha})f(\vec{r})$$

The form of this transformation is what we're seeking to find.

Rotations in Three Dimensions – Scalar Function

- We already know the form of this transformation for rotations about the *z*-axis, as this is just a two dimensional transformation of a scalar function, and takes the form we saw in SO(2).
- The generator of an infinitesimal transformation about the *z*-axis is $L_z = xp_y - yp_x$
- It turns out that this generalises and the three generators of SO(3) for a scalar function are L_i.

Rotations in Three Dimensions – Scalar Function

- We could also derive this by looking at the Taylor series.
  - Reminder: the generator describes infinitesimal transformations.

$$f(\mathbf{R}^{-1}\vec{r}) \approx f(\vec{r}) + \left.\frac{\partial f}{\partial \alpha_i}\right|_{\vec{\alpha}=0} \delta\alpha_i$$

So the generator will be:

$$-iX_i = \left.\frac{\partial f}{\partial \alpha_i}\right|_{\vec{\alpha}=0}$$

$$\begin{aligned}
\left.\frac{\partial f}{\partial \alpha_i}\right|_{\vec{\alpha}=0} &= \left.\frac{\partial f}{\partial r_j}\frac{\partial r_j}{\partial \alpha_i}\right|_{\vec{\alpha}=0} \\
&= \left.\frac{\partial f}{\partial r_j} r'_k \frac{\partial (R^{-1})_{jk}}{\partial \alpha_i}\right|_{\vec{\alpha}=0} \\
&= \left.\frac{\partial f}{\partial r_j} r_k \frac{\partial (R^{-1})_{jk}}{\partial \alpha_i}\right|_{\vec{\alpha}=0} \\
&= \left.\frac{\partial f}{\partial r_j} r_k \frac{\partial (R^{T})_{jk}}{\partial \alpha_i}\right|_{\vec{\alpha}=0}
\end{aligned}$$

$r'_k = r_k$ when $\vec{\alpha} = 0$

We have orthogonal matrices

Rotations in Three Dimensions – Scalar Function

$$\left.\frac{\partial f}{\partial \alpha_i}\right|_{\vec{\alpha}=0} = \left.\frac{\partial f}{\partial r_j} r_k \frac{\partial (R^T)_{jk}}{\partial \alpha_i}\right|_{\vec{\alpha}=0}$$

- But we know:

$$\left.\frac{\partial \mathbf{R}}{\partial \alpha_i}\right|_{\vec{\alpha}=0} = -i\mathbf{X}_i$$

This is the definition of how we found X_i - the infinitesimal rotation of the position vector

- And since:

$$\left.\frac{\partial \mathbf{R}^{-1}}{\partial \alpha_i}\right|_{\vec{\alpha}=0} = \left.\frac{\partial \mathbf{R}^T}{\partial \alpha_i}\right|_{\vec{\alpha}=0} = -i\mathbf{X}_i^T = i\mathbf{X}_i$$

$$(X_i)_{jk} = -i\varepsilon_{ijk}$$

$$\left.\frac{\partial f}{\partial \alpha_i}\right|_{\vec{\alpha}=0} = i(X_i)_{jk} r_k \frac{\partial f}{\partial r_j} = -(\vec{r} \times \vec{\nabla} f)_i$$

- The generators are $X_i = i \left.\frac{\partial f}{\partial \alpha_i}\right|_{\vec{\alpha}=0} = -i(\vec{r} \times \vec{\nabla})_i = (\vec{X} \times \vec{P})_i = L_i$

Rotations in Three Dimensions – Scalar Function

- So, we have found that for a scalar function,

$$f'(\vec{r'}) = f(\mathbf{R}^{-1}(\vec{\alpha})\vec{r'}) = \mathbf{U}(\vec{\alpha})f(\vec{r})$$

$$U(\vec{\alpha}) = e^{-i\vec{\alpha}\cdot\vec{L}}$$

- The generators of the group are the angular momentum operators.
- Note that these operators obey the same commutation relation as the three generators did for the defining representation:

$$[L_i, L_j] = i\epsilon_{ijk}L_k$$

- We've worked here in Gaussian coordinates, but could have also used Spherical Polars.

Rotations in Three Dimensions – Scalar Function

- What is the conservation argument here? What's the symmetry, and what's the conserved quantit(ies) if the symmetry commutes with the Hamiltonian?

Rotations in Three Dimensions – Scalar Function

- What is the conservation argument here? What's the symmetry, and what's the conserved quantit(ies) if the symmetry commutes with the Hamiltonian?

- If the Hamiltonian is invariant under rotations in 3 dimensions, then, for scalar functions/scalar fields, the quantity is $\langle L_i \rangle$ does not change with time; we see conservation of angular momentum about each axis.

Rotations in Three Dimensions – Vector Field

- Now let's consider what happens if we rotate a vector field.
- This is a field that takes different values at each point in space, and, crucially, has a direction at each point in space.
- We could think of this therefore as a (different) vector at each position in space.
- Examples of Vector Fields in Classical Fields include the Electric Field, and the Magnetic Field.
- In Particle Physics, we use Vector Fields to describe spin-1 particles – the Vector Bosons, which have Spin 1.

Rotations in Three Dimensions – Vector Field

- Consider a general vector field, $\vec{A}(\vec{r})$.
- We can consider a rotation of this field in two steps: first we rotate the space (to get what we will call *A'*), then we rotate the vector at each point in space (to get *A''*).
- For now, let's just consider rotations about the *z*-axis.
- For the first stage, the transformation of the space, each component of *A* transforms like the scalars functions we have just considered.

$$A_i' = A_i - i\delta\alpha_3 L_3 A_i$$

$$\begin{pmatrix} A_x' \\ A_y' \\ A_z' \end{pmatrix} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} - i\delta\alpha_3 \begin{pmatrix} L_3 & 0 & 0 \\ 0 & L_3 & 0 \\ 0 & 0 & L_3 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Note: we still need to rotate the vector itself at each point in space.

Rotations in Three Dimensions – Vector Field

- We can then apply the rotation of the vector at each point in space.
- This is the same as rotating a vector that we saw before in the defining representation.

$$\begin{pmatrix} A''_x \\ A''_y \\ A''_z \end{pmatrix} = \mathbf{R}_3(\delta\alpha_3) \begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix}$$

$$= \begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} - i\delta\alpha_3 \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix}$$

Rotations in Three Dimensions – Vector Field

- We can put both pieces together:

$$\begin{pmatrix} A''_x \\ A''_y \\ A''_z \end{pmatrix} = \begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} - i\delta\alpha_3 \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix}$$

$$= \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} - i\delta\alpha_3 \begin{pmatrix} L_3 & 0 & 0 \\ 0 & L_3 & 0 \\ 0 & 0 & L_3 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_3 \end{pmatrix} - i\delta\alpha_3 \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$= \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} - i\delta\alpha_3 \left(\mathbf{I} L_3 + \mathbf{X}_3 \right) \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \left[\mathbf{I} - i\delta\alpha_3 \left(\mathbf{I} L_3 + \mathbf{X}_3 \right) \right] \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

- So the generator of infinitesimal rotations about *z* is: $\mathbf{J}_3 = \mathbf{I} L_3 + \mathbf{X}_3$

Rotations in Three Dimensions – Vector Field

- Similarly, rotations about the other two axes yield similar results.

$$\mathbf{J}_\mathrm{i} = \mathbf{I}\mathrm{L}_\mathrm{i} + \mathbf{X}_\mathrm{i}$$

- For a rotation of a vector field, the generator is the angular momentum operator added to the operator describing the rotation of the vector at each point.
- The unitary operator describing the general rotation is $e^{-i\alpha_i \mathbf{J}_i}$
- Does anyone want to tell us what this means in terms of conservation laws?

Rotations in Three Dimensions – Vector Field

- If I have a Hamiltonian describing a vector field that is invariant under rotations in three dimensions, then there is a conserved quantity associated with this field.
- This conserved quantity is <u>not</u> the orbital angular momentum alone, but has an additional component attached to the direction of the vector itself at each point in space.

Rotations in Three Dimensions – Vector Field

- We can now see mathematically why spin behaves like an additional source of angular momentum (though spin ½ particles aren't described by a vector field).
- For a rotationally invariant Hamiltonian in 3D, vector fields (which do describe spin 1 particles) have a conserved quantity which is the orbital angular momentum plus the spin.

- Aside: we know that vector fields are needed for spin 1 particles, because the **X** operator has 3 eigenvalues – one for each of s=1,0,-1.

Rotations in Three Dimensions – Vector Fields

- Let's consider the commutation relations of the generators for the representation of SO(3) that describes transformations of vector fields.
- The two separate parts of the generator (L, X) clearly commute themselves, so:

$$
\begin{aligned}
[\mathbf{J}_i, \mathbf{J}_j] &= [\mathbf{I}L_i + \mathbf{X}_i, \mathbf{I}L_j + \mathbf{X}_j] \\
&= \mathbf{I}[L_i, L_j] + [\mathbf{X}_i, \mathbf{X}_j] \\
&= i\mathbf{I}\epsilon_{ijk}L_k + i\epsilon_{ijk}\mathbf{X}_k \\
&= i\epsilon_{ijk}\mathbf{J}_k
\end{aligned}
$$

- Once again, the generators obey the same commutation relation as they did in the defining representation.
- This is generally true: the commutation relation of generators is the same in all representations, and can also be used to define the group.

QUESTIONS

and

BREAK

SO(3) Representations

- When we consider the effect of rotations in 3D it's useful to describe an arbitrary wavefunction with the spherical harmonic basis states Y_{lm}
- Just considering the angular part of a wavefunction, we can write

$$\psi(\theta, \phi) = \sum_{l.m} a_{lm} Y_{lm}(\theta, \phi)$$

- Since the functional form of the Y_{lm} is independent of any rotation, when we make a rotation, we simply change the values of a_{lm} associated with a state.
- Since rotation cannot increase the total angular momentum (*l*) the rotations will mix states of different *m*, for a given *l.*

SO(3) Representations

$$\begin{pmatrix} a'_{0\,0} \\ a'_{1\,1} \\ a'_{1\,0} \\ a'_{1\,-1} \\ a'_{2\,2} \\ a'_{2\,1} \\ a'_{2\,0} \\ a'_{2\,-1} \\ a'_{2\,-2} \\ \vdots \end{pmatrix} = \begin{pmatrix} M^{(0)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & M^{(1)}_{11} & M^{(1)}_{12} & M^{(1)}_{13} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & M^{(1)}_{21} & M^{(1)}_{22} & M^{(1)}_{23} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & M^{(1)}_{31} & M^{(1)}_{32} & M^{(1)}_{33} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & M^{(2)}_{11} & M^{(2)}_{12} & M^{(2)}_{13} & M^{(2)}_{14} & M^{(2)}_{15} & \cdots \\ 0 & 0 & 0 & 0 & M^{(2)}_{21} & M^{(2)}_{22} & M^{(2)}_{23} & M^{(2)}_{24} & M^{(2)}_{25} & \cdots \\ 0 & 0 & 0 & 0 & M^{(2)}_{31} & M^{(2)}_{32} & M^{(2)}_{33} & M^{(2)}_{34} & M^{(2)}_{35} & \cdots \\ 0 & 0 & 0 & 0 & M^{(2)}_{41} & M^{(2)}_{42} & M^{(2)}_{43} & M^{(2)}_{44} & M^{(2)}_{45} & \cdots \\ 0 & 0 & 0 & 0 & M^{(2)}_{51} & M^{(2)}_{52} & M^{(2)}_{53} & M^{(2)}_{54} & M^{(2)}_{55} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_{0\,0} \\ a_{1\,1} \\ a_{1\,0} \\ a_{1\,-1} \\ a_{2\,2} \\ a_{2\,1} \\ a_{2\,0} \\ a_{2\,-1} \\ a_{2\,-2} \\ \vdots \end{pmatrix}$$

- If we write this out, the form is already in block diagonal form.
- Each sub-matrix provides another representation of SO(3), for objects of different angular momentum.
- The matrices are of size $(2l + 1)\times(2l + 1)$.
- In SO(3), it turns out <u>only odd dimensions matrices provide representations</u>.

Structure Constants

- In SO(3) we found that the matrices in the defining representation had generators which obeyed:

$$[\mathbf{X}_i, \mathbf{X}_j] = i\epsilon_{ijk}\mathbf{X}_k$$

- Likewise, we saw this in other representations, and also when we considered operators:

$$[L_i, L_j] = i\epsilon_{ijk}L_k$$

- And we noted that this was a general rule associated with a group, that the commutation relation of the generators was independent of the representation.

- We call the numbers ϵ_{ijk} the <u>structure constants</u> of the group.

Structure Constants

- Any multiplicative combination of group elements is a member of the group (from the group rules).
- Therefore any commutation relation must be able to be written as a sum over the group elements.
- The idea of structure constants is therefore common to all groups.

Structure Constants

- Let’s consider the the more general case of a Lie group in general.
- For simplicity, let’s take the same parameter α as multiplying two generators.
- Let’s apply a sequence of small transformations:

$$g_i g_j g_i^{-1} g_j^{-1} = e^{i\,\delta\alpha\,X_i} e^{i\,\delta\alpha\,X_j} e^{-i\,\delta\alpha\,X_i} e^{-i\,\delta\alpha\,X_j}$$

- To second order this is (remembering to expand <u>each</u> term to second order, as cross terms cancel with individual expansions):

$$e^{i\,\delta\alpha\,X_i} e^{i\,\delta\alpha\,X_j} e^{-i\,\delta\alpha\,X_i} e^{-i\,\delta\alpha\,X_j} = 1 - \delta\alpha^2 \left[X_i, X_j\right]$$

- But we also know that this must be able to be expressed by a sum of group members itself:

$$1 - \delta\alpha^2 \left[X_i, X_j\right] = e^{-i\,\Sigma\delta\beta_k\,X_k}$$

- Where this $\delta\beta_k\ X_k$ will depend on *i* and *j*.

Structure Constants

$$1 - \delta\alpha^2 [X_i, X_j] = e^{-i\,\Sigma\delta\beta_k\,X_k}$$

- Where this $\delta\beta_k\ X_k$ will depend on *i* and *j*.

- Let us write: $\delta\beta_k = \delta\alpha^2\ f_{ijk}$ (we know that this effect is second order in $\delta\alpha$), and we use this to define f_{ijk}.

$$1 - \delta\alpha^2 [X_i, X_j] = e^{-i\,\Sigma\delta\beta_k\,X_k} = e^{-i\,\delta\alpha^2\,\Sigma\,f_{ijk}\,X_k}$$

$$= 1 - i\ \delta\alpha^2\ \Sigma\ f_{ijk}\ X_k$$

- So we can then write (though as a definition):

$$[X_i, X_j] = i\ \Sigma\ f_{ijk}\ X_k$$

where we expect f_{ijk} to be O(1).

- This applies to all Lie groups, and each group has its own f_{ijk}

Structure Constants

$$[X_i, X_j] = i \, \Sigma \, f_{ijk} \, X_k$$

- By definition the structure constants are anti-symmetric under a swap of the first two indices, *i* and *j*.
- In SO(3), the structure constants are anti-symmetric under a swap of any two indices.
- For the groups we will discuss here, we will have structure constants that are anti-symmetric in swaps of any indices.

Structure Constants

- Once the structure constants are known we can directly find a representation of a group.
- This follows from the Jacobi identity:

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$$

- We can apply this to the group generators:

$$[[X_i, X_j], X_k] + [[X_j, X_k], X_i] + [[X_k, X_i], X_j] = 0$$

$$\implies \sum_l if_{ijl}[X_l, X_k] + if_{jkl}[X_l, X_i] + if_{kil}[X_l, X_j] = 0$$

Applying the definition of the structure constants; we sum over *l* so it is a dummy variable.

$$\implies -\sum_{l,m} f_{ijl} f_{lkm} X_m + f_{jkl} f_{lim} X_m + f_{kil} f_{ljm} X_m \ = 0$$

We sum over *m* so it is a dummy variable.

$$\implies \quad -\sum_{l,m} (f_{ijl} f_{lkm} + f_{jkl} f_{lim} + f_{kil} f_{ljm}) X_m = 0$$

Structure Constants

$$-\sum_{l,m}(f_{ijl}f_{lkm} + f_{jkl}f_{lim} + f_{kil}f_{ljm})X_m = 0$$

$$\Longrightarrow \sum_l f_{ijl}f_{lkm} + f_{jkl}f_{lim} + f_{kil}f_{ljm} = 0$$

This is generally true for all X_m

$$\Longrightarrow \sum_l f_{ijl}f_{lkm} - f_{jkl}f_{ilm} + f_{ikl}f_{jlm} = 0$$

Swapping indices

- Define matrices $(T_i)_{jk} = -if_{ijk}$ (or $f_{ijk} = i(T_i)_{jk}$)

$$\Longrightarrow \sum_l if_{ijl}(T_l)_{km} + (T_j)_{kl}(T_i)_{lm} - (T_i)_{kl}(T_j)_{lm} = 0$$

$$\Longrightarrow \quad \mathbf{T}_i\mathbf{T}_j - \mathbf{T}_j\mathbf{T}_i = [\mathbf{T}_i, \mathbf{T}_j] = \sum_l if_{ijl}(T_l)$$

The last two terms are matrix multiplication

- The matrices **T** obey the group commutation relation, so are a representation.

Structure Constants

- The matrices $(T_i)_{jk} = -if_{ijk}$ form a representation of the group.

- This specific representation of the group uses $n \times n$ matrices.

- We label this representation the regular representation.

- For SO(3), the regular representation is the defining representation. This is not necessary however.

Rank of a Group

- For a group with N generators, the number that mutually commute is the rank R. (If none commute, then the group has rank 1.)
- SO(3) has three generators, but none commute. The group has rank 1.
- The rank of a group determines some of the physical properties of a system.

Casimirs

- In QM we define the state of a particle by its quantum numbers: $\mathrm{E}, q_2, \dots, q_n$
- For these quantum numbers to be "good labels" they won't change over time.
- The associated operators must commute with the Hamiltonian, and we have a mutual eigenstate of $n-1$ operators (in addition to the Hamiltonian).
- In non-relativistic QM (in 3D), we can label the system with the Energy and J_z (once we choose J_z we do not have good quantum numbers from J_x and J_y).
- However, we also have another label, $J^2 = J_x^2 + J_y^2 + J_z^2$
- Let's explore this operator and label in terms of group theory.

Casimirs

- For a spinless particle in a central potential, the state is labelled by: *n, l*, and *m.*
- *l* is determined from the operator L^2, which takes eigenvalues $l(l+1)$.
- For this system, $[H, L^2] = 0$

- $L^2 = L_x^2 + L_y^2 + L_z^2$ is not a generator of SO(3), and is non-linear in the generators.
- We call this operator a Casimir operator.
- SO(3) has only one Casimir operator, and this is the Casimir operator of SO(3).

Casimirs

- Casimir Operators are typically formed from even powers of the generators of the group.
- Casimir Operators commute with all the generators of a group, so provide another label for a state.

- The rank of a group is related to the number of Casimir Operators associated with a group.
- SO(3) is rank 1 and therefore has 1 Casimir Operator.

Casimirs

- The Casimirs commute with all the generators of a group.
- Consider the commutator of the Hamiltonian with a (quadratic) Casimir operator:

$$\begin{aligned}[H, X_i X_j] &= H X_i X_j - X_i X_j H \\ &= H X_i X_j - X_i X_j H + X_i H X_j - X_i H X_j \\ &= X_i [H, X_j] + [H, X_i] X_j\end{aligned}$$

- So the Casimir will commute with the Hamiltonian.
- So not only does the Casimir provide a label for the state when considering the symmetry, it is also a commuting operator with the Hamiltonian, and provides a label for the eigenstate in the overall QM system.

Casimirs

- For a Hamiltonian symmetric under a symmetry group of rank R, there will be 2R conserved variables (and labels) associated with each specific symmetry group: R from the generators, and R from the Casimirs.

- The R conserved variables that arise from the Casimirs (and not the generators) are invariant under the group symmetry operations.
- This is because the Casimirs commute with the group generators.
- e.g. in SO(3), a rotation does not change the value of *l*, but does change the value of *m*.

Casimirs

- We've discussed Casimirs as quadratic operators.
- If these operators represent observables, then they must be Hermitian.
- We must therefore form them from Hermitian combinations of Hermitian operators:

$$i[X_i, X_j] \text{ or } \{X_i, X_j\}$$

- The first of these is only linear in the group elements (from the structure constants); Hermitian Casimirs are formed out of anti-commutators of group elements.

Summary

Lecture 6:

- Rotations in 3D – SO(3). We looked at this in many different representations.
- Interpretations in terms of conserved quantities – we see why "spin is a form of angular momentum"
- The commutation relation between generators in a group is the same in all representations, and can be used to define the group.

Summary

Lecture 7:

- Spherical Harmonics and SO(3)
- Structure Constants
- Regular Representation
- Rank of a group, and Casimir Operators