Topic 2: Imaging Properties

2.1 Basic Image Formation

Before we can consider processing an image by a computer we must first understand the basis of image formation, and in particular find a method of representing this by a mathematical model. Most images are formed by an optical system, the simplest being a *camera*, with forms a *two-dimensional* intensity image, $f(x,y)$ of a *three-dimensional* intensity scene, $o(x,y,z)$ as shown in figure 1.

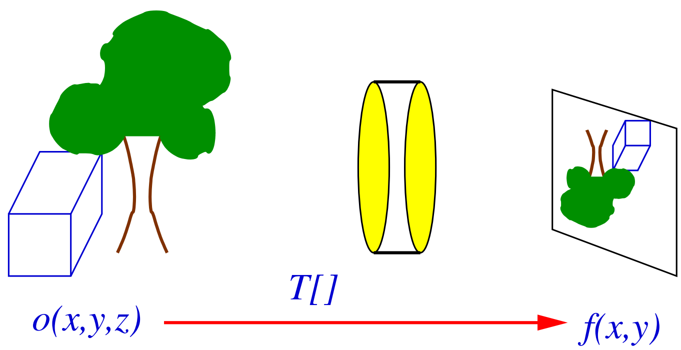

Figure 1: Basic schematic of image formation.

We can then write this as,

$$f(x,y) = T[o(x,y,z)]$$

where we describe the image formation process by the operator $T[]$, which images the three dimensional object to the two dimensional image plane, it is this plane that we will considering as the input to our digital processing system. The full three-dimensional problem is not solvable, so we will start with the simpler case where the object in two-dimensional[1] as shown in figure 2, so we reformulate the imaging process as

$$f(x,y) = T[o(x,y)]$$

where $T[]$ is an operator that take a two-dimensional intensity object and forms a two-dimensional intensity image.

2.1.1 Linearity Condition

All practical imaging systems to be considered will be assumed to be linear so that the image of a scene containing two separate objects, $o_1(x,y)$ and $o_2(x,y)$ of brightness α and β is just the a weighted sum sum of the images of the two individual objects. given by

$$\begin{aligned} f(x,y) &= T[\alpha o_1(x,y) + \beta o_2(x,y)] \\ &= \alpha T[o_1(x,y)] + \beta T[o_2(x,y)] \\ &= \alpha f_1(x,y) + \beta f_2(x,y) \end{aligned}$$

where $f_1(x,y)$ and $f_2(x,y)$ are the images of o_1 and o_2, as shown in figure 3 where we have assumed that the two object to not overlap.

[1]This assumption will result in problems that we will consider later in the course when we look at true three-dimensional objects.

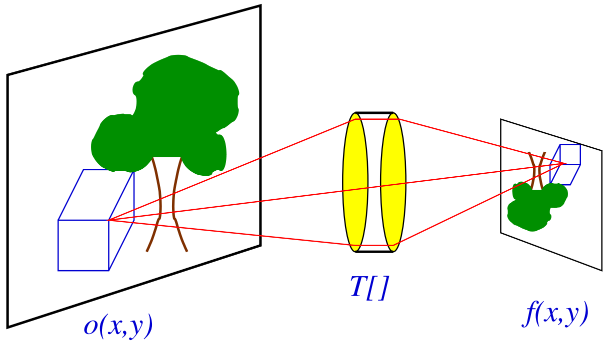

Figure 2: Image formation with a two-dimensional object

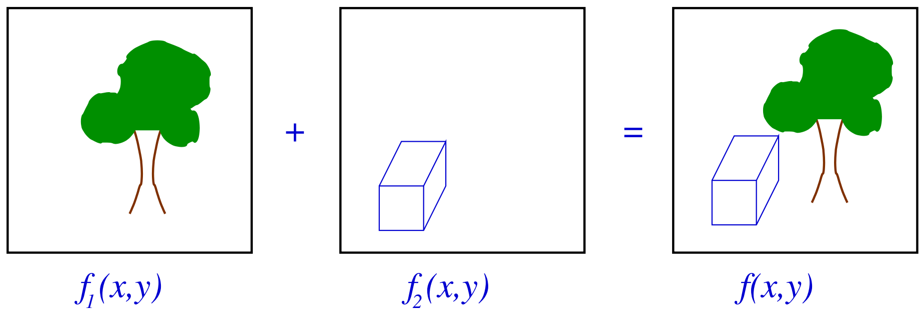

Figure 3: Two non-overlapping images added together in a linear imaging systems.

We can now consider the input object $o(x,y)$ to be a grid of closely packed *points* of different brightness as shown in figure 4, then if the system in *linear* we can consider *each* point as a separate object, and can form the image of each *point* separately, with the final image being an intensity sum of the if the individual images. It is this property that allows us to built the mathematical model. At the moment we will assume that these individual points are *close enough* together to fully represent object and will revisit what we mean by *close enough* in the next section when we consider sampling the detected image.

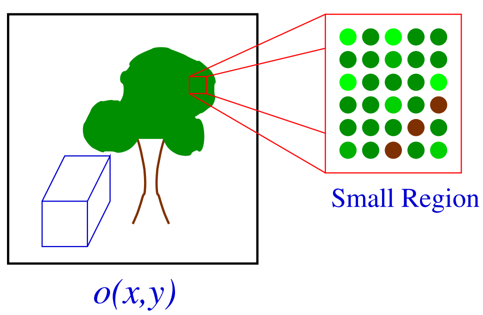

Figure 4: Sampled region of the image.

We can consider each point on the object as a two-dimensional δ-function, so a point at position a,b of is represented by

$$\delta(x-a,y-b)$$

then using the shifting properties of δ-function, we can write

$$o(x,y) = \int\!\!\int o(s,t)\,\delta(x-s,y-t)\,\mathrm{d}s\,\mathrm{d}t$$

which can also be through of as being convolution with δ-function which is a null operation. We have that

$$f(x,y) = T[o(x,y)]$$

So substitution for $o(x,y)$ from above, gives,

$$f(x,y) = T\left[\int\!\!\int o(s,t)\,\delta(x-s,y-t)\,\mathrm{d}s\,\mathrm{d}t\right]$$

Now we can us the property that $T[]$ is *linear*, so we can re-arrange the order of the integration to give that

$$f(x,y) = \int\!\!\int o(s,t)\,T[\delta(x-s,y-t)]\,\mathrm{d}s\,\mathrm{d}t$$

We can then write

$$T[\delta(x-s,y-t)] = h(x,s,y,t)$$

which we can interpret as at being the *image* of a δ-function located at position s,t on the object, as shown in figure 5, thus $h(x,s,y,t)$ is known as the *Point Spread Function*.

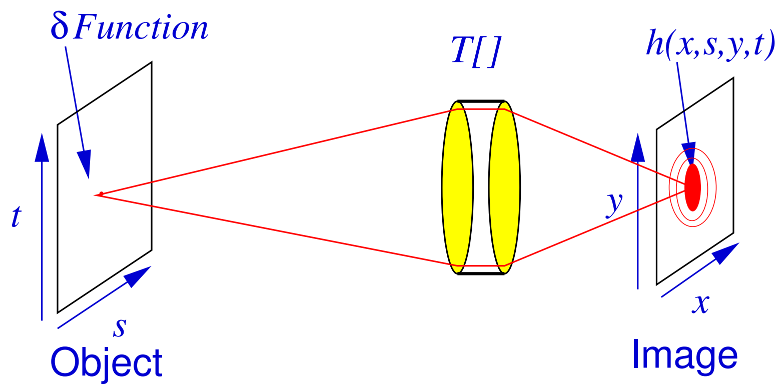

Figure 5: Image of a δ-function giving the *point spread function*.

2.1.2 Space Invariance Condition

Now if we further assume that the system is *Space Invariant*, so shape of $h(x,s,y,t)$ does *not* depend on s,t, they only give it location as shown in figure 6, then we can write

$$h(x,s,y,t) = h(x-s,y-t)$$

so we have that

$$f(x,y) = \int\!\!\int o(s,t)\,h(x-s,y-t)\,\mathrm{d}s\,\mathrm{d}t$$

which is just the *convolution* integral in two dimensions between the object $o(x,y)$ and the *point spread function* $h(x,y)$, which we will write as

$$f(x,y) = o(x,y) \odot h(x,y)$$

The important thing to notice is that $h(x,y)$ is purely a property of the imaging system characterised by the transformation $T[]$, and it thus the *same* for all objects. Thus is we have system where we either know, calculate or measure $h(x,y)$, then we know how this system will image any object, and more importantly, we can then digitally correct for this defect. It is this property that is central to image reconstruction considered later in this course.

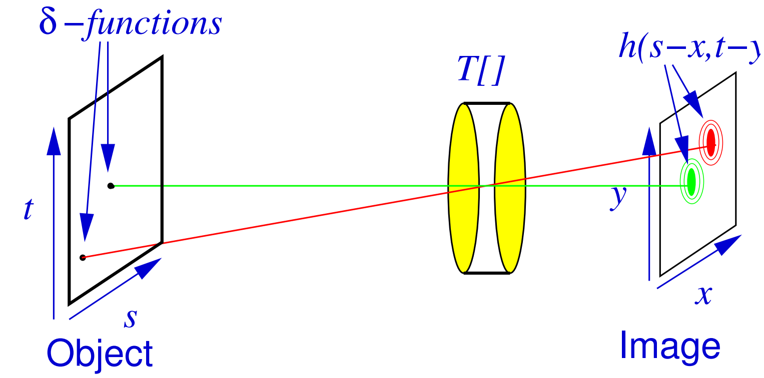

Figure 6: Space invariance of the *point spread function*

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This formulation assume unit magnification between the *object* and *image* planes. In most system this is not the case, but can easily be corrected for by linear scaling the coordinates of the image by the magnification of the imaging systems. This is again simply given by the system geometry, being the ratio of the object and image distances[2].

2.2 Analysis in the Fourier Domain

We have just seen that in normal, or real space, that the image is a convolution of the *object* and the *point spread function* of the imaging system given by

$$f(x,y) = o(x,y) \odot h(x,y)$$

it is therefore natural to consider the system in Fourier space, since

$$F(u,v) = O(u,v)\, H(u,v)$$

where from the *convolution theorem* we have that

$$\begin{array}{rcll} F(u,v) & = & \mathcal{F}\{f(x,y)\} & \text{Fourier transform of the Image} \\ O(u,v) & = & \mathcal{F}\{o(x,y)\} & \text{Fourier transform of the Object} \\ H(u,v) & = & \mathcal{F}\{h(x,y)\} & \text{Fourier filter Function} \end{array}$$

for an imaging systems we have that $h(x,y)$ is *point spread function*, and as seen above $H(u,v)$ acts like a Fourier space filter, and when derived from a *point spread function*, then,

$$H(u,v) \rightarrow \text{ Optical Transfer Function}$$

If we know *either* $h(x,y)$ *or* $H(u,v)$ we are able to fully characterise the system. For a practical systems it is easiest to measure $h(x,y)$ being the image of a *star*, and then calculate $H(u,v)$ digitally.

[2]See JH optics course for details.

2.3 The Ideal Imaging System

If we consider the simple optical system of an *ideal lens* forming an image of a distant point source, as shown in figure 7, then *it can be shown*[3], that the the *point spread function* is given by diffraction from a circular aperture, and is given by

$$h(x,y) = \left[\frac{\mathrm{J}_1(\alpha\kappa r)}{\alpha\kappa r}\right]^2$$

where we have that,

$$\begin{array}{rcll} \alpha & = & \sin\theta & \text{The numerical aperture} \\ \kappa & = & 2\pi/\lambda & \lambda \text{ is wavelength of light} \\ r & = & \sqrt{x^2+y^2} & \text{Radial position} \\ \mathrm{J}_1() & & & \text{First order Bessel function} \end{array}$$

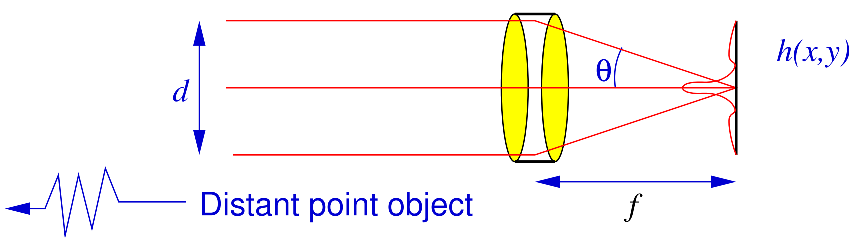


Figure 7: Imaging of a distant point by an ideal lens.

It is useful to introduce the measure of F_{No}, being defined as

$$\mathrm{F}_{\mathrm{No}} = \frac{f}{d} = \frac{\text{Focal Length}}{\text{Diameter}}$$

so that for optical systems where θ is small[4], we can take the approximation that

$$\alpha \approx \frac{d}{2f} = \frac{1}{2\mathrm{F}_{\mathrm{No}}}$$

so that the *ideal point spread function* of a system is given by it F_{No} and *Wavelength* of light only.

The shape of the function $(\mathrm{J}_1(x)/x)^2$ is shown in figure 8, being similar in shape to the $\mathrm{sinc}()^2$ function with a large central peak that contains 88% of the energy and a series of decreasing secondary peaks with the zero occurring at

$$\begin{array}{rcccl} x_0 & = & 3.832 & = & 1.22\pi \\ x_0 & = & 7.016 & = & 2.23\pi \\ x_0 & = & 10.174 & = & 3.24\pi \\ x_0 & = & 13.324 & = & 4.24\pi \end{array}$$

In two-dimensions we get circular central peak with a series of rings with the radius of the rings given by the zeros of the Bessel function.

[3] Not part of this course, see Optics courses or standard optics textbooks for details

[4] True in most optical imaging system *except* microscopes.

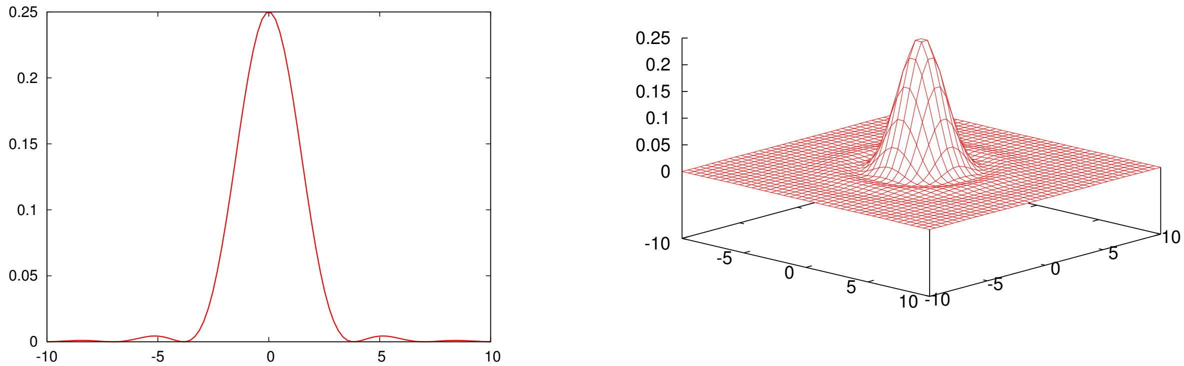


Figure 8: The shape of the function $(\mathrm{J}_1(x)/x)^2$ and $(\mathrm{J}_1(r)/r)^2$ as a two-dimensional surface.

Therefore for the ideal optical system above, the radius of the first zero will be given by

$$r_0 = 1.22\lambda \mathrm{F}_{\mathrm{No}}$$

so a system with $\mathrm{F}_{\mathrm{No}} = 8$ using green light of $\lambda \approx 550\,\mathrm{nm}$ will result in the first zero being at $4.4\,\mu\mathrm{m}$. This initially looks small, but is comparable to the spacing on a modern CCD camera chip, so is of direct relevance in digital imaging.

In the case of an *ideal* imaging system, the *Optical Transfer Function* also has an analytic solution, and *can-be-shown*[5] to be given by:

$$H(u,v) = \frac{2}{\pi}\left[\cos^{-1}\left(\frac{w}{v_0}\right) - \frac{w}{v_0}\left(1-\left(\frac{w}{v_0}\right)^2\right)^{\frac{1}{2}}\right]$$

where $w = \sqrt{u^2+v^2}$ and

$$v_0 = \frac{d}{\lambda f} = \frac{1}{\lambda \mathrm{F}_{\mathrm{No}}}$$

which is the spatial frequency limit of the system, again for the *ideal* system, given only by its physical size and the wavelength of the illumination used. The shape of $H(u,v)$ is a *tent* shape shown in figure 9 in one and two dimensional plotted for $v_0 = 100$.

As we have seen above the *Optical Transfer Function* acts as a *Fourier Filter*, so modifies the Fourier transform of the detected image. It is therefore a measure of the *fidelity* with which each spatial frequency is passed, and of most significance is that

$$H(u,v) = 0 \quad \text{for } u^2+v^2 > v_0^2$$

so there is a *Spatial Frequency limit*, which for the ideal systems is just given by the physical dimensions of the system.

- Not all spatial frequencies passed with the same fidelity

[5] It is the auto-correlation of a circular aperture.

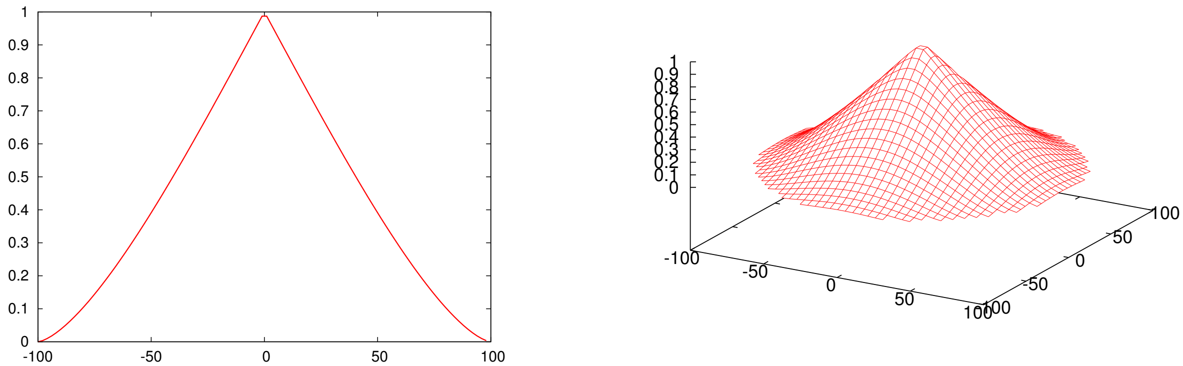


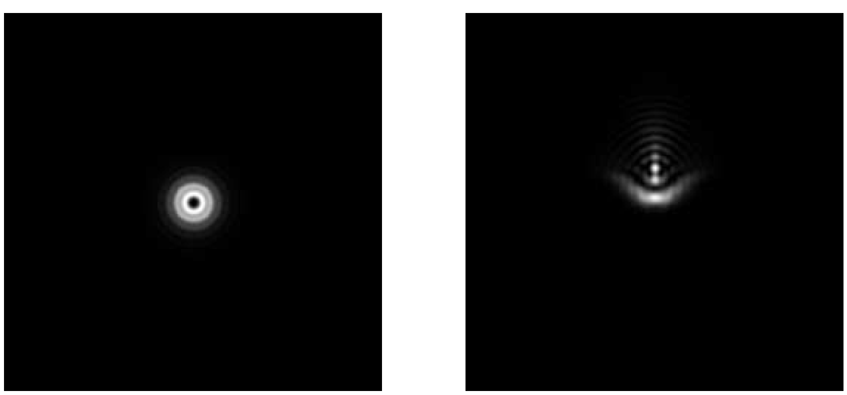
Figure 9: Shape of the *Optical Transfer Function* of idea optical system with maximum frequency of 100.

- Spatial Frequency limit even for ideal system, so all imaging systems *band-width limited*.

These properties allow us to represent an image in a computer and we will consider them again when we come to sample an image in the next section.

2.4 The Non-ideal Imaging System

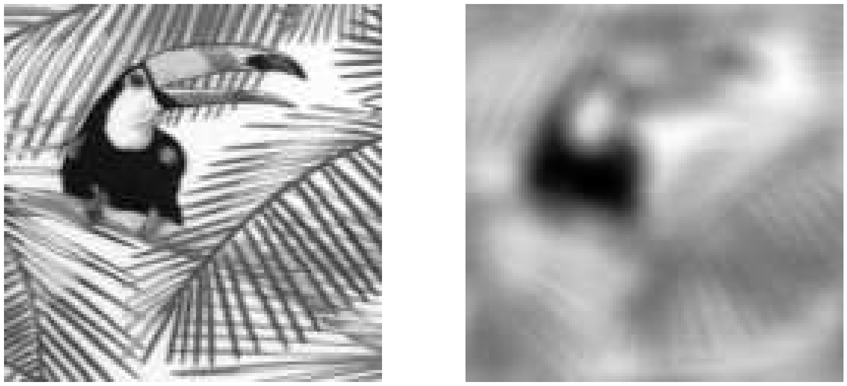
A real imaging system will have defects, or *aberration*, which reduce it performance and quality of the formed image. If the aberration in known, then the aberrated point spread function can be digitally generated. Two typical point spread functions are shown in figure 10 showing the affect of (a) defocus and (b) mixed aberrations. The affect of convolving an image with the digitally generated defocus point spread function is shown in figure 11 with the original image is (a) and the digitally defocused in (b).



(a) (b)

Figure 10: Digitally calculated *point spread function* for a system system with a) defocus, and b) mixed aberrations.

For real imaging system we are able to measure the *point spread function* since it image of a



(a) (b)

Figure 11: (a) unmodified image, (b) digitally defocused with *point spread function* in figure 10.a

distant point. In addition it is also possible to measure the *Optical Transfer Transfer* but this requires more optical background than expected in this course.

We well see towards the end of the course that this is the basic scheme for image reconstruction, where if we know the PFS, then for the *detected* aberrated image we can *reconstruct* an *ideal* image.

2.5 Validity of Assumptions

To allow use of *Convolution Theorem* we have assumed:

Linearity: Valid for most system, but problems with

1. 3-D scenes, where one object obscures the other.
2. Photographic film and video systems are frequently non-linear in intensity, but can usually be allowed in the processing
3. Saturation in sensors, particularly in CCD systems.
4. Low light level when the quantum nature of light makes the system non-linear in intensity. This is more of a problem in statistical analysis, and will be considered again in the section on noise models

Space Invariance: Valid for most *good* imaging systems, problems with:

1. Large telescopes, where the main aberration is coma which means the off-axis points spread function is different as shown in figure 12.
2. Geometric distortions, for example perspective distortions, which we will consider in a later section.

In general, most normal imaging systems, such as video cameras, digital cameras, remote sensing satellite system, microscopes and low-power telescopes all obey these assumptions.

(a) (b)

Figure 12: Typical point spread functions of a large telescope, (a) on-axis, ideal PSF, (b) off-axis, PSF showing coma.

2.6 Summary

In the section we have covered:

1. Basics of image formation in optical system.
2. Used the assumptions of linearity and space invariance for form a Fourier based model of imaging.
3. Considered the implications of this model for an ideal, perfect, lens in both real and Fourier space.
4. Outlines the affect of aberrations and the resultant degraded imaging quality.
5. Outlined the validity of the underlying assumptions with reference to real imaging systems.

Workshop Questions

2.1 Fourier Transform Theory

See tutorial questions and solutions in *The Fourier Transform, (What you need to know)*. Questions 4, 7, 8 and 10 are particularly useful.

2.2 Linear Imaging

State all the conditions required of an imaging system for it be be described by

$$f(x,y) = h(x,y) \odot o(x,y)$$

where $h(x,y)$ is the intensity point spread function.

2.3 Use of the OTF

Optical imaging theory shows that the OTF of an ideal lens is given by:

$$H(w) = \frac{2}{\pi}\left[\cos^{-1}\left(\frac{w}{w_0}\right) - \frac{w}{w_0}\sqrt{1-\frac{w^2}{w_0^2}}\right]$$

where $w^2 = u^2 + v^2$, and $w_0 = 1/\lambda \mathrm{F_{No}}$. If for an $\mathrm{F_{No}} = 8$ imaging system using green light, ideal image of the object in the image plane is

$$o(x,y) = 1 + \cos(2\pi a x)$$

calculate the detected image $f(x,y)$ when (i) $a = 100\mathrm{mm}^{-1}$, (ii) $a = 200\mathrm{mm}^{-1}$, (iii) $a = 300\mathrm{mm}^{-1}$

2.4 The Spot Satellite

The *Spot* satellite has a ground resolution in the near infra-red (at 1.3μm) of approximately 10 m from an orbit of 832 km. Assuming that this resolution limit is due to the point spread function of the imaging telescope estimate its diameter.

If the CCD sensors has a pixel size of $10 \times 10\mu$m, estimate the focal length and diameter of the optical system.

Look up the technical information on the Spot satellite system, and see how close there estimates are.