Topic 3: Operation of Simple Lens

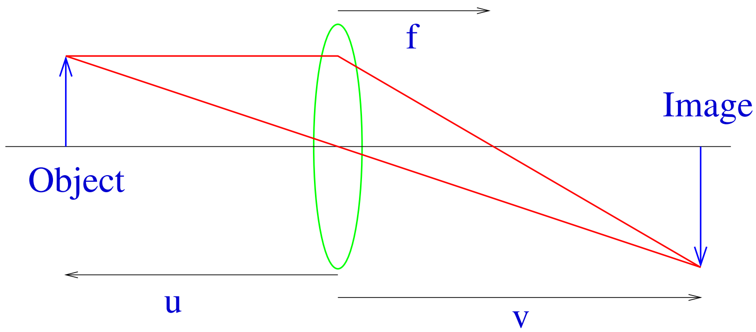
Aim: Covers imaging of simple lens using Fresnel Diffraction, resolution limits and basics of aberrations theory.

Contents:

1. Phase and Pupil Functions of a lens
2. Image of Axial Point
3. Example of Round Lens
4. Diffraction limit of lens
5. Defocus
6. The Strehl Limit
7. Other Aberrations

Ray Model

Simple Ray Optics gives



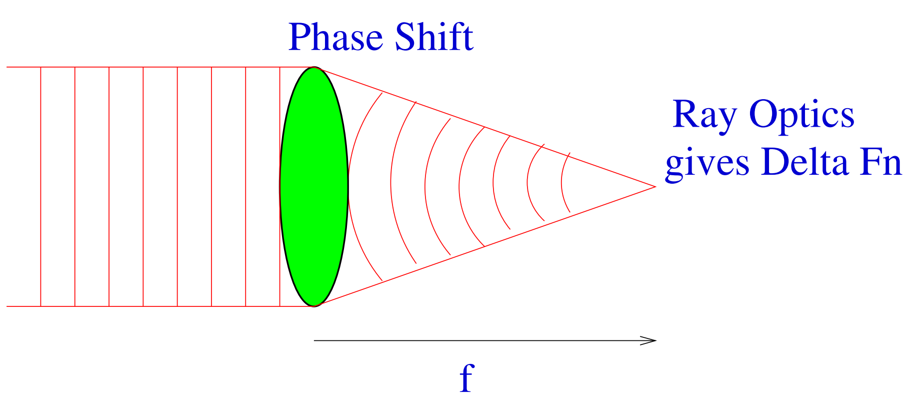
Imaging properties of

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

The focal length is given by

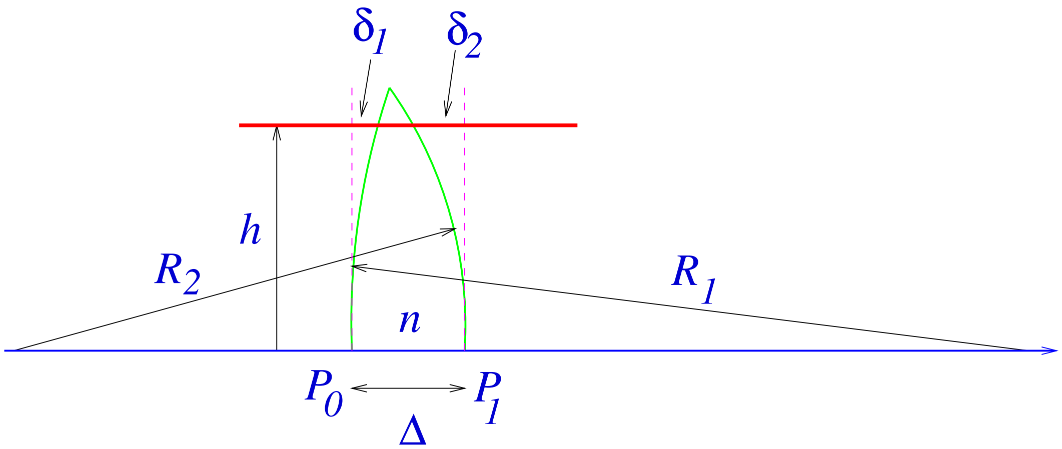
$$\frac{1}{f} = (n-1)\left[\frac{1}{R_1} + \frac{1}{R_2}\right]$$

For Infinite object



Lens introduces a path length difference, or PHASE SHIFT.

Phase Function of a Lens



With **NO** lens, Phase Shift between , $P_0 \rightarrow P_1$ is

$$\Phi = \kappa\Delta \quad \text{where } \kappa = \frac{2\pi}{\lambda}$$

with lens in place, at distance h from optical,

$$\Phi = \kappa\left(\underbrace{\delta_1 + \delta_2}_{\text{Air}} + n(\underbrace{\Delta - \delta_1 - \delta_2}_{\text{Glass}})\right)$$

which can be arranged to give

$$\Phi = \kappa n\Delta - \kappa(n-1)(\delta_1 + \delta_2)$$

where δ_1 and δ_2 depend on h, the ray height.

Parabolic Approximation

Lens surfaces are **Spherical**, but:

If R_1 & $R_2 \gg h$, take parabolic approximation

$$\delta_1 = \frac{h^2}{2R_1} \quad \text{and} \quad \delta_2 = \frac{h^2}{2R_2}$$

So that

$$\Phi(h) = \kappa \Delta n + \frac{\kappa h^2}{2}(n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

substituting for focal length, we get

$$\Phi(h) = \kappa \Delta n - \frac{\kappa h^2}{2f}$$

So in 2-dimensions, $h^2 = x^2 + y^2$, so that

$$\Phi(x, y; f) = \kappa \Delta n - \kappa \frac{(x^2 + y^2)}{2f}$$

the phase function of the lens.

Note: In many cases $\kappa \Delta n$ can be ignored since absolute phase not usually important.

The difference between Parabolic and Spherical surfaces will be considered in the next lecture.

Pupil Function

The pupil function is used to define the physical size and shape of the lens.

$$p(x,y) \Rightarrow \text{Shape of lens}$$

so the total effect of the lens of focal length is

$$p(x,y)\exp\left(\imath\Phi(x,y;f)\right)$$

For a circular lens of radius a,

$$\begin{aligned} p(x,y) &= 1 \quad \text{if } x^2+y^2 \le a^2 \\ &= 0 \quad \text{else} \end{aligned}$$

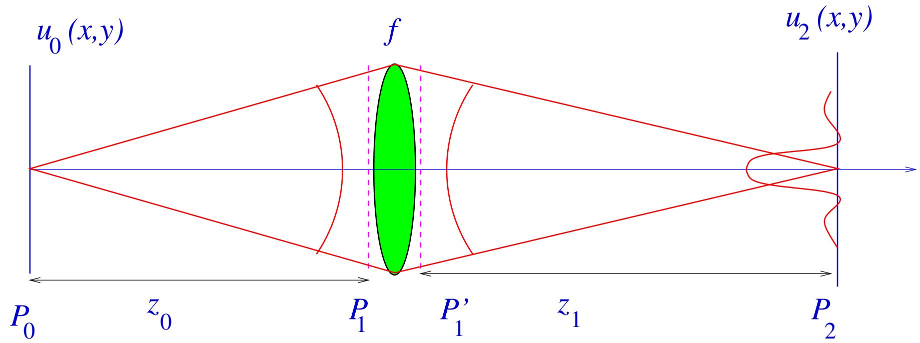
The circular lens is the most common, but all the following results apply equally well for other shapes.

Pupil Function of a simple lens is real and positive, but it will be used later to include aberrations, and will become complex.

Fresnel Image of Axial Point

We will now consider the imaging of an axial point using the Fresnel propagation equations from the last lecture.

Consider the system.



In plane P_0 we have that

$$u_0(x,y) = A_0 \delta(x,y)$$

so in plane P_1 a distance z_0, we have that,

$$\begin{aligned} u(x,y;z_0) &= h(x,y;z_0) \odot A_0 \delta(x,y) \\ &= A_0 h(x,y;z_0) \end{aligned}$$

where $h(x,y;z)$ is the free space impulse response function.

If we now assume that the lens is **thin**, then there is no diffraction between planes, P_1 and P'_1, so in P'_1 we have

$$u'(x,y;z_0) = A_0 h(x,y;z_0) p(x,y) \exp(\imath \Phi(x,y;f))$$

so finally in P_2 a further distance z_1 we have that

$$\begin{aligned} & u_2(x,y) = u'(x,y;z_0) \odot h(x,y;z_1) \\ = \; & A_0 h(x,y;z_0) p(x,y) \exp(\imath \Phi(x,y;f)) \odot h(x,y;z_1) \end{aligned}$$

Take the Fresnel approximation, where

$$h(x,y;z) = \frac{\exp(\imath\kappa z)}{\imath\lambda z}\exp\left(\imath\frac{\kappa}{2z}(x^2+y^2)\right)$$

and

$$\Phi(x,y;f) = \kappa n\Delta - \frac{\kappa}{2f}(x^2+y^2)$$

so we get that

$$u(x,y;z_0) = \frac{A_0\exp(\imath\kappa z_0)}{\imath\lambda z_0}\exp\left(\imath\frac{\kappa}{2z_0}(x^2+y^2)\right)$$

then by more substitution,

$$\begin{aligned} u'(x,y;z_0) &= \frac{A_0\exp\left(\imath\kappa(z_0+n\Delta)\right)}{\imath\lambda z_0}p(x,y) \\ &\quad \exp\left(\imath\frac{\kappa}{2}\left(\frac{1}{z_0}-\frac{1}{f}\right)(x^2+y^2)\right) \end{aligned}$$

Finally!, we can substitute and expand to get

$$\begin{aligned} u_2(x,y) &= \overbrace{\frac{A_0}{\lambda^2 z_0 z_1}\exp\left(\imath\kappa(z_0+z_1+n\Delta)\right)}^{1} \\ &\quad \overbrace{\exp\left(\imath\frac{\kappa}{2z_1}(x^2+y^2)\right)}^{2} \\ &\quad \iint p(s,t)\overbrace{\exp\left(\imath\frac{\kappa}{2}(s^2+t^2)\left(\frac{1}{z_0}+\frac{1}{z_1}-\frac{1}{f}\right)\right)}^{3} \\ &\quad \underbrace{\exp\left(-\imath\frac{\kappa}{z_1}(sx+ty)\right)}_{4}\mathrm{d}s\mathrm{d}t \end{aligned}$$

Look at the term:

1. Constant, amplitude gives absolute brightness, phase not measurable.
2. Quadratic phase term in plane P_2, no effect on intensity, and usually ignored.
3. Quadratic phase term across the Pupil, depends on both z_0 and z_1.
4. Scaled Fourier Transform of Pupil Function.

If we select the location of plane P_2 such that

$$\frac{1}{z_0} + \frac{1}{z_1} - \frac{1}{f} = 0$$

Note same expression as Ray Optics, then we can write

$$\begin{aligned} u_2(x,y) &= B_0 \exp\left(\imath \frac{\kappa}{2z_1}(x^2+y^2)\right) \\ & \iint p(s,t) \exp\left(-\imath \frac{\kappa}{z_1}(sx+ty)\right) \mathrm{d}s\mathrm{d}t \end{aligned}$$

so in plane P_2 we have the scaled Fourier Transform of the Pupil Function, (plus phase term).

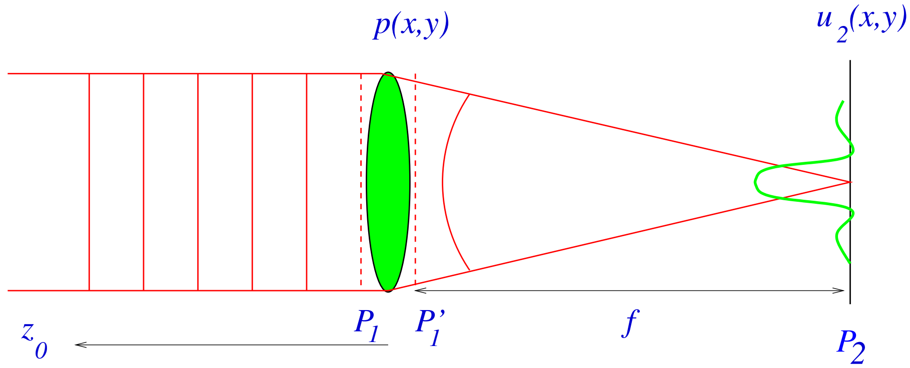
Intensity in plane P_2 is thus

$$\begin{aligned} g(x,y) &= |u_2(x,y)|^2 \\ &= B_0^2 \left| \iint p(s,t) \exp\left(-\imath \frac{\kappa}{z_1}(sx+ty)\right) \mathrm{d}s\mathrm{d}t \right|^2 \end{aligned}$$

Being the scaled power spectrum of the Pupil Function.

Image of a Distant Object

For a distant object $z_0 \to \infty$ and $z_1 \to f$



Then the amplitude in P_2 becomes,

$$u_2(x,y) = B_0 \exp\left(\imath \frac{\kappa}{2f}(x^2+y^2)\right) \iint p(s,t) \exp\left(-\imath \frac{\kappa}{f}(sx+ty)\right) \mathrm{d}s\mathrm{d}t$$

which being the scaled Fourier Transform of the Pupil function.

The intensity is therefore:

$$g(x,y) = B_0^2 \left| \iint p(s,t) \exp\left(-\imath \frac{\kappa}{f}(sx+ty)\right) \mathrm{d}s\mathrm{d}t \right|^2$$

which is known as the **Point Spread Function** of the lens

Key Result

Note on Units: x, y, s, t all have units of length m. Scaler Fourier kernal is:

$$\exp\left(-\imath \frac{2\pi}{\lambda f}(sx+ty)\right)$$

Simple Round Lens

Consider the case of round lens of radius a, amplitude in P_2 then becomes,

$$u_2(x,y) = \hat{B}_0 \iint_{s^2+t^2 \leq a^2} \exp\left(-\imath\frac{\kappa}{f}(sx+ty)\right) \mathrm{d}s\mathrm{d}t$$

the external phase term has been absorbed into $\hat{B}_0$

This can be integrated, using standard results (see Physics 3 Optics notes or tutorial solution), to give:

$$u_2(x,0) = 2\pi\hat{B}_0 a^2 \frac{J_1\left(\frac{\kappa a}{f}x\right)}{\frac{\kappa a}{f}x}$$

where J_1 is the first order Bessel Function.

This is normally normalised so that $u_2(0,0) = 1$, and noting that the output is circularly symmetric, we get that,

$$u_2(x,y) = 2\frac{J_1\left(\frac{\kappa a}{f}r\right)}{\frac{\kappa a}{f}r}$$

where $x^2 + y^2 = r^2$.

The intensity PSF is then given by,

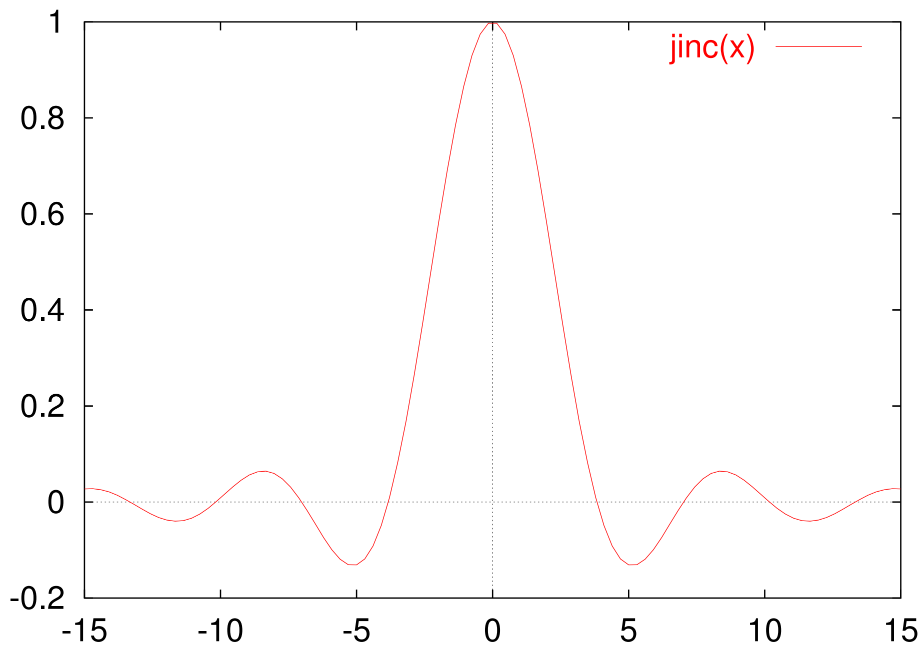
$$g(x,y) = 4\left|\frac{J_1\left(\frac{\kappa a}{f}x\right)}{\frac{\kappa a}{f}x}\right|^2$$

which is known as the Airy Distribution, first derived in 1835.

Shape of jinc

The function

$$\mathrm{jinc}(x) = \frac{2\mathrm{J}_1(x)}{x}$$



Similar shape to the $\mathrm{sinc}()$ function, except

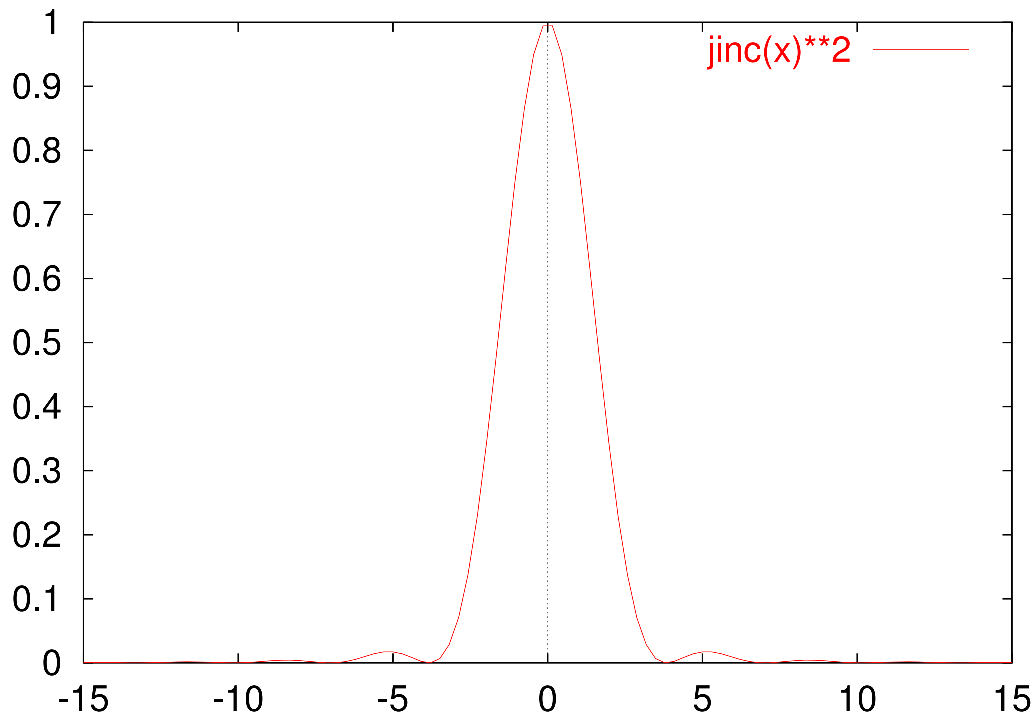
- Zeros at different locations.
- Lower secondary maximas

Zeros of $\mathrm{jinc}()$ located at

x_0	3.832	1.22π
x_0	7.016	2.23π
x_0	10.174	3.24π
x_0	13.324	4.24π

Shape of $\text{jinc}^2()$

The PSF function is the square of the $\text{jinc}()$,



We get 88% of power in the central peak.

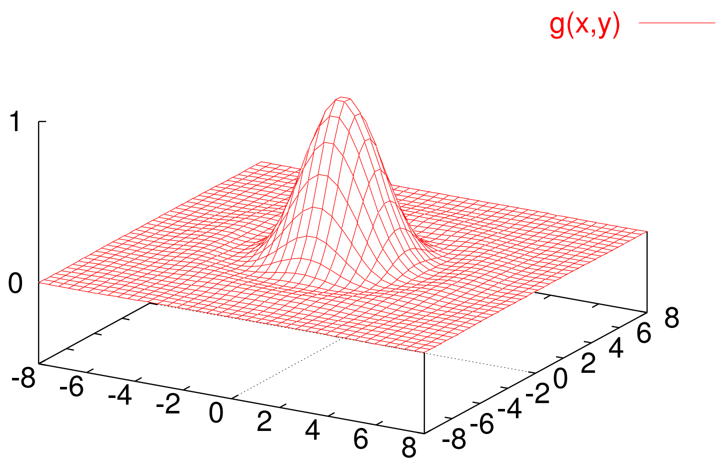
Height of secondary maximas

Order	Location	Height
1	5.136	0.0175
2	8.417	0.0042
3	11.620	0.0016

Shape of PSF

The PSF is circular symmetry, being

$$g(x,y) = \text{jinc}^2\left(\frac{\kappa a}{f} r\right)$$



Radius of first zero occurs at

$$\frac{\kappa a}{f} r_0 = 1.22\pi \quad \Rightarrow \quad r_0 = \frac{0.61\lambda f}{a}$$

Define: $\mathrm{F_{No}}$ as

$$\mathrm{F_{No}} = \frac{f}{2a} = \frac{\text{Focal Length}}{\text{Diameter}}$$

Then the zero of the PSF occur at

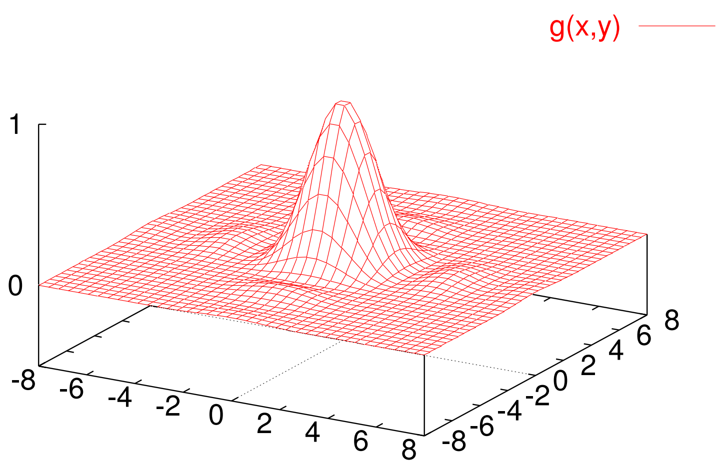
$$\begin{aligned} r_0 &= 1.22\lambda \mathrm{F_{No}} \\ r_1 &= 2.23\lambda \mathrm{F_{No}} \\ r_2 &= 3.24\lambda \mathrm{F_{No}} \end{aligned}$$

All lenses with the same $\mathrm{F_{No}}$ have the same Point Spread Function.

Other Shaped Lenses

This method of calculating the PSF is valid for any Pupil Function, for example a "square" lens of size $2a \times 2a$ will have a PSF

$$g(x,y) = \text{sinc}^2\left(\frac{\kappa a}{f}x\right)\text{sinc}^2\left(\frac{\kappa a}{f}y\right)$$



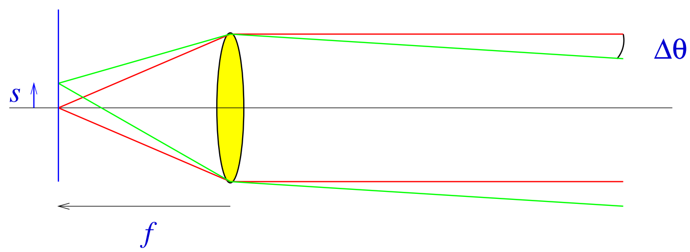
with the first zeros in the x direction at:

$$\frac{\kappa a}{f}x_0 = \pi \quad \Rightarrow \quad x_0 = \frac{0.5\lambda f}{a}$$

See tutorial questions for important case of annular lens and lens with Guassian transmission function.

Diffraction Limit of Lens

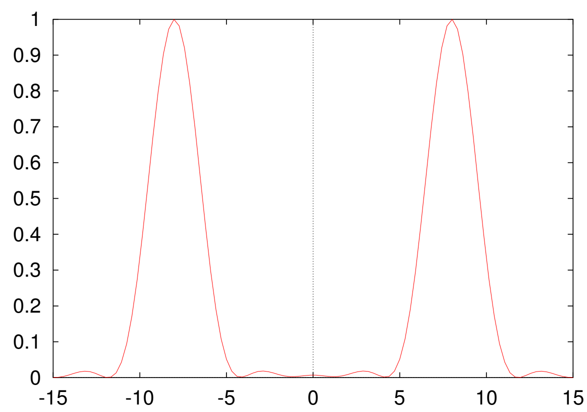
Angular Resolution: Take two point sources (stars) at infinity separated by $\Delta\theta$



$$s = f \tan \Delta\theta \approx f \Delta\theta \quad \text{Small } \Delta\theta$$

For *large* $\Delta\theta$, then

$$s \gg r_0 \quad \text{Width of PSF}$$



Then two stars resolved.

While is

$$\Delta\theta f \ll r_0 \quad \text{See one star}$$

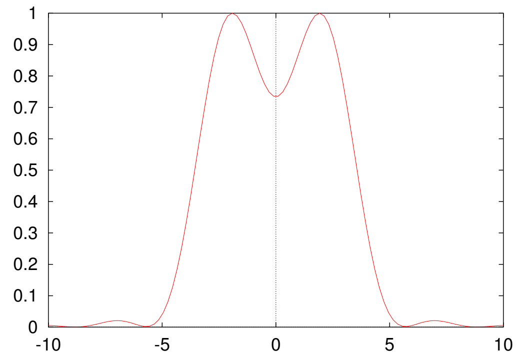
Then star NOT resolved.

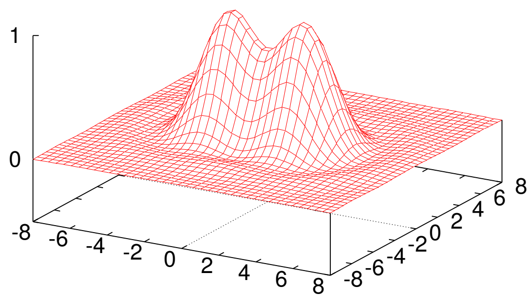
Note: Light from two distant stars, so Intensities will sum.

Rayleigh Limit

Limit when start "just" resolved. For stars of equal brightness, when

$$s = r_0 \rightarrow 27\% \text{ "Dip" between peaks}$$





Stars are said to be "just" resolved.

$$\Delta\theta_0 f = r_0 = \frac{0.61\lambda f}{a}$$

giving that

$$\Delta\theta_0 = \frac{0.61\lambda}{a} = \frac{1.22\lambda}{d}$$

So angular resolution limit of a lens depend ONLY on it diameter.

Key Result

See tutorial questions for other configurations and lenses with different Pupil Functions.

Examples

Telescope: Medium size,

$$d = 10\text{cm} \quad \& \quad \lambda = 550\text{nm} \quad \text{Green}$$

Then resolution limit

$$\Delta\theta_0 = 6.71 \times 10^{-6}\text{Rad} = 1.4'' \text{ of arc}$$

Human Eye: in normal sunlight

$$d = 2\text{mm} \quad \& \quad \lambda = 550\text{nm}$$

Then resolution limit

$$\Delta\theta = 3.3 \times 10^{-4}\text{Rad} \approx 1' \text{ of arc}$$

Actual resolution limit of eye is really limited by the spacing of cones on the retina, and is typically 5×10^{-4} Rad (1 mm divisions on a ruler just resolved at 2 m).

Rule of Thumb

- Resolution of eye 1'→2' of arc.
- Resolution of telescope 1" of arc.

Aside: For all telescopes bigger than 10 cm resolution limited by atmospheric movement, so resolution of 1" of arc is true for all optical earth bound telescopes.

Angular Measures

Many systems still specified in degrees, and fractions of degrees

$$
\begin{aligned}
1^\circ &= \frac{2\pi}{360}\mathrm{Rad} &= 0.0174\mathrm{Rad}\\
1' &= \frac{1}{60}^\circ &= 2.91\times10^{-4}\mathrm{Rad}\\
1'' &= \frac{1}{60}' &= 4.85\times10^{-6}\mathrm{Rad}
\end{aligned}
$$

These measure are still in use on spectrometers, telescopes, astronomical tables and maps and charts.

Defocus of Optical System

Consider point source imaged by a lens

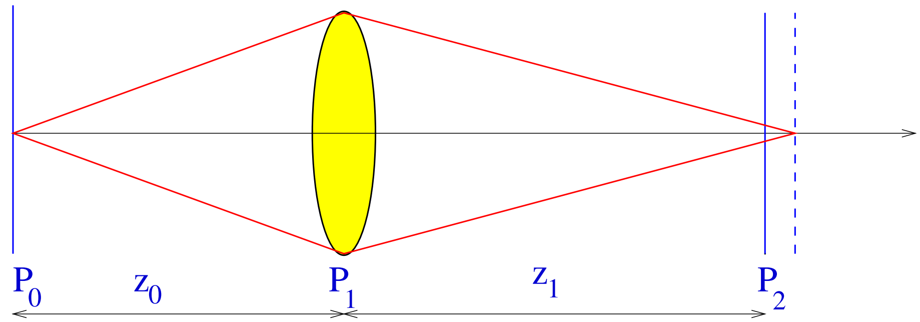


Image is "In Focus" if

$$\frac{1}{z_0} + \frac{1}{z_1} = \frac{1}{f}$$

Move P_2 system is "Defocused".

Define **Defocus Parameter**, D as:

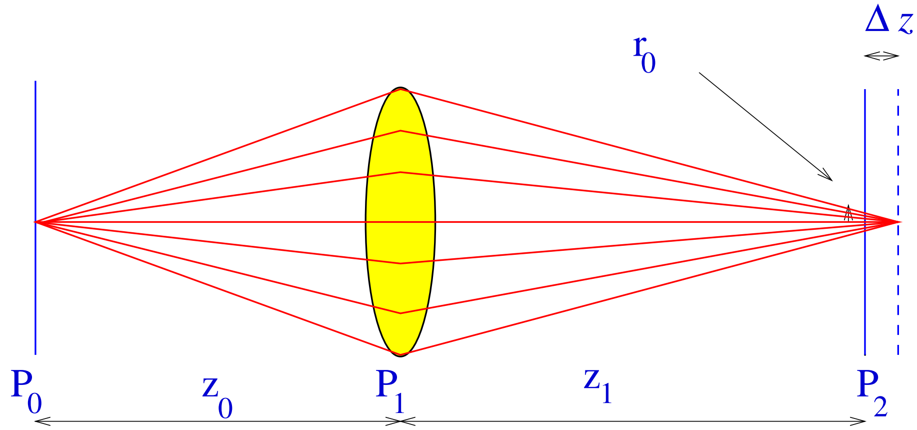
$$D = \frac{1}{z_0} + \frac{1}{z_1} - \frac{1}{f}$$

Then if

$D < 0$ Negative Defocus, (z_1 too large)

$D > 0$ Positive Defocus, (z_1 too small)

Ray Optics: Defocus system by Δz



Radius of the spot is given by similar triangles to be

$$r_0 = \frac{\Delta z d}{2 z_1}$$

where the lens is of diameter d. So larger defocus, large PSF. OK for **Large Defocus**

Scalar Theory: From previous, if $D \neq 0$, then in P_2,

$$u_2(x,y) = \hat{B}_0 \iint \overbrace{p(s,t)\exp\left(\imath\frac{\kappa}{2}D(s^2+t^2)\right)}^{\text{EffectivePupilFunction}} \exp\left(-\imath\frac{\kappa}{z_1}(sx+ty)\right)\mathrm{d}s\mathrm{d}t$$

Which is the Fourier Transform of the "Effective Pupil Function",

$$q(x,y) = p(x,y)\exp\left(\imath\frac{\kappa}{2}D(x^2+y^2)\right)$$

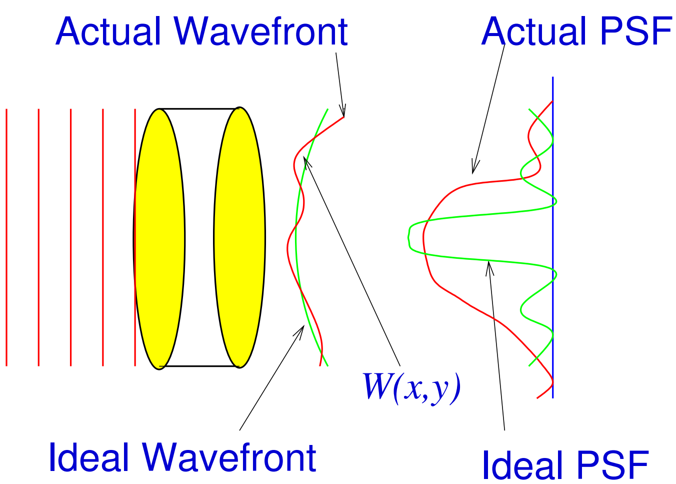
Pupil function goes complex under defocus.

Note: Pupil function is product so PSF is a convolution, which will be "wider" than ideal focus.

Wavefront Aberration

To get ideal PSF (sharp focus), we need Parabolic Wave front behind the lens.

Actual wavefront may vary from this ideal.



Define Wavefront Aberration Function as deviation from ideal parabolic wavefront.

System Pupil Function then becomes,

$$q(x,y) = p(x,y)\exp\left(\imath\kappa W(x,y)\right)$$

where

$W(u,v)$ is Wavefront Aberration Function

The Effective Pupil Function is now Complex, with the PSF given by

$$g(x,y) = B_0^2 \left| \iint q(s,t)\exp\left(-\imath\frac{\kappa}{z_1}(sx+ty)\right)\mathrm{d}s\mathrm{d}t\right|^2$$

This is a general method of dealing with all types of aberrations.

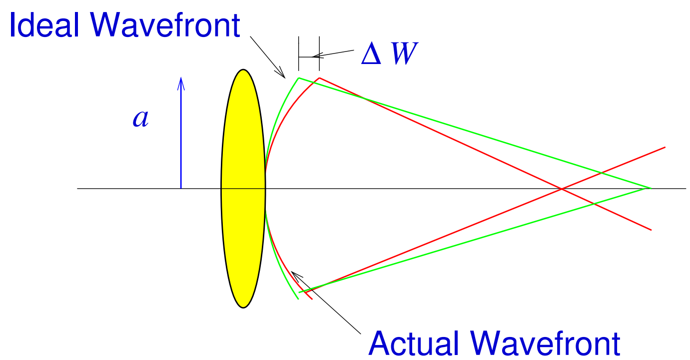
Defocus as an Aberration

Under defocus, the wavefront aberration is

$$W(x,y) = \frac{D}{2}(x^2 + y^2)$$

Measure the Defocus as the extent of the wavefront aberration at the edge of the lens, at

$$x^2 + y^2 = a^2$$



Denote wavefront aberration at edge by ΔW, so wavefront aberration is:

$$W(x,y) = \Delta W \left(\frac{x^2 + y^2}{a^2}\right)$$

so

$$D = \frac{2\Delta W}{a^2}$$

No easy solutions for PSF under defocus.

Strehl Limit

For **small** phase shifts, the PSF retains it $|\mathrm{J}(r)/r|^2$ shape but

- Zero do not move
- Peak value drops
- Subsidiary maximas rise

Define the **Strehl Limit** when central peak drops to 80% of ideal. This occurs when phase difference

$$|\Delta\Phi(r)| \leq \frac{\pi}{4}$$

over the whole pupil function.

Key Result

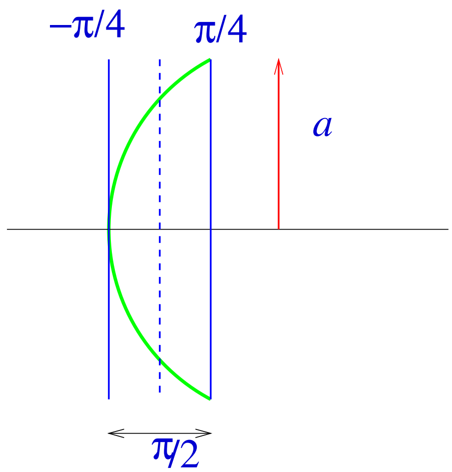
Systems that obey the Strehl limit are have "good" imaging properties, and is the standard design criteria for most good optical systems.

In terms of the Wavefront Aberration function,

$$\Delta\Phi(x,y) = \kappa W(x,y)$$

Strehl Limit for Defocus

For defocus,



The Strehl Limit is that

$$\Delta\Phi_{\max} \leq \frac{\pi}{2}$$

Max phase error occurs at $r = a$, so

$$\Delta\Phi_{\max} = \kappa\Delta W = \frac{2\pi}{\lambda}\Delta W$$

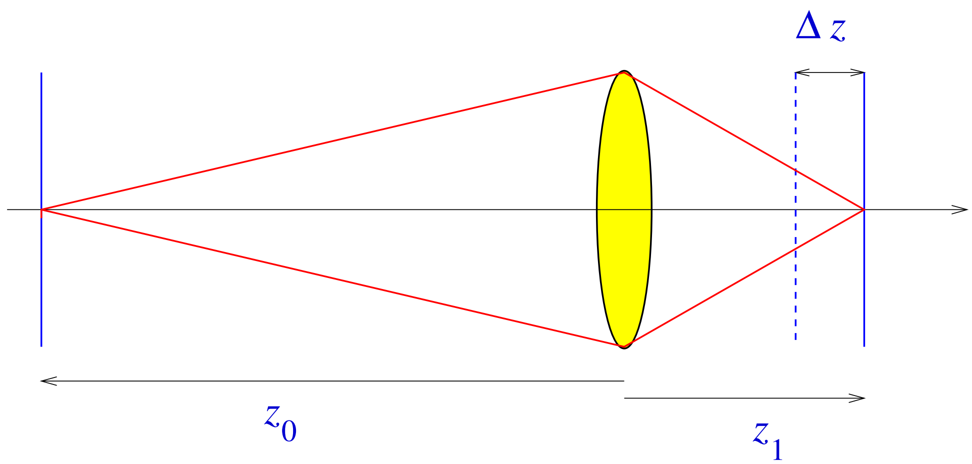
So the Stehl limit for defocus is

$$\Delta W < \frac{\lambda}{4}$$

This is equivalent to a max Optical Path Difference of $\lambda/4$ over the aperture

Object at a Finite Distance

Image is sharp focus with Image distance z_1



Move image plane to $z_1 - \Delta z$, ($\Delta z \ll z_1$)

$$D = \frac{1}{z_0} - \frac{1}{f} + \frac{1}{z_1 - \Delta z} = \frac{1}{z_1 - \Delta z} - \frac{1}{z_1} \approx \frac{\Delta z}{z_1^2}$$

This gives that

$$\Delta W = \frac{1}{2} D a^2 = \frac{\Delta z}{2} \left(\frac{a}{z_1} \right)^2 < \frac{\lambda}{4}$$

so the Strehl Limit gives that

$$\Delta z \leq \frac{1}{2} \left(\frac{z_1}{a} \right)^2 \lambda$$

Special case when object at infinity, $z_1 \to f$

$$\Delta z \leq \frac{1}{2} \left(\frac{f}{a} \right)^2 \lambda = 2 \mathrm{F_{No}}^2 \lambda$$

Example:
Pocket camera, with $f = 35\mathrm{mm}$ and $\mathrm{F_{No}} = 3.5$, $\Delta z = 13.5\,\mu\mathrm{m}$ (about half the thickness of a human hair.)

Other Aberrations

For On-axis points, system is cylinderically symmetric, to that

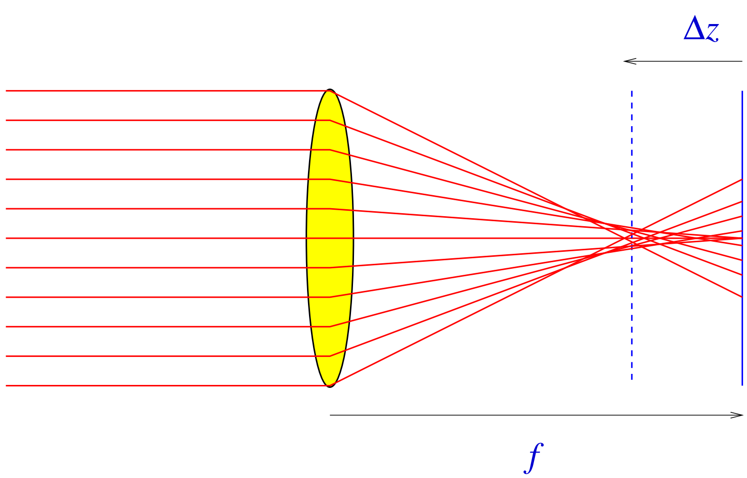
$$W(x,y) \quad \text{Even powers or } r$$

Taking terms to r^4, we get

$$W(x,y) = \frac{1}{8} S_1 \frac{(x^2+y^2)^2}{a^4} + \Delta W \frac{(x^2+y^2)}{a^2}$$

The S_1 term is known as Spherical Aberration.

Physical Explanation: Lens surfaces are Spherical, NOT parabolic so outer rays focused "short"



Should be able to "improve" PSF by moving the image plane short of the ideal (paraxial) focus.

Strehl Limit for Spherical Aberration

1) **No Defocus**:

$$W(r) = \frac{1}{8} S_1 \left(\frac{r}{a}\right)^4$$

Phase error then equals

$$\Delta\Phi(r) = \kappa W(r) = \frac{\pi}{4\lambda} S_1 \left(\frac{r}{a}\right)^4$$

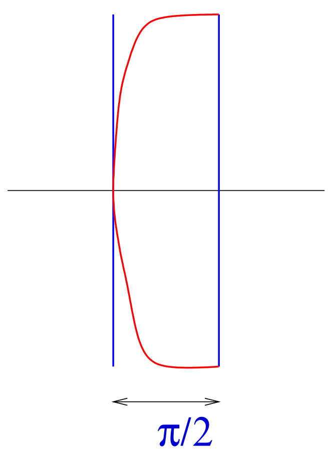
As for defocus, Strehl limit is that

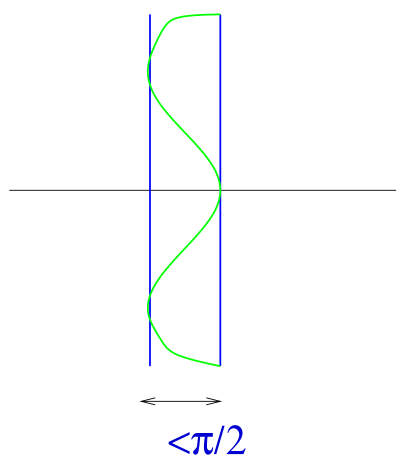
$$\Delta\Phi_{\max} \leq \frac{\pi}{2}$$

so that the limit for Spherical Aberration is

$$S_1 \leq 2\lambda$$

2) **With Defocus:**
Able to "cancel" some of the Spherical Aberration with defocus





cont: We can find optimal defocus by least squares minimisation of

$$\int_0^a |rW(r)|^2 \,\mathrm{d}r$$

which "can-be-shown" to give the best PSF at

$$\Delta W = -\frac{7}{72} S_1$$

Minimum and maximum of phase function occurs at

$$r = \sqrt{\frac{7}{18}} a \quad \& \quad r = a$$

This gives a Strehl Limit of

$$S_1 \leq 5.36\lambda$$

Which is more than twice the limit if there is no defocus.

Aside: If viewed from a purely ray optics model, we get that

$$\Delta W = -\frac{1}{8} S_1$$

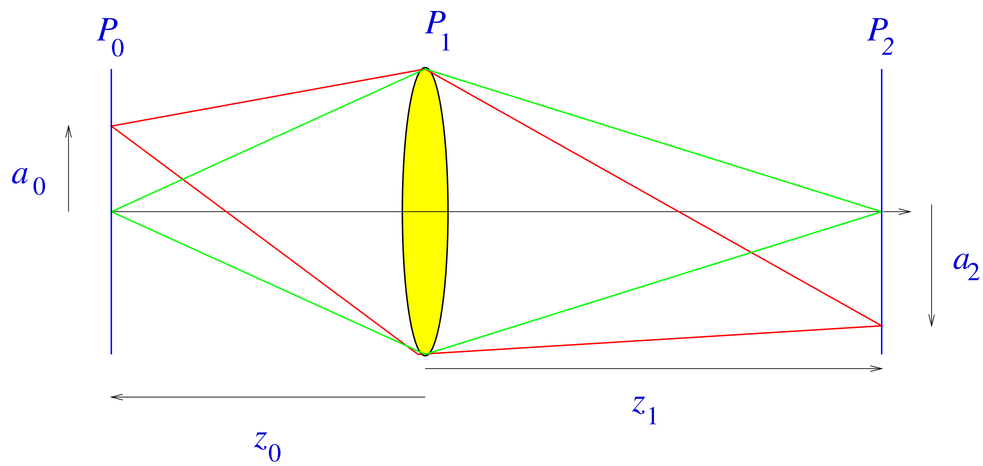
and the Strehl Limit for Spherical Aberration is

$$S_1 \leq 7.6\lambda$$

which is a similar result.

Off Axis Points

1) **Ideal Case:**
PSF moves linearly and does not change shape



System is said to be **Space Invariant**, and

$$a_2 = -\frac{z_1}{z_0} a_0 = -M a_0$$

Where $M = z_1/z_2$ is the magnification of the system.

If the Object is a δ-function at (a_0, b_0) then in plane P_2 we get amplitude

$$u_2(x - a_2, y - b_2)$$

where $u_2(x, y)$ is amplitude for δ-function on axis and

$$a_2 = -\frac{z_1}{z_0} a_0 \quad \& \quad b_2 = -\frac{z_1}{z_0} b_0$$

Practical Case

Shape of Pupil function will change,

- On-axis $P(x,y)$ is circular.
- Off-axis $P(x,y)$ is an ellipse.

So the PSF will change.

Compound Lenses: Effective shape of Pupil Function will change much more rapidly due to three dimensional nature of lens.

Result known as **Vignetting**. Major problem with very wide angle lenses which leads to edges of image being dull.

Off-Axis Aberrations

No cylinderical symmetry, so much more complicated aberration problems.

Range of aberrations that depend on the object location, full form for First Order aberrations become,

$$
\begin{aligned}
W(x,y;\eta) &= \frac{1}{2}S_0\left(\frac{r^2}{a^2}\right) + \frac{1}{8}S_1\left(\frac{r^4}{a^4}\right) + \frac{1}{2}S_2\frac{y\,r^2}{a^3}\eta + \\
& \frac{1}{2}S_3\frac{y^2}{a^2}\eta^2 + \frac{1}{4}(S_3+S_4)\left(\frac{r^2}{a^2}\right)\eta^2 + \frac{1}{2}S_5\frac{y}{a}\eta^3
\end{aligned}
$$

where the terms are

1. η Off-Axis angle as fraction of maximum
2. S_0 Defocus, same a $2\Delta W$
3. S_1 Spherical Aberration
4. S_2 Coma
5. S_3 Astigmatism
6. S_4 Field Curvature
7. S_5 Distortion.

Shape of PSF under these aberrations is difficult to calculate.