3 Dirac Delta Function

A frequently used concept in Fourier theory is that of the *Dirac Delta Function*, which is somewhat abstractly defined as:

$$\begin{aligned} \delta(x) &= 0 \qquad \text{for } x \neq 0 \\ \int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x &= 1 \end{aligned} \tag{1}$$

This can be thought of as a very "*tall*-and-*thin*" spike with unit area located at the origin, as shown in figure 1.

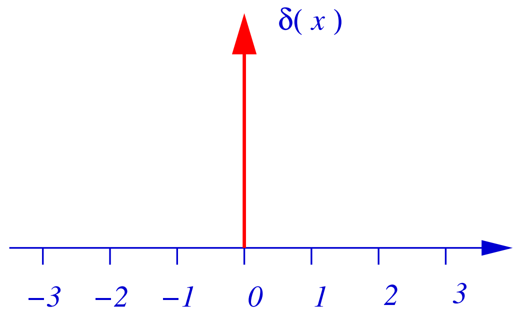

Figure 1: The δ-function.

NOTE: The δ-functions should **not** be considered to be an *infinitely high* spike of *zero* width since it scales as:

$$\int_{-\infty}^{\infty} a\,\delta(x)\,\mathrm{d}x = a$$

where a is a constant.

The *Delta Function* is not a true function in the analysis sense and if often called an *improper function*. There are a range of definitions of the *Delta Function* in terms of *proper function*, some of which are:

$$\begin{aligned} \Delta_\varepsilon(x) &= \frac{1}{\varepsilon\sqrt{\pi}} \exp\left(\frac{-x^2}{\varepsilon^2}\right) \\ \Delta_\varepsilon(x) &= \frac{1}{\varepsilon}\Pi\left(\frac{x - \frac{1}{2}\varepsilon}{\varepsilon}\right) \\ \Delta_\varepsilon(x) &= \frac{1}{\varepsilon}\mathrm{sinc}\left(\frac{x}{\varepsilon}\right) \end{aligned}$$

being the Gaussian, Top-Hat and Sinc approximations respectively. All of these expressions have the property that,

$$\int_{-\infty}^{\infty} \Delta_\varepsilon(x)\,\mathrm{d}x = 1 \quad \forall\varepsilon \tag{2}$$

and we may form the approximation that,

$$\delta(x) = \lim_{\varepsilon \to 0} \Delta_\varepsilon(x) \tag{3}$$

which can be interpreted as making any of the above approximations $\Delta_\varepsilon(x)$ a very "*tall*-and*thin*" spike with unit area.

In the field of optics and imaging, we are dealing with two dimensional distributions, so it is especially useful to define the *Two Dimensional Dirac Delta Function*, as,

$$\begin{aligned} \delta(x,y) &= 0 \qquad \text{for } x \neq 0 \;\&\; y \neq 0 \\ \iint \delta(x,y)\,\mathrm{d}x\,\mathrm{d}y &= 1 \end{aligned} \tag{4}$$

which is the two dimensional version of the $\delta(x)$ function defined above, and in particular:

$$\delta(x,y) = \delta(x)\,\delta(y). \tag{5}$$

This is the two dimensional analogue of the *impulse* function used in signal processing. In terms of an imaging system, this function can be considered as a single bright spot in the centre of the field of view, for example a single bright star viewed by a telescope.

3.1 Properties of the Dirac Delta Function

Since the *Dirac Delta Function* is used extensively, and has some useful, and slightly perculiar properties, it is worth considering these are this point. For a function $f(x)$, being integrable, then we have that

$$\int_{-\infty}^{\infty} \delta(x)\,f(x)\,\mathrm{d}x = f(0) \tag{6}$$

which is often taken as an alternative definition of the Delta function. This says that integral of any function multiplied by a δ-function located about zero is just the value of the function at zero. This concept can be extended to give the *Shifting Property*, again for a function $f(x)$, giving,

$$\int_{-\infty}^{\infty} \delta(x-a)\,f(x)\,\mathrm{d}x = f(a) \tag{7}$$

where $\delta(x-a)$ is just a δ-function located at $x = a$ as shown in figure 2.

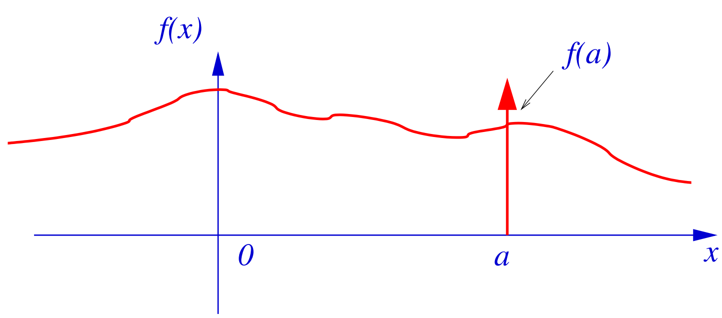

Figure 2: Shifting property of the δ-function.

In two dimensions, for a function $f(x,y)$, we have that,

$$\iint \delta(x-a, y-b)\;f(x,y)\,\mathrm{d}x\,\mathrm{d}y = f(a,b) \tag{8}$$

where $\delta(x-a, y-b)$ is a δ-function located at position a,b. This property is central to the idea of convolution, which is used extensively in image formation theory, and in digital image processing.

The Fourier transform of a Delta function is can be formed by direct integration of the definition of the Fourier transform, and the shift property in equation 6 above. We get that,

$$\mathcal{F}\{\delta(x)\} = \int_{-\infty}^{\infty} \delta(x)\,\exp(-\imath 2\pi u x)\,\mathrm{d}x = \exp(0) = 1 \tag{9}$$

and then by the *Shifting Theorem*, equation 7, we get that,

$$\mathcal{F}\{\delta(x-a)\} = \exp(-\imath 2\pi a u) \tag{10}$$

so that the Fourier transform of a shifted Delta Function is given by a phase ramp. It should be noted that the modulus squared of equation 10 is

$$|\mathcal{F}\{\delta(x-a)\}|^2 = |\exp(-\imath 2\pi a u)|^2 = 1$$

saying that the power spectrum a *Delta Function* is a constant independent of its location in real space.

Now noting that the Fourier transform is a linear operation, then if we consider two *Delta Function* located at $\pm a$, then from equation 10 the Fourier transform gives,

$$\mathcal{F}\{\delta(x-a)+\delta(x+a)\} = \exp(-\imath 2\pi a u) + \exp(\imath 2\pi a u) = 2\cos(2\pi a u) \tag{11}$$

while if we have the *Delta Function* at $x=-a$ as negative, then we also have that,

$$\mathcal{F}\{\delta(x-a)-\delta(x+a)\} = \exp(-\imath 2\pi a u) - \exp(\imath 2\pi a u) = -2\imath\sin(2\pi a u). \tag{12}$$

Noting the relations between forward and inverse Fourier transform we then get the two useful results that

$$\mathcal{F}\{\cos(2\pi a x)\} = \frac{1}{2}\left[\delta(u-a)+\delta(u+a)\right] \tag{13}$$

and that

$$\mathcal{F}\{\sin(2\pi a x)\} = \frac{1}{2\imath}\left[\delta(u-a)-\delta(u+a)\right] \tag{14}$$

So that the Fourier transform of a cosine or sine function consists of a single frequency given by the period of the cosine or sine function as would be expected.

3.2 The Infinite Comb

If we have an infinite series of Delta functions at a regular spacing of Δx, this is described as an *Infinite Comb*. The the expression for a *Comb* is given by,

$$\mathrm{Comb}_{\Delta x}(x) = \sum_{i=-\infty}^{\infty} \delta(x - i\Delta x). \tag{15}$$

A short section of such a *Comb* is shown in figure 3.

Since the Fourier transform is a linear operation then the Fourier transform of the infinite comb is the sum of the Fourier transforms of shifted Delta functions, which from equation (29) gives,

$$\mathcal{F}\{\mathrm{Comb}_{\Delta x}(x)\} = \sum_{i=-\infty}^{\infty} \exp(-\imath 2\pi i \Delta x u) \tag{16}$$

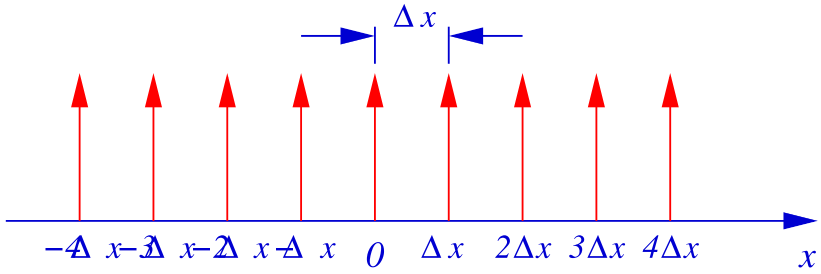

Figure 3: Infinite Comb with separation Δx

Now the exponential term,

$$\exp(-\imath 2\pi i \Delta x u) = 1 \quad \text{when } 2\pi\Delta x u = 2\pi n$$

so that:

$$\begin{aligned}\sum_{i=-\infty}^{\infty} \exp(-\imath 2\pi i \Delta x u) &\rightarrow \infty \quad \text{when } u = \frac{n}{\Delta x} \\ &= 0 \quad \text{else}\end{aligned}$$

which is an infinite series of δ-function at a separation of $\Delta u = \frac{1}{\Delta x}$. So that an *Infinite Comb* Fourier transforms to another *Infinite Comb* or reciprocal spacing,

$$\mathcal{F}\{\text{Comb}_{\Delta x}(x)\} = \text{Comb}_{\Delta u}(u) \quad \text{with } \Delta u = \frac{1}{\Delta x} \tag{17}$$

This is an important result used in Sampling Theory in the DIGITAL IMAGE ANALYSIS and IMAGE PROCESSING I courses.

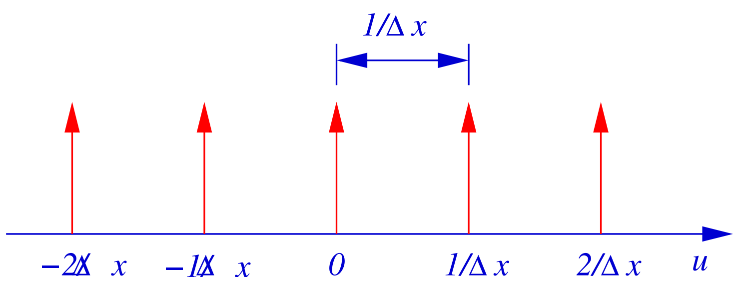

Figure 4: Fourier Transform of comb function.