DIA Course Summary

Aim

This final lecture contains a summary of the course material concentrating on the key results and interconnections between the various topics.

Topics 2: Image Formation

If imaging system is **Linear** and **Space Invariant**, detected image is:

$$f(x,y) = o(x,y) \odot h(x,y)$$

where $o(x,y)$ is "Object" and $h(x,y)$ is system *Point Spread Function* (PSF).

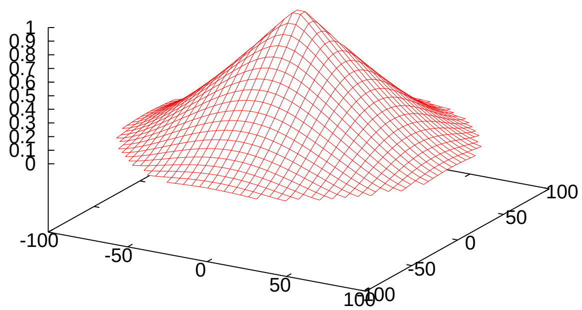
In **Fourier Space** this given,

$$F(u,v) = O(u,v)\,H(u,v)$$

where $H(u,v)$ is the *Optical Transfer Function* (OTF).

Image Formation I

For ideal imaging system OTF has shape of:



and goes to Zero at **frequency**

$$v_0 = \frac{d}{\lambda f} = \frac{1}{\lambda \mathrm{F_{No}}}$$

where d is the diameter of the lens, f is its focal length and $\mathrm{F_{No}}$ is the f-number.

This is the "best-you-can-do" so showing that all detected image have a *maximum* spatial frequency v_0 and are thus *Band Limited*.

Topic 3: Digital Sampling

Sample image in **Real Space** at intervals $\Delta x, \Delta y$ to get digital image

$$f(i,j) \quad \text{for } i \,\&\, j = 0,1,2,\ldots,N-1$$

Equivalent in sampling is **Fourier Space** at intervals $\Delta u, \Delta v$ to get

$$F(k,l) \quad \text{for } k \,\&\, l = 0,1,2,\ldots,N-1$$

where

$$\Delta u = \frac{1}{N\Delta x} \quad \& \quad \Delta v = \frac{1}{N\Delta y}$$

where the two-dimensional Discrete Fourier Transform is given by

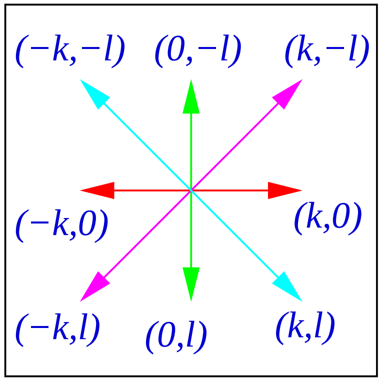
$$F(k,l) = \frac{1}{N^2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} f(i,j)\exp\left[-\imath 2\pi\left(\frac{ki}{N}+\frac{lj}{N}\right)\right]$$

and the inverse is given by:

$$f(i,j) = \frac{1}{N^2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} F(k,l)\exp\left[\imath 2\pi\left(\frac{ki}{N}+\frac{lj}{N}\right)\right]$$

Digital Sampling

If the *Real Space* function is **real**, then DFT will have point symmetry,



When can be written as:

$$\begin{aligned} F_R(k,l) &= F_R(-k,-l) \\ F_R(-k,l) &= F_R(k,-l) \\ F_I(k,l) &= -F_I(-k,-l) \\ F_I(-k,l) &= -F_I(k,-l) \end{aligned}$$

which for an $N \times N$ image (N even), gives that

$$\frac{N^2}{2} + 2 \rightarrow \text{Real Values} \quad \text{and} \quad \frac{N^2}{2} - 2 \rightarrow \text{Imaginary Values}$$

being dependent on the data, the rest being given by symmetry properties.

Digital Sampling

Shannon Sampling Theorem: If function $f(x)$ has FT of bandwidth a, so that:

$$F(u) = 0 \quad \text{for } |u| > a/2$$

then $f(x)$ **completely** specified by samples at

$$\Delta x = \frac{1}{a}$$

$\Delta x > 1/a$: Undersampled and information lost due to overlaps of Fourier orders

$\Delta x < 1/a$: Oversampled and NO MORE information gained, (ignoring the effect of noise)

Same process in two dimensions, usually take $\Delta x = \Delta y$. For ideal lens, we have bandwidth limit set by $H(w)$. We have that

$$H(w) = 0 \quad \text{for } |w| > v_0 \quad \Rightarrow a = 2v_0$$

so that,

$$\Delta x = \Delta y = \frac{1}{2v_0} = \frac{\lambda F_{No}}{2}$$

Ditigal Sampling

To reconstruct continuous function from sampled data must isolate one order in *Fourier Space*. In real space:

$$f(x) = f(i) \odot h(x)$$

where $h(x)$ is interpolation function.

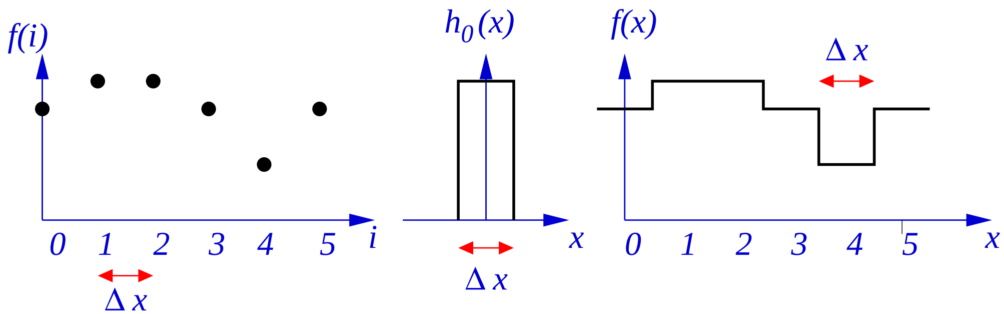
Ideal interpolation function,

$$h(x) = \mathrm{sinc}\left(\frac{\pi x}{\Delta x}\right)$$

Ditigal Sampling

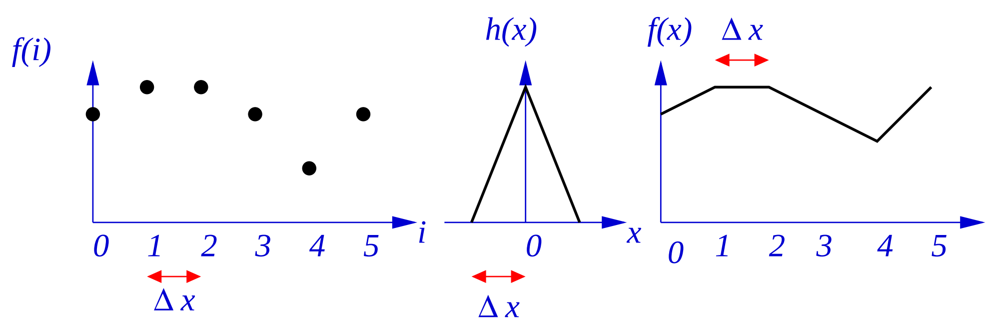
but in practice have to make approximations,

Zero Order: nearest neighbour rule.



Typical “Stair-Case” effect.

First Order: linear interpolation,



gives smoother reconstruction.

Topic 4: Point Processing

Histograms: Really a measure of *Probability Density Function*, in particular

$$p(f) = \frac{h(f)}{N^2}$$

where $h(f)$ is the grey-level histogram.

Note that the mean and variance of the image is given by

$$\mu = \frac{1}{N^2} \sum_{f=0}^{f_{max}} f h(f)$$

and

$$\sigma^2 = \frac{1}{N^2} \sum_{f=0}^{f_{max}} (f-\mu)^2 h(f)$$

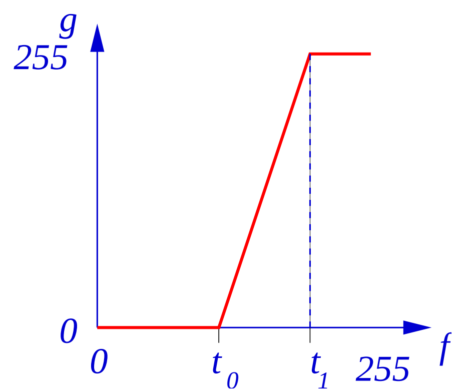
so when you change the histogram, you also change the *first order* statistics.

Point Processing

Modify image pixels depending on single pixel value *only*:

$$g = T(f)$$

usually display in graphical form.



Simple scheme to *Stretch Grey Levels*, *Gamma Correction*, and *Binary Thresholding*.

Can also be extended to *colour* images to colour balance and correct for in-correct exposures.

Simplest form of Digital Image Processing

Histogram Equalisation

Techniques to “flatten” the histogram, so distribute pixels evenly over the whole available range of displable grey levels.

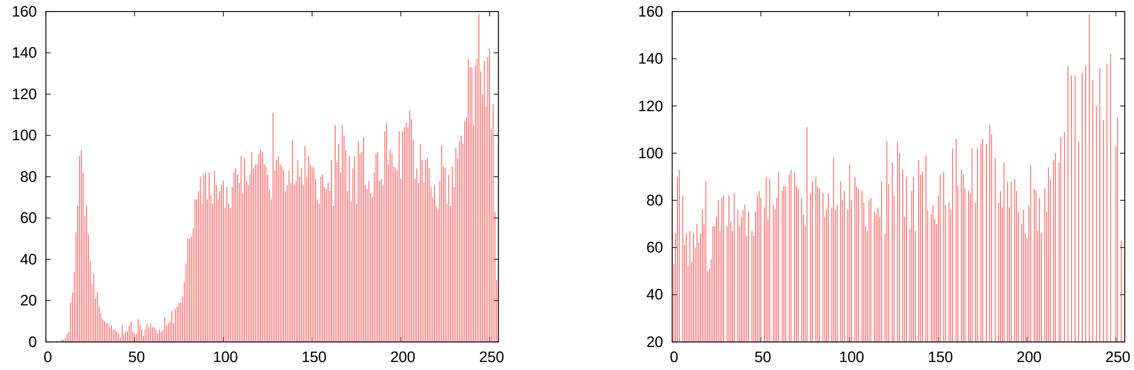
Transformation that depends on the PDF of the image, giving that

$$g = T(f) = \beta \int_0^f p_f(a)\,\mathrm{d}a$$

where $p_f(f)$ is the PDF if image $f(i,j)$ and β is a constant.

We can *estimate* $p_f(f)$ from simple grey-level histogram, and *estimate* integral by summation.

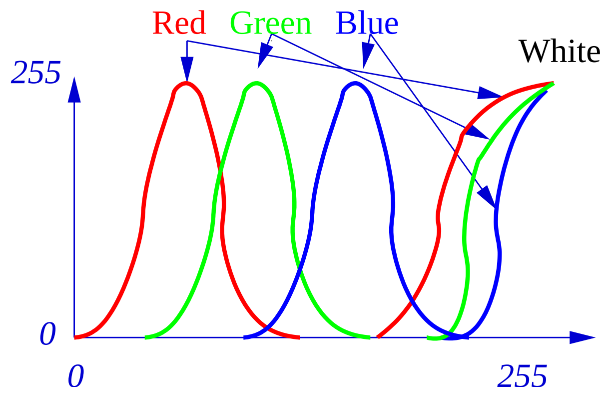
Typical result of:



Note: If information **not** associated with more frequent pixels, we can degrade the image.

False Colour

Make use of colour display capability with three transformations, (for Red, Green and Blue), displayed graphically as:



but care has to be taken to match the information displayed to the colour response of the eye.

Topic 5: Noise in Images

Removable Noise: If has identifiable character, we can identify it and “remove” it.

Data Dropout: Differs from surrounding pixels. Remove by average threshold or median filter.

Fixed Pattern: Characteristic of CCD. Subtract off known patter.

Sensor Stripes: Calibrate sensor from line average of image data. Then subtract the effect.

Non-Removable Noise: Detector or shot noise is fundamental part of image detection process. Not able to “remove” it, but able to reduce its effect.

If we assume the system is **linear**, **high brightness** and **low contrast** we get the noise to be additive, signal independent with a Gaussian PDF.

Properties of Additive Noise

For additive noise

$$f(i,j) = s(i,j) + n(i,j)$$

Noise does not depend on the signal, also each point is independent, so we get that

$$|N(k,l)|^2 \approx \text{Constant}$$

which is known as White Noise.

Signal to Noise Ratio

Range of definitions, one used here is

$$\text{SNR} = \frac{\sigma_s}{\sigma_n}$$

which can be estimated from a single image with “blank” areas, or from multiple images of the same scene by cross-correlation.

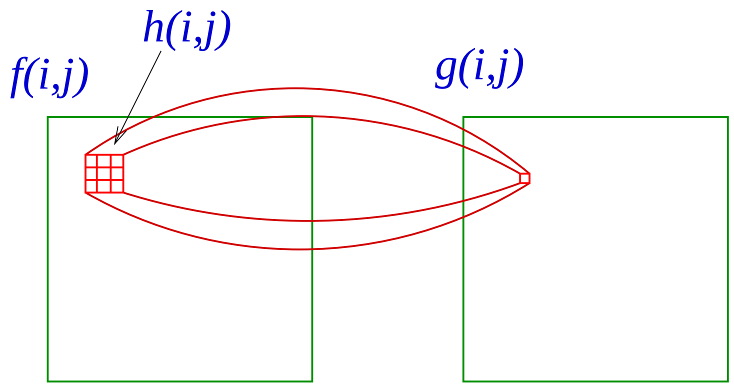
Topic 6: Linear Digital Filtering

Mainstay of Image Processing. Idea to convolve image with filter,

$$g(i,j) = h(i,j) \odot f(i,j) \quad \Longleftrightarrow \quad G(k,l) = H(k,l)\,F(k,l)$$

Fourier Space: Need two DFTs size of image, constant computational overhead.

Real Space: Use “shift-and-multiply”



Computational cost $\propto$ size of filter.

For filter size bigger than 9×9 faster to use Fourier method.

Fourier Space Filtering

Define filter by $H(k,l)$ is Fourier Space. To retain real output filter $H(k,l)$ **must** have central point symmetry. (usually restricted to Real and Symmetric filters).

Filter Types:

- **Lowpass:** Cuts of reduced High Spatial frequencies. Useful in reducing noise, especially additive Gaussian.
- **Highpass:** Cuts or reduces Low Spatial frequencies. Enhances edges and take differentials. Used as part of edge detection schemes.
- **Bandpass:** Combination of the two, allows through range of frequencies. User to enhance/detect edges in the presence of noise.

Normally use “smoothed” versions of filters, such a Gaussians to prevent “ringing”

Real Space Filters

Define $h(i, j)$ is Real Space, usually over small window. Same mathematics as Fourier Space implementation.

Each real space has an equivalent Fourier space equivalent.

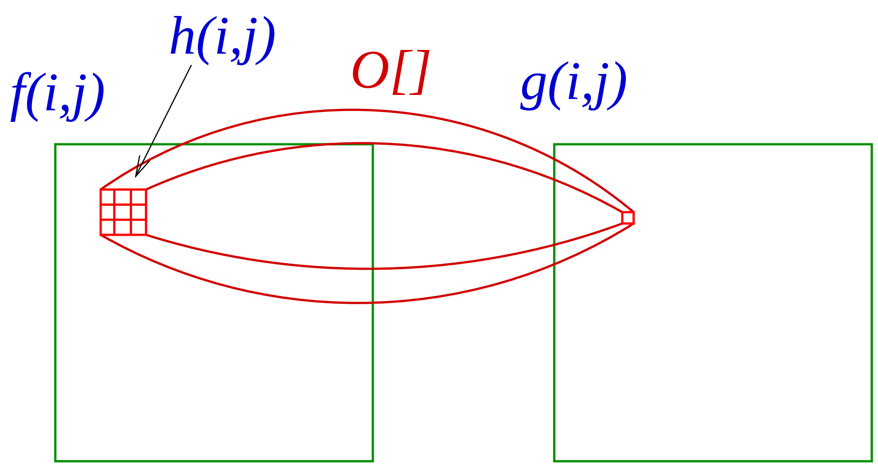
Filter Types:

- Averaging: Equivalent to Lowpass, used to smooth image and reduce effects of noise.
- Differentials: Equivalent to Highpass, used to enhance edges.

Use multiple filters by combining them by *convolution*, to give bandpass operations.

Non-Linear Filters

Extend to non-linear real space by:



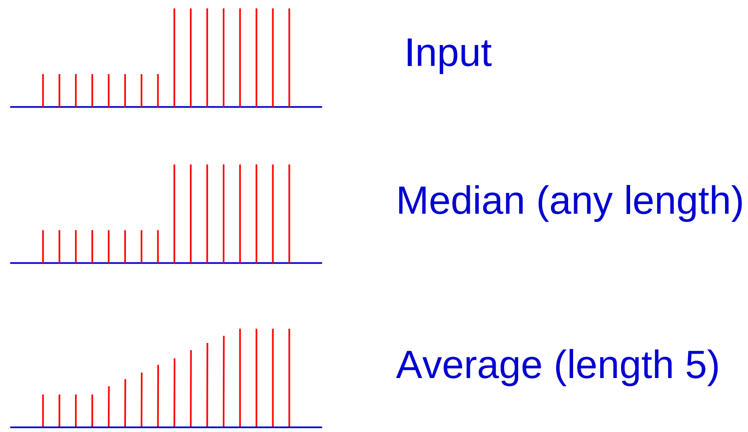
Most filter have $h(i,j) = 1$ so the filtering operation is controlled by the operation $O[]$ and the size of the window over which operator acts.

Non-Linear Operations

Shrink and Expand: Use MIN/MAX operations to shrink and expand images to remove isolated noise points from segmented images.

Threshold Average Filter: Selectively search for disparate points and replace them. Used for Data-Dropout removal.

Median Filter: Non-linear smoothing filter that smoothes regions while retaining edges.



Median is the most important non-linear filter. Problem with high computational cost.

Topic 7: Image Reconstruction

Aim is to correct, or remove, the effect of aberrated PSF.

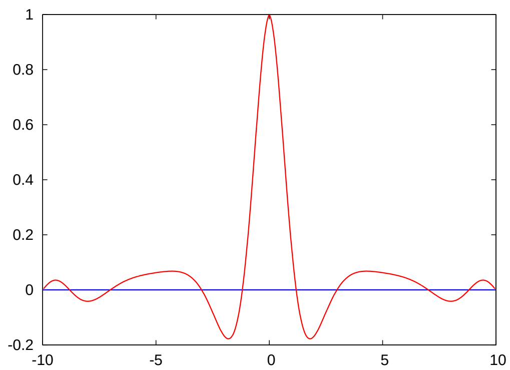
$$g(i,j) = f(i,j) \odot h(i,j) + n(i,j)$$

so after collecting $g(i,j)$ we want to reconstruct $f(i,j)$ knowing what $h(i,j)$ is.

In Fourier space we have

$$G(k,l) = H(k,l)\,F(k,l) + N(k,l)$$

we have the problem that $H(k,l)$ is typically small at high spatial frequency.



so simple *Inverse Filter* of

$$\tilde{F}(k,l) = \frac{G(k,l)}{H(k,l)} = F(k,l) + \frac{N(k,l)}{H(k,l)}$$

fails since $|N(k,l)|^2 \approx \text{Constant}$.

Wiener Filter

Least Squares estimate to the reconstruction being:

$$\left\langle |\tilde{f}(i,j) - f(i,j)|^2 \right\rangle \quad \text{Minimum}$$

Define an optimal filter $y(i,j)$, and by performing the minimisation in Fourier space we get a solution that

$$Y(k,l) = \frac{H^*(k,l)}{|H(k,l)|^2 + \frac{|N(k,l)|^2}{|F(k,l)|^2}}$$

with

$H(k,l)$	System OTF
$\|N(k,l)\|^2$	Power spectrum of Noise
$\|F(k,l)\|^2$	Power spectrum of Ideal image

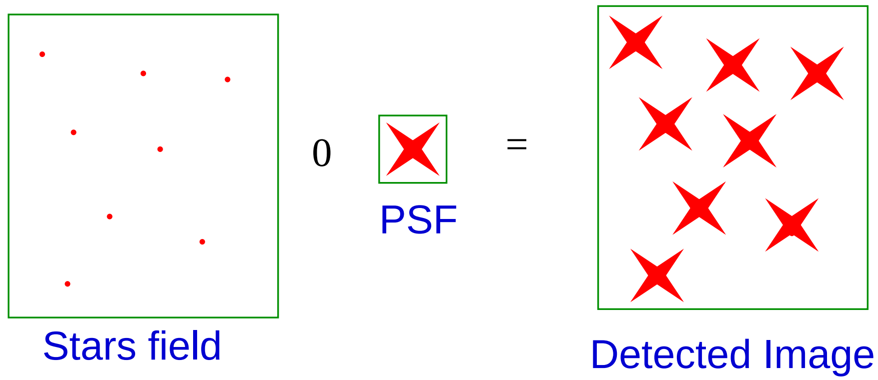
Usually take the approximation that $|F(k,l)|^2 \approx \text{constant}$ to give

$$Y(u,v) = \frac{H^*(k,l)}{|H(k,l)|^2 + \frac{1}{\mathrm{SNR}^2}}$$

Problem of Lowpass effect reducing edges at low SNR. Can improve with Modified Wiener Filter.

Clean Algorithm

Simple model assuming "star" image of isolated points,



able to implement with iterative scheme of:

1. Locate Maximum value in image.
2. Record location and height of PSF.
3. Subtract scaled PSF from image at that location.
4. If any peaks left, go to (1)

Scheme works where part(s) of Fourier Plane is missing, for example Tomographic images.

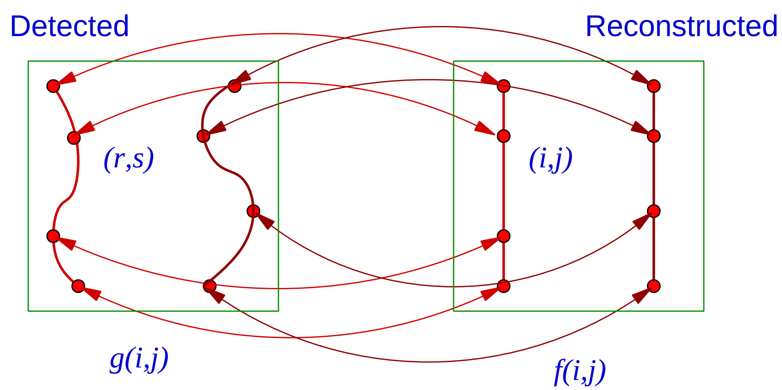
Maximum Entropy

Iterative scheme to form the “smoothest possible” reconstruction given the collected image data.

Excellent reconstructions but computationally very slow.

Geometric Correction

Deal the space variant systems by defining non-linear sampling grid to correct geometric errors,



Reconstruction is then

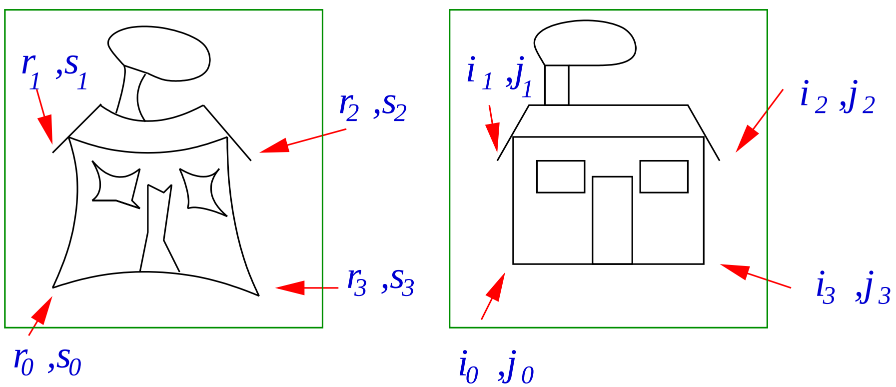
$$f(i,j) = g(r,s)$$

where to second order

$$\begin{aligned} r &= a_0 + a_1 i + a_2 j + a_3 i^2 + a_4 j^2 + a_5 i j \\ s &= b_0 + b_1 i + b_2 j + b_3 i^2 + b_4 j^2 + b_5 i j \end{aligned}$$

Geometric Correction

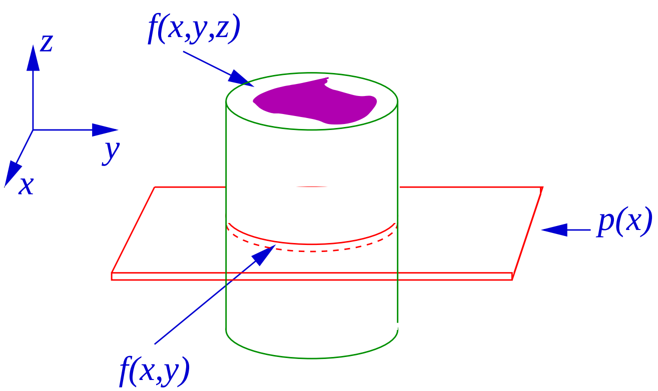
The parameters being obtained from known control points



Used extensively in satellite imagery and automatic map-making

Topic 9: Tomographic Imaging

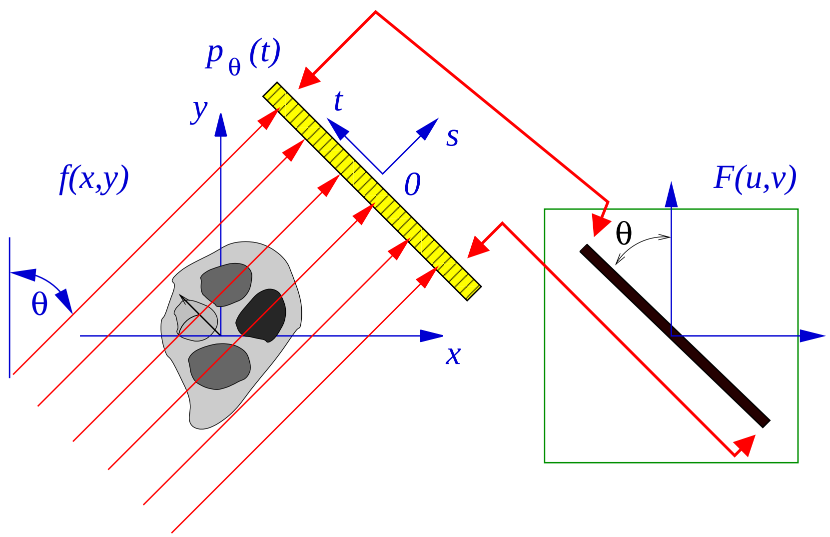
Imaging scheme to image internal structure of objects, usually the human body, by imaging “slides”



which can then be extended to form full three-dimensional reconstruction.

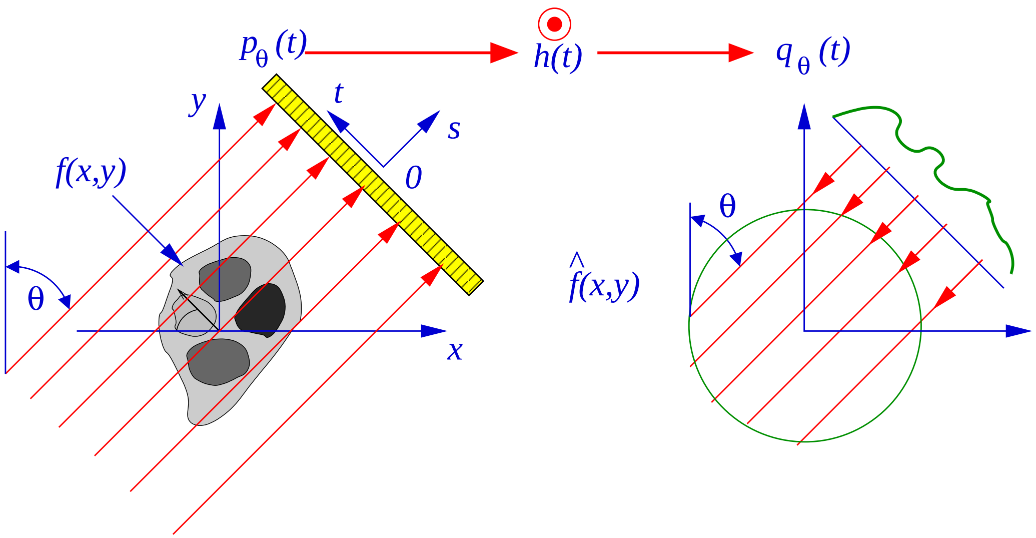
Tomographic Imaging

Collimated Illumination: Under collimated geometry we detect projections that given “strips” in the Fourier plane,



Back Projection

Fourier technique can be reformulated into the Back Projection scheme



which is mathematically identical.

This Back Projection scheme can then be extended to the Fan Beam geometry which is used in practical tomographic systems to produce “Sections” or full “three-dimensional” images.