

# Topic 4: Point Processing

## Workshop Solutions

### Workshop Questions

#### 4.1 Zero mean in Fourier Space

If we have an  $N \times N$  image  $f(i, j)$  and subtract the mean to form

$$\tilde{f}(i, j) = f(i, j) - \langle f(i, j) \rangle$$

what is the difference between the Fourier transform of  $\tilde{f}(i, j)$  and  $f(i, j)$ .

#### Solution

The Fourier transform is a linear operation, so that

$$\tilde{F}(k, l) = F(k, l) - \mathcal{F}\{\mu\}$$

where  $\mu = \langle f(i, j) \rangle$ . The Fourier transform of a constant is a  $\delta$ -function, so from the definition of the digital Fourier transform, for a  $N \times N$  image,

$$\mathcal{F}\{\mu\} = N^2 \mu \delta_{k,l}$$

also from the definition of the digital Fourier transform,

$$F(0, 0) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} f(i, j) = N^2 \mu$$

so that  $\tilde{F}(0, 0) = 0$ , while all other values are identical. So that  $\tilde{F}(k, l)$  and  $F(k, l)$  are identical, *except* at the DC, or central point.

#### 4.2 Range of Statistics

Calculate the *mean* and *variance* if an image with a constant histogram (or probability density function) where the image values are between  $0 \rightarrow f_{\max}$ .



Calculate the maximum possible variance of an image with values between  $0 \rightarrow f_{\max}$ . What is special about this image.

#### Solution

For an image  $f(x, y)$  with probability density function  $p(f)$  then the mean and variance are given by:

$$\begin{aligned} \mu &= \int_0^{f_{\max}} f p(f) df \\ \sigma^2 &= \int_0^{f_{\max}} (f - \mu)^2 p(f) df \end{aligned}$$

where  $f_{\max}$  is the maximum value of the image.

We know that  $p(f)$  is normalised, so that

$$\int_0^{f_{\max}} p(f) \, df = 1$$

so if the image has been histogram equalised then

$$p(f) = \text{Constant} = \frac{1}{f_{\max}}$$

The mean is thus:

$$\mu = \frac{1}{f_{\max}} \int_0^{f_{\max}} f \, df = \frac{1}{2} f_{\max}$$

and the variance:

$$\begin{aligned} \sigma^2 &= \frac{1}{f_{\max}} \int_0^{f_{\max}} \left( f - \frac{1}{2} f_{\max} \right)^2 \, df \\ &= \frac{1}{f_{\max}} \left[ \int_0^{f_{\max}} f^2 \, df - f_{\max} \int_0^{f_{\max}} f \, df + \frac{1}{4} f_{\max}^2 \int_0^{f_{\max}} \, df \right] \\ &= \frac{1}{f_{\max}} \left[ \frac{1}{3} f_{\max}^3 - \frac{1}{2} f_{\max}^3 + \frac{1}{4} f_{\max}^3 \right] \\ &= \frac{1}{12} f_{\max}^2 \end{aligned}$$

Take the real example of the toucan.



Original Picture



Histogram Equalised

| Statistic | Value  |
|-----------|--------|
| Minimum   | 9.0    |
| Maximum   | 255.0  |
| Mean      | 161.3  |
| Variance  | 3762.2 |

| Statistic | Value  |
|-----------|--------|
| Minimum   | 0.0    |
| Maximum   | 255.0  |
| Mean      | 126.3  |
| Variance  | 5397.9 |

The predicted mean of the histogram equalised image  $\mu = 127.5$  and the variance  $\sigma^2 = 5418.75$ . The experimental results differ slightly due to the discrete nature of the pixel levels giving a histogram that is not ideally constant.

The maximum variance occurs when  $(f - \mu)^2$  is maximised for each pixel. This occurs when half the pixels are 0 and the other half are  $f_{\max}$ , the image is binary. For such an image,

$$\mu = \frac{1}{2} f_{\max}$$

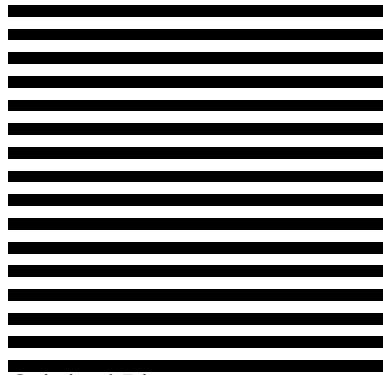
The variance is then

$$\begin{aligned}\sigma^2 &= \int_0^{f_{\max}} \left( f - \frac{1}{2}f_{\max} \right) p(f) df \\ &= \frac{1}{4}f_{\max}^2 \int_0^{f_{\max}} p(f) df \\ &= \frac{1}{4}f_{\max}^2\end{aligned}$$

where we note that  $p(f)$  is normalised. This is  $\times 3$  the variance for the histogram equalised image.

(This is not a formal proof of the maximum of  $\sigma^2$  but it does give the correct answer.)

Example of a binary grating with “bars” 4 pixels wide:



Original Picture

| Statistic | Value   |
|-----------|---------|
| Minimum   | 0.0     |
| Maximum   | 255.0   |
| Mean      | 127.5   |
| Variance  | 16255.8 |

This gives the correct predicted mean, but a slightly difference in the variance which has a theoretical value of  $\sigma^2 = 16256.25$ . This small difference is due to numerical rounding errors in the calculation of the variance from the  $\langle |f|^2 \rangle - \langle f \rangle^2$  formulation that results in the difference to two large numbers.

### 4.3 Grey Level Stretch and Threshold

Use the image `wjh/dia/images/junction.pgm` with the `xv` utility to investigate the following operations. All operations are obtainable via the *Color Edit* control panel.

Use the *Intensity* edit display bottom of the middle panel to implement:

- Grey level linear stretch with bound at 40 and upper at 175. (You need 4 control points at the linear interpolation scheme).
- Optimise a grey level stretch to make the “Give-Way” signs readable.
- Perform a binary threshold to try and extract the light car entering the junction from the left. (You need seven control points).

## Solution

There are too many pictures to display here. Try it for yourself.

### 4.4 Histogram Equalisation

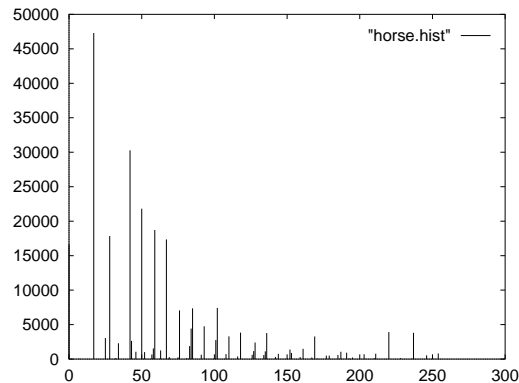
Use the utility `pgmhist` to form the histogram of `horseshoe.pgm` file, and display this histogram using one of the standard graph plotting utilities, (`gnuplot`, or `xmgrace`).

**Note:** you will have to edit the output of `pgmhist` to remove the first three header lines.

Use `xv` to form the histogram equalised version, save this as a `pgm` file and plot the equalised histogram. Explain what you get.

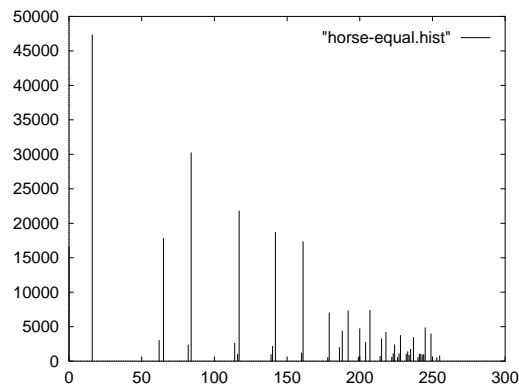
## Solution

The  $512 \times 512$  image `horseshoe.pgm` has only 75 discrete grey levels, it is actually extracted from a fully colour image of the horseshoe nebula. This histogram is thus rather sparse.



Most of the image is dark, (black sky), with relatively small regions of “nebula” and “stars”.

The histogram equalised histogram is again rather sparse, being:



The “dark” areas where there are many pixels are spread out while the light areas, where there are few pixels are closed together. This is as predicted. However the equalised histogram is *far* from flat since the input image has a finite number of grey levels, and all pixels of a particular grey level on the input are transformed to the same grey level in the equalised pixel.

For example in the original image there are 47,294 pixels with grey level 17. In the histogram equalised image all these pixels have value 16.

## 4.5 False Colour Maps

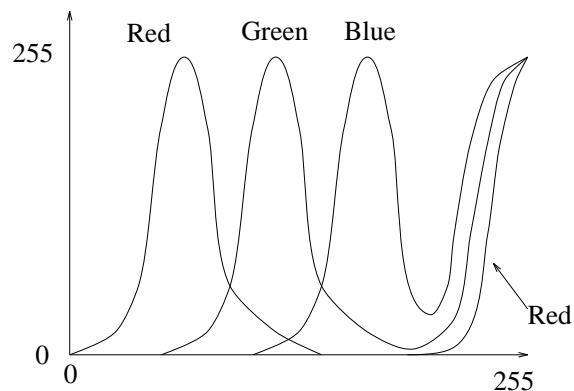
The most common “colour map” for false colour display is called the “Temperature” map. This is based on the ideal of heating a piece of material so that

| Values    | Colour |
|-----------|--------|
| Low       | Red    |
| Medium    | Green  |
| High      | Blue   |
| Very High | White  |

Design and implement such a false colour transformation using xv and display a range of images.

### Solution

There are three colour outputs for each single grey level input. Use the three *RGB Modification* panels to produce false colour look-up of the form.



You will need 7 control points on each panel.

**Note:** it is difficult to get a good colour display since the human eye is very sensitive to small changes in the green part of the spectrum and relatively in-sensitive to changes in red and blue.