Topic 2: Scalar Diffraction

Aim: Covers Scalar Diffraction theory to derive Rayleigh-Sommerfled diffraction. Take approximations to get Kirchhoff and Fresnel approximations.

Contents:

1. Preliminary Theory.
2. General propagation between two planes.
3. Kirchhoff Diffraction
4. Fresnel Diffraction
5. Summary

Scalar Wave Theory

Light is really a vector electomagnetic wave with $\mathbf{E}$ and $\mathbf{B}$ field linked by Maxwell's Equations.

Full solution only possible in limited cases, so we have to make assumptions and approximations.

Assume:
Light field can be approximated by a complex scalar potential. (amplitude)

Valid for:
Apertures and objects $\gg \lambda$, (most optical systems).

NOT Valid for:
Very small apertures, Fibre Optics, Planar Wave Guides, Ignores Polarisation.

Also Assume:
Scalar potential is a *linear* super-position of monochromatic components. So theory only valid for Linear Systems, (refractive index does not depend on wavelength).

Scalar Potentials

For light in free space the $\mathbf{E}$ and $\mathbf{B}$ field are linked by

$$\begin{aligned} \nabla.\mathbf{E} &= 0 \\ \nabla.\mathbf{B} &= 0 \\ \nabla \wedge \mathbf{B} &= \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \\ \nabla \wedge \mathbf{E} &= \Leftrightarrow \frac{\partial \mathbf{B}}{\partial t} \end{aligned}$$

which, for free space, results in the "Wave Equation" given by

$$\nabla^2 \mathbf{E} \Leftrightarrow \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

Assume light field represented by scalar potential $\Phi(\mathbf{r}, t)$ which MUST also obey the "Wave Equation", so:

$$\nabla^2 \Phi \Leftrightarrow \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = 0$$

Write the Component of scalar potential with angular frequency ω as

$$\Phi(\mathbf{r}, t) = u(\mathbf{r}) \exp(\imath \omega t)$$

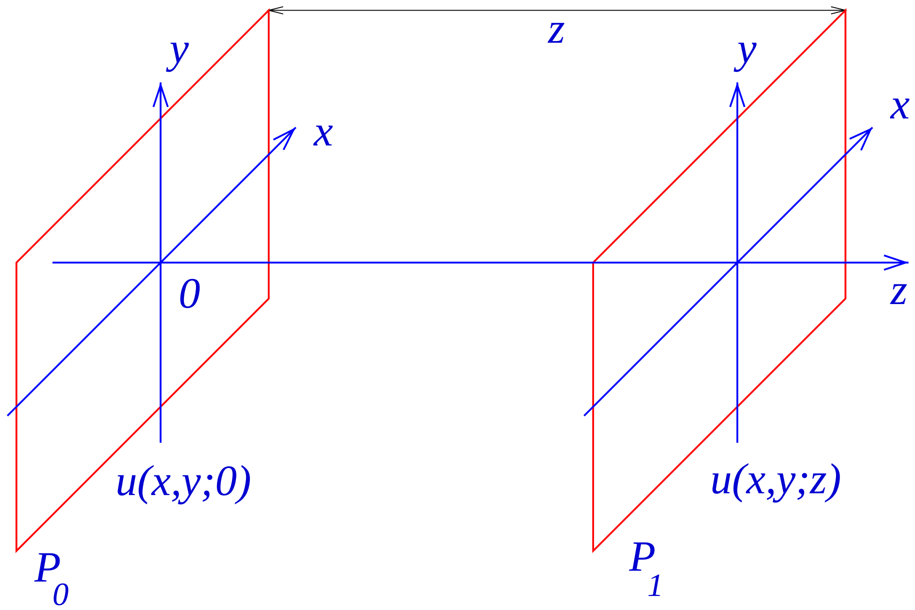
then substituting for Φ we get that,

$$[\nabla^2 + \kappa^2] u(\mathbf{r}) = 0$$

where $\kappa = 2\pi/\lambda$ or *wave number*.

So that $u(\mathbf{r})$ must obey Helmholtz Equation, (starting point for Scalar Wave Theory).

Diffraction Between Planes



In P_0 the **2-D** scalar potential is:

$$u(x,y;0) = u_0(x,y)$$

where in plane P_1 the scalar potential is:

$$u(x,y;z)$$

where the planes are separated by distance z.

Problem: Given $u_0(x,y)$ in plane P_0 we want to calculate in $u(x,y;z)$ is any plane P_1 separated from P_0 by z.

Looking for a **2-D Scalar Solution** to the full wave equation.

Fourier Transform Approach

Take the 2-D Fourier Transform in the plane, wrt x, y. so in plane P_1 we get

$$U(u,v;z) = \iint u(x,y;z)\exp(\Leftrightarrow\imath 2\pi(ux+vy))\,\mathrm{d}x\,\mathrm{d}y$$

since we have that

$$u(x,y;z) = \iint U(u,v;z)\exp(\imath 2\pi(ux+vy))\,\mathrm{d}u\,\mathrm{d}v$$

then if we know $U(u,v;z)$ in any plane, then we can easily find $u(x,y;z)$ the required amplitude distribution.

The amplitude $u(x,y;z)$ is a Linear Combination of term of

$$U(u,v;z)\exp(\imath 2\pi(ux+vy))$$

These terms are **Orthogonal** (from Fourier Theory),

So **each** of these terms **must** individually obey the Helmholtz Equation.

Note: the term:

$$U(u,v;z)\exp(\imath 2\pi(ux+vy))$$

in a **Plane Wave** with Amplitude $U(u,v;z)$ and Direction $(u/\kappa, v/\kappa)$, where $\kappa = 2\pi/\lambda$.

Propagation of a Plane Wave

From Helmholtz Equation we have that

$$[\nabla^2 + \kappa^2] U(u,v;z) \exp\left(\imath 2\pi(ux + vy)\right) = 0$$

Noting that the $\exp()$ terms cancel, then we get that

$$\frac{\partial^2 U(u,v;z)}{\partial z^2} + 4\pi^2 \left(\frac{1}{\lambda^2} \Leftrightarrow u^2 \Leftrightarrow v^2 \right) U(u,v;z) = 0$$

so letting

$$\gamma^2 = \frac{1}{\lambda^2} \Leftrightarrow u^2 \Leftrightarrow v^2$$

then we get the relation that

$$\frac{\partial^2 U}{\partial z^2} + (2\pi\gamma)^2 U = 0 \quad \forall u, v$$

With the condition that

$$U(u,v;0) = U_0(u,v) = \mathcal{F}\,\{u_0(x,y)\}$$

we get the solution that

$$U(u,v;z) = U_0(u,v)\exp(\imath 2\pi\gamma z)$$

This tells us how each component of the Fourier Transform propagates between plane P_0 and plane P_1, so:

$$u(x,y;z) = \iint U_0(u,v)\exp(\imath 2\pi\gamma z)\exp\left(\imath 2\pi(ux + vy)\right) \mathrm{d}u\,\mathrm{d}v$$

which is a general solution to the propagation problems valid for **all** z.

Free Space Propagation Function

Each Fourier Component (or Spatial Frequency) propagates as:

$$U(u,v;z) = U_0(u,v)\exp(\imath 2\pi\gamma z)$$

Define: *Free Space Propagation Function*

$$H(u,v;z) = \exp(\imath\pi\gamma z)$$

so we can write:

$$U(u,v;z) = U_0(u,v)H(u,v;z)$$

Look at the form of $H(u,v;z)$.

$$u^2+v^2+\gamma^2 = \frac{1}{\lambda^2}$$

Case 1: If $u^2+v^2 \leq 1/\lambda^2$ then γ is **REAL**

$$H(u,v) \quad \text{Phase Shift}$$

so **all** spatial frequency components passed with Phase Shift.

Case 2: If $u^2+v^2 > 1/\lambda^2$ them γ is **IMAGINARY**

$$H(u,v) = \exp(\Leftrightarrow 2\pi|\gamma|z)$$

So Fourier Components of $U_0(u,v)$ with $u^2+v^2 > 1/\lambda^2$ decay with Negative Exponential. (evanescent wave)

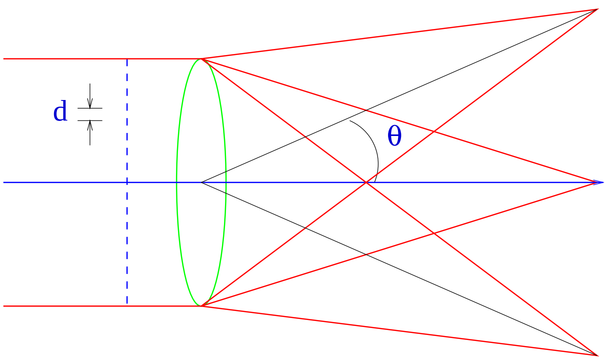
Frequency Limit for Propagation

In plane P_1 where $z \gg \lambda$ then the negative exponential decay will remove high frequency components.

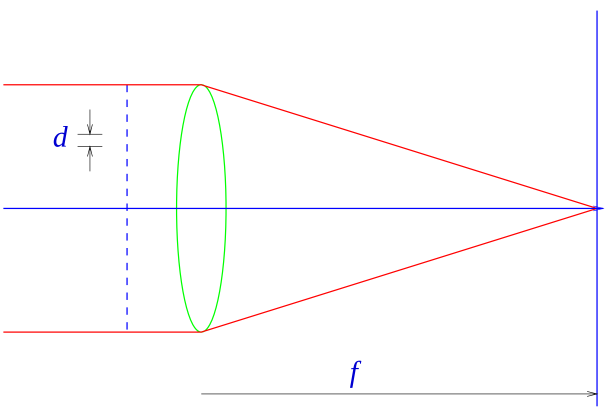
$$U(u,v;z) = 0 \quad u^2 + v^2 > 1/\lambda^2$$

so Fourier Transform of $u(x,y;z)$ is of limited extent.

Maximum spatial frequency when $u = 1/\lambda$, this corresponds to a grating with period λ.



$$d \sin\theta = n\lambda \quad \text{for } d > \lambda$$



No diffraction when $d < \lambda$. Information not transferred to plane P_1.

Convolution Relation

We have that

$$U(u,v;z) = H(u,v;z)\,U_0(u,v;z)$$

so by the Convolution Theorem, we have that

$$u(x,y;z) = h(x,y;z) \odot u_0(x,y;z)$$

With $u(x,y;z)$ the distribution in P_1 due to $u_0(x,y;z)$ in P_0.

We then have that

$$h(x,y;z) = \iint_{u^2+v^2<1/\lambda^2} \exp(\imath 2\pi\gamma z)\exp(\imath 2\pi(ux+vy))\,\mathrm{d}u\mathrm{d}v$$

where

$$\gamma = \sqrt{\frac{1}{\lambda^2} \Leftrightarrow u^2 + v^2}$$

"It-Can-Be-Shown" (with considerable difficulty), that

$$h(x,y;z) = \Leftrightarrow \frac{2\pi}{\lambda^2}\frac{\partial}{\partial z}\left(\frac{\exp(\imath\kappa r)}{\kappa r}\right)$$

where be have that

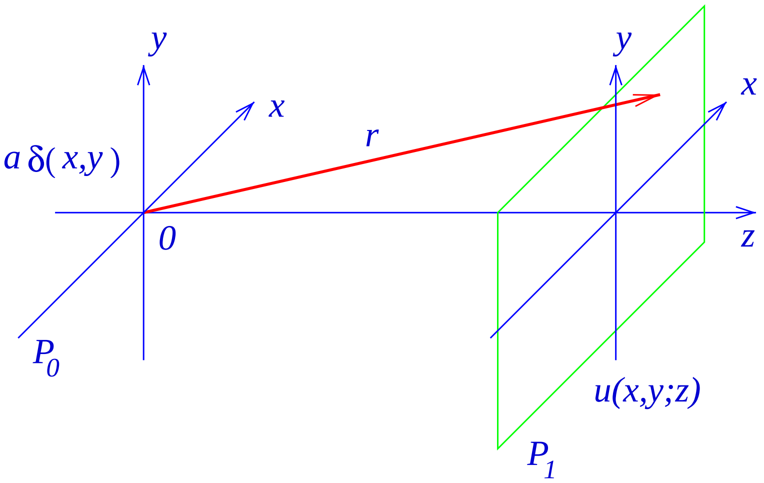
$$r^2 = x^2 + y^2 + z^2 \quad \text{and} \quad \kappa = \frac{2\pi}{\lambda}$$

we therefore get that

$$h(x,y;z) = \frac{1}{\lambda}\left[\frac{1}{\kappa r} \Leftrightarrow \imath\right]\left(\frac{z}{r}\right)\frac{\exp(\imath\kappa r)}{r}$$

which is known as "The Impulse Response Function for Free Space Propagation"

Point Object



In P_0 we have a Delta Function, so:

$$u_0(x,y) = a\delta(x,y)$$

So in plane P_1 we have

$$\begin{aligned} u(x,y;z) &= a\,h(x,y;z) \\ &= \frac{a}{\lambda}\left[\frac{1}{\kappa r} \Leftrightarrow \imath\right]\left(\frac{z}{r}\right)\frac{\exp(\imath\kappa r)}{r} \end{aligned}$$

where r is the distance from:

$$(0,0;0) \Rightarrow (x,y;z)$$

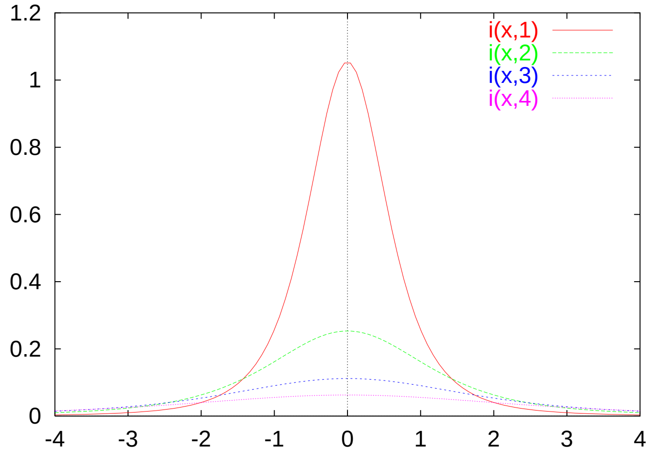
So the intensity in plane P_1 is given by:

$$i(x,y;z) = \frac{b^2}{r^2}\left[\frac{1}{\kappa^2 r^2} + 1\right]\left(\frac{z}{r}\right)^2$$

where the λ^2 has been incorperated into the constant b.

Shape of Function

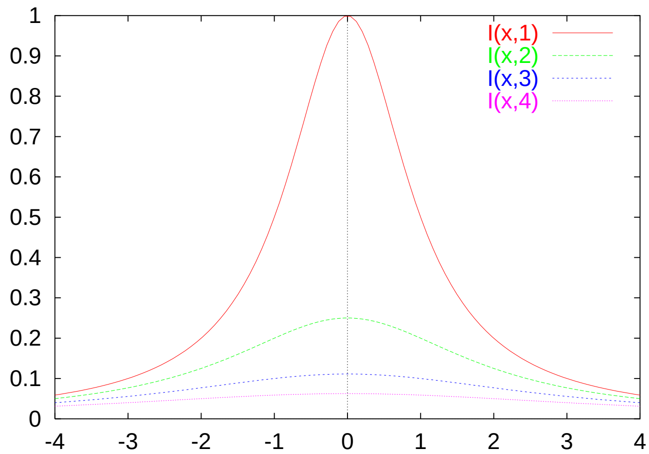
The shape of $i(x,0;z)$ is shown below:



for $z = \lambda, 2\lambda, 3\lambda, 4\lambda$, x-scale in λs.

Compare with a isolated 3-D point source, spherical expanding waves, so intensity in plane at distance z of:

$$I(x,y;z) = \frac{b^2}{r^2}$$



Note We have a 2-D Delta Function, (whole in a screen), and NOT an 3-D point source.

Full Expression

The full convolution expression is

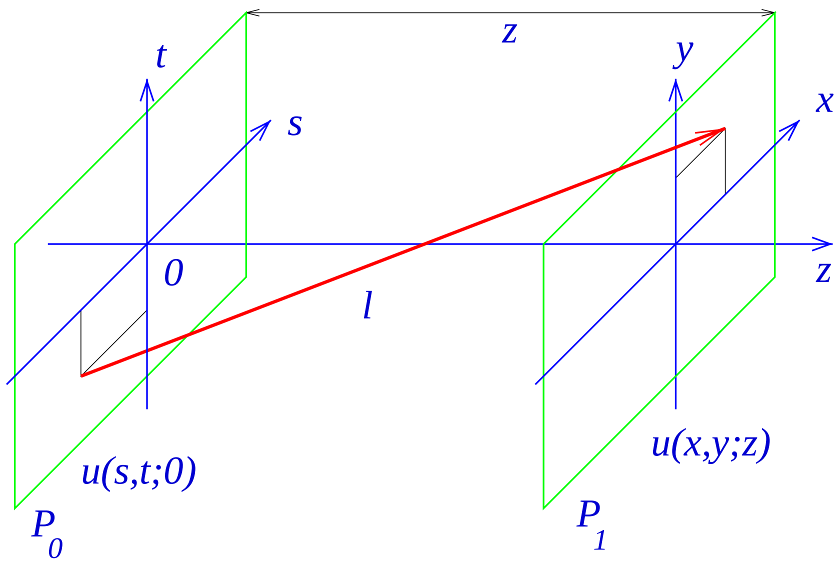
$$u(x,y;z) = \iint_{P_0} u_0(s,t)\, h(x \Leftrightarrow s, y \Leftrightarrow t; z)\, \mathrm{d}s\mathrm{d}t$$

where s,t are variables in plane P_0

The $h()$ term will contain terms of the form

$$(x \Leftrightarrow s)^2 + (y \Leftrightarrow t)^2 + z^2 = l^2$$

so l is Distance from $(s,t;0) \Rightarrow (x,y;z)$



Full Expression is

$$u(x,y;z) = \frac{1}{\lambda} \iint_{P_0} u_0(s,t) \left[\frac{1}{\kappa l} \Leftrightarrow \imath \right] \frac{\exp(\imath \kappa l)}{l} \left(\frac{z}{l} \right) \mathrm{d}s\mathrm{d}t$$

Rayleigh-Sommerfeld Diffraction Equation

Kirchoff Diffraction

Look at Impulse Response Function:

$$h(x,y;z) = \frac{1}{\lambda}\left[\frac{1}{\kappa r} \Leftrightarrow \imath\right]\left(\frac{z}{r}\right)\frac{\exp(\imath \kappa r)}{r}$$

Most Practical cases, P_0 and P_1 separated by MANY wavelength,

$$\Rightarrow z \gg \lambda \quad \Rightarrow r \gg \lambda$$

Approximate the term:

$$\left[\frac{1}{\kappa r} \Leftrightarrow \imath\right] \approx \Leftrightarrow \imath$$

so that

$$h(x,y;z) \approx \frac{1}{\imath\lambda}\left(\frac{z}{r}\right)\frac{\exp(\imath \kappa r)}{r}$$

Look at Terms

$$\frac{\exp(\imath \kappa r)}{r}$$

Spherically expanding wave from point $(0,0;0)$

$$\frac{z}{r}$$

"Obliquity Factor" which forces propagation in z direction. (In Goodman's notation the obliquity is written as a $\cos()$ term).

The other terms are just constants.

Note: For 3-D point source, get expanding spherical wave obliquity factor results from 2-D source in a plane.

Impulse Response Function: Spherical Wave with directional weighting term.

Model: Each point in P_0 acts a source of impulse response functions, that sum is P_1.

$$h(x,y;z) = \text{Spherical Wave} \left[\frac{z}{r}\right]$$

This is **Hygen's Secondary Wavelet** (weighted by obliquity factor).

Hygen's Secondary Wavelets

Model: Each point on the wavefront gives rise to Spherical Waves,

$$h(x,y;z) = \frac{\exp(\imath\kappa r)}{r}$$

Add postulate that "wave propagate in positive z direction".

$$\text{Kirchhoff} \Leftrightarrow \text{Hygen's} \left[\frac{z}{r}\right]$$

Kirchhoff Diffraction Integral

In Rayleigh-Sommerfeld integral,

$$z \gg \lambda \quad \Rightarrow \quad l \gg \lambda$$

make the approximation that

$$\left[\frac{1}{\kappa l} \Leftrightarrow \imath\right] \approx \Leftrightarrow \imath$$

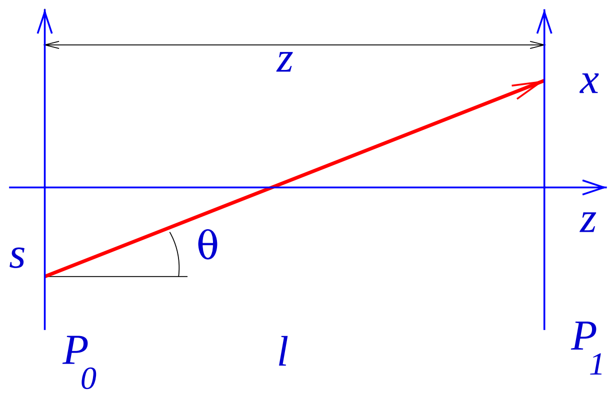
so we get that

$$u(x,y;z) = \frac{1}{\imath\lambda} \iint_{P_0} u_0(s,t) \frac{\exp(\imath\kappa l)}{l} \left(\frac{z}{l}\right) \mathrm{d}s\mathrm{d}t$$

which is valid (1%), $z > 20\lambda$.

Typical starting point for optical calculations.

Look at z/l factor:



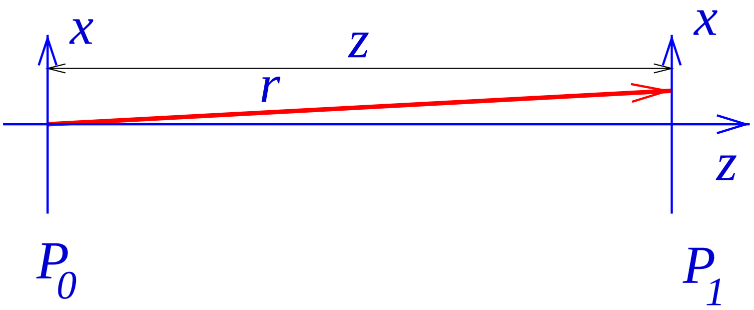
so that

$$\frac{z}{l} = \cos\theta$$

Same expression as in books [eg. Goodman page 52, equation (3-51)]

Fresnel Approximation

Assume extend of P_0 and $P_1 \ll z$



So we have that

$$|x| \,\&\, |y| \ll z$$

We have Kirchhoff impulse response

$$h(x,y;z) = \frac{1}{\imath\lambda}\left(\frac{z}{r}\right)\frac{\exp(\imath\kappa h r)}{r}$$

Which we can write as:

$$h(x,y;z) = A(x,y;z)\exp(\imath\kappa r)$$

where the amplitude

$$A(x,y;z) = \frac{1}{\imath\lambda}\frac{z}{r^2}$$

Since $x \,\& y << z$, we can expand r as

$$r = z\left(1 + \frac{x^2+y^2}{z^2}\right)^{\frac{1}{2}} \approx z + \frac{x^2+y^2}{2z}$$

Amplitude Term: First Order approx:

$$r \approx z \quad \Rightarrow \quad A(x,y;z) \approx \frac{1}{\imath\lambda z}$$

Phase Term: Second Order approx:

$$r \approx z + \frac{x^2+y^2}{2z}$$

$$\exp(\imath\kappa r) \approx \exp(\imath\kappa z)\exp\left(\imath\frac{\kappa}{2z}(x^2+y^2)\right)$$

So the Fresnel Approximation is that

$$h(x,y;z) \approx \frac{\exp(\imath\kappa z)}{\imath\lambda z}\exp\left(\imath\frac{\kappa}{2z}(x^2+y^2)\right)$$

Fresnel Approximations

1) Replace Spherical waves by Parabolic Waves

2) Ignore Obliquity factor.

NOTE: only valid

$$|x| \;\&\; |y| \ll z \quad \Rightarrow \text{Small Objects}$$

This is also known as "Paraxial Approximation". Useful in many practical systems.

Use in the rest of the course

Fresnel Diffraction

We have that

$$
\begin{aligned}
& u(x,y;z) = h(x,y;z) \odot u_0(x,y) \\
= \quad & \frac{\exp(\imath\kappa z)}{\imath\lambda z} \iint u_0(s,t) \exp\left[\imath \frac{\kappa}{2z}\left((x \Leftrightarrow s)^2 + (y \Leftrightarrow t)^2\right)\right] \mathrm{d}s\,\mathrm{d}t
\end{aligned}
$$

This we can expand to get the

$$
\overbrace{\frac{1}{\imath\lambda z}\exp(\imath\kappa z)}^{1}\ \overbrace{\exp\left[\imath\frac{\kappa}{2z}(x^2+y^2)\right]}^{2} \times
$$

$$
\underbrace{\iint \overbrace{u_0(s,t)\exp\left[\imath\frac{\kappa}{2z}(s^2+t^2)\right]}^{4} \exp\left[\Leftrightarrow\imath\frac{\kappa}{z}(sx+ty)\right] \mathrm{d}s\mathrm{d}t}_{3}
$$

Look the **FOUR** terms

1. Absolute amplitude and phase, depends only on z (constant which is not normally important).
2. Parabolic phase term, no effect on intensity.
3. Fourier Transform scaled by κ/z
4. Scalar distribution in P_0 weighted by parabolic phase term.

Fresnel Diffraction $\Rightarrow$ Fourier Transform weighted by parabolic phase term [+ Extra phase and constants]

In Fourier Space

In Fourier Space we have

$$U(u,v;z) = H(u,v;z)\,U_0(u,v)$$

so we have that

$$H(u,v;z) = \frac{1}{\imath\lambda z}\exp(\imath\kappa z)$$
$$\iint \exp\left[\imath\frac{\kappa}{2z}(x^2+y^2)\right]\exp\left(\Leftrightarrow\imath 2\pi(ux+vy)\right)\mathrm{d}x\mathrm{d}y$$

This can be separated into 2 integrals in x and y, and with identity

$$\int_{-\infty}^{\infty}\exp(\Leftrightarrow bx^2)\exp(\imath ax)\,\mathrm{d}x = \sqrt{\frac{\pi}{b}}\exp\left[\Leftrightarrow\frac{a^2}{4b}\right]$$

It-Can-be-Shown that

$$H(u,v;z) = \exp(\imath\kappa z)\exp\left(\imath\pi\lambda z(u^2+v^2)\right)$$

which is again a parabolic term.

Fresnel diffraction $\Rightarrow$ Multiplication in Fourier plane by parabolic phase term.

Summary

Scalar Wave solution for prapagation between two plane:

Full Solution $\Rightarrow$ Rayleigh-Sommerfeld

If distance between planes $>$ a "few wavelengths" then

Scaler Kirchoff Diffraction

This is Hygen's Secondary Spherical wavelets plus obliquity factor due to 2-D whole in plane.

If distance between planes *Large* **and** planes are *Small* make small angle approximation to get

Fresnel Diffraction

which replaces Spherical waves with Parabolic and ignores obliquity factor.

Fresnel Diffraction will be used for the rest of this course.